

Question 6

Let $a_n = n^{1/n}$ for $n \geq 1$.

- (a) Use logarithms to find $\lim_{n \rightarrow \infty} a_n$.
- (b) For which integers n is $a_n > 1$?
- (c) Use the continuous extension $f(x) = x^{1/x}$ to explain why the terms first rise and then decrease, even though their limit is 1.

Question 6 – Solution

Step 1: Transform the variable exponent.

Because $a_n = n^{1/n} > 0$, logarithms are valid. Let $b_n = \ln a_n$. Then

$$b_n = \ln(n^{1/n}) = \frac{\ln n}{n}.$$

Step 2: Evaluate the logarithmic limit.

Extend the quotient to $g(x) = \ln x/x$. It has the indeterminate form ∞/∞ , and L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Restricting this continuous limit to integer values gives $b_n \rightarrow 0$. By continuity of the exponential function, $a_n = e^{b_n} \rightarrow e^0 = 1$.

Step 3: Determine when the terms exceed the limit.

For every integer $n > 1$, the base $n > 1$ and exponent $1/n > 0$, so $a_n > 1$; also $a_1 = 1$. A sequence may approach a limit entirely from one side, so convergence to 1 does not require later terms to equal or cross 1.

Step 4: Explain the rise and fall.

For the continuous extension $f(x) = x^{1/x}$, logarithmic differentiation gives

$$\frac{f'(x)}{f(x)} = \frac{1 - \ln x}{x^2}.$$

Because $f(x) > 0$, the sign of $f'(x)$ is the sign of $1 - \ln x$. Thus f increases for $x < e$ and decreases for $x > e$. Among integer indices, the maximum occurs at $n = 3$, after which the sequence decreases toward 1 while remaining above it.