

Question 7

Let $a_n = \frac{\ln n}{\sqrt{n}}$ for $n \geq 1$.

- (a) Use a continuous extension and L'Hopital's Rule to find the limit.
- (b) Explain why taking logarithms of a_n is not the most direct method.
- (c) Use the derivative of $f(x) = \ln x / \sqrt{x}$ to explain the initial increase of the sequence and identify where the eventual decrease begins.

Question 7 – Solution

Step 1: Choose a continuous extension.

Let $f(x) = \ln x / \sqrt{x}$ for $x \geq 1$. Both numerator and denominator tend to infinity, so the quotient has the indeterminate form ∞/∞ and L'Hopital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Restricting the continuous limit to integer inputs gives $a_n \rightarrow 0$.

Step 2: Compare possible methods.

Taking $\ln(a_n) = \ln(\ln n) - \frac{1}{2} \ln n$ would introduce an $\infty - \infty$ expression and still require comparing logarithmic growth. Applying L'Hopital's Rule directly to the original quotient is cleaner.

Step 3: Determine where the sequence increases and decreases.

Differentiate the extension:

$$f'(x) = x^{-3/2} \left(1 - \frac{1}{2} \ln x \right).$$

Since $x^{-3/2} > 0$, $f'(x) > 0$ for $x < e^2$, equals 0 at $e^2 \approx 7.39$, and is negative afterward. The integer terms rise through $n = 7$ and then decrease very slowly. This initial increase is consistent with the eventual limit of zero.