

Question 9

Let $a_n = \frac{(n!)^{1/n}}{n}$ for $n \geq 1$.

- (a) Apply Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ to rewrite a_n as a dominant constant times a factor tending to 1.
- (b) Find $\lim_{n \rightarrow \infty} a_n$ and give its decimal value.
- (c) Explain why the first dozen terms alone do not reveal the limit accurately.

Question 9 – Solution**Step 1: Insert Stirling's approximation.**

Stirling's formula states

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Step 2: Apply the n th root and normalize.All quantities are positive, so taking n th roots and dividing by n gives

$$a_n = \frac{1}{n} (2\pi n)^{1/(2n)} \frac{n}{e} (1 + o(1))^{1/n} = \frac{1}{e} (2\pi n)^{1/(2n)} (1 + o(1))^{1/n}.$$

Step 3: Evaluate the correction factors.

The extra factors tend to 1. For the first factor, take logarithms:

$$\ln \left((2\pi n)^{1/(2n)} \right) = \frac{\ln(2\pi n)}{2n} \longrightarrow 0,$$

so exponentiating gives $(2\pi n)^{1/(2n)} \rightarrow 1$. Also, if $u_n = o(1)$, then $(1 + u_n)^{1/n} \rightarrow 1$ because its logarithm is $\ln(1 + u_n)/n \rightarrow 0$. Consequently,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{e} \approx 0.367879.$$

Step 4: Interpret the numerical evidence.At $n = 12$, $a_{12} \approx 0.4407$, still about 0.0729 above $1/e$. The slowly decaying correction factor $(2\pi n)^{1/(2n)}$ explains why a short numerical table is suggestive but cannot identify the hidden constant reliably.