

Question 10

Let $a_n = \cos(1/n)$, where angles are measured in radians.

- (a) Find $\lim_{n \rightarrow \infty} a_n$ using continuity.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain an explicit upper bound for the error $|a_n - L|$.
- (c) Find a simple integer N that guarantees $|a_n - L| < 10^{-4}$ for every $n \geq N$.

Question 10 – Solution

Step 1: Find the limit by continuity.

Since $1/n \rightarrow 0$ and cosine is continuous at 0,

$$L = \lim_{n \rightarrow \infty} \cos(1/n) = \cos 0 = 1.$$

Step 2: Express and bound the error.

Because $\cos(1/n) \leq 1$, the absolute error simplifies to

$$|a_n - L| = 1 - \cos(1/n).$$

Apply $0 \leq 1 - \cos x \leq x^2/2$ with $x = 1/n$:

$$0 \leq |a_n - 1| \leq \frac{1}{2n^2}.$$

Step 3: Solve for a sufficient index.

It is enough to make the upper bound smaller than the desired tolerance:

$$\frac{1}{2n^2} < 10^{-4} \iff n^2 > 5000 \iff n > \sqrt{5000} \approx 70.71.$$

Step 4: State the guarantee precisely.

Thus $N = 71$ guarantees $|a_n - 1| < 10^{-4}$ for every $n \geq 71$, because $1/(2n^2)$ decreases with n . This is a sufficient value obtained from an upper bound; it need not be the smallest possible N for the exact cosine error.