

Question 8

Define $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

- (a) Prove by induction that $0 \leq a_n < 2$ for every n .
- (b) Prove that (a_n) is increasing, and conclude that it converges.
- (c) Find its limit by solving the appropriate fixed-point equation, and explain why the other algebraic root is inadmissible.

Question 8 – Solution**Step 1: Prove an invariant bound.**

We prove $0 \leq a_n < 2$ by induction. The base case holds because $a_1 = 0$. Assume $0 \leq a_n < 2$. Then

$$0 < 2 + a_n < 4 \implies 0 < a_{n+1} = \sqrt{2 + a_n} < 2.$$

Thus the bound holds for a_{n+1} , completing the induction.

Step 2: Prove monotonicity.

Use induction again. Since $a_2 = \sqrt{2} > a_1$, the base case holds. If $a_n > a_{n-1}$, the square-root function is strictly increasing, so

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{2 + a_{n-1}} = a_n.$$

Thus (a_n) is increasing. Together with the upper bound 2, the Monotone Convergence Theorem guarantees $a_n \rightarrow L$ for some $0 \leq L \leq 2$.

Step 3: Solve the fixed-point equation.

Only after proving convergence may we pass to the limit in the recurrence. Continuity of the square root gives

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0.$$

The algebraic candidates are $L = 2$ and $L = -1$. Because every term and hence its limit are non-negative, $L = -1$ is inadmissible. Therefore $L = 2$. Solving this equation alone would identify only possible limits; boundedness and monotonicity are what prove a limit actually exists.