

Question 5

Use the Ratio Test, without Stirling's formula, to determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges. Show all algebra used to find the ratio limit.

Question 5 – Solution

Step 1: Form and simplify the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}.\end{aligned}$$

Step 2: Use the standard exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

Conclusion.

The series **converges**. This gives the decisive geometric-rate factor $1/e$ directly, without an asymptotic formula for $n!$.