

Question 4

Let $a_n = \left(1 + \frac{2}{n}\right)^n$.

- (a) Interpret a_n as the balance produced by compounding a principal of 1 at a total rate of 2 over one time unit, using n periods.
- (b) Evaluate $\lim_{n \rightarrow \infty} a_n$ using the standard exponential limit.
- (c) Give a decimal approximation of the limiting balance.

Question 4 – Solution**Step 1: Interpret the compounding model.**

The principal is $P = 1$. Dividing the total rate 2 into n equal periods gives rate $2/n$ per period; applying that growth factor n times produces $1(1 + 2/n)^n$.

Step 2: Rewrite in the standard exponential form.

Put $m = n/2$. Then $m \rightarrow \infty$ and

$$a_n = \left[\left(1 + \frac{1}{m} \right)^m \right]^2.$$

Step 3: Apply the standard limit.

Using $\lim_{m \rightarrow \infty} (1 + 1/m)^m = e$ and continuity of the squaring function gives

$$\lim_{n \rightarrow \infty} a_n = e^2 \approx 7.38906.$$

Step 4: Interpret and verify.

This agrees with the continuous-compounding formula $Pe^{rt} = e^2$ for $P = 1$, $r = 2$, and $t = 1$. The standard exponential limit supplies the proof.