

Question 3

Let $a_n = \frac{n}{n+1}$ for $n \geq 1$.

- (a) Prove algebraically (without derivatives) that (a_n) is strictly increasing.
- (b) Find an upper bound and use the Monotone Convergence Theorem to explain why a limit exists.
- (c) Find the limit and express the error $|a_n - L|$ exactly.

Question 3 – Solution**Step 1: Prove monotonicity algebraically.**

Compare consecutive terms by subtraction:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

The denominator is positive for $n \geq 1$, so $a_{n+1} - a_n > 0$ and (a_n) is strictly increasing.**Step 2: Find an upper bound.**

Rewrite the term as

$$a_n = 1 - \frac{1}{n+1} < 1,$$

Thus $a_n < 1$ for every n , so the sequence is bounded above by 1.**Step 3: Establish convergence and calculate the limit.**

An increasing sequence bounded above converges by the Monotone Convergence Theorem. Using the rewritten formula,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Step 4: Express the error exactly.Since $a_n < 1$, the exact error is $|a_n - 1| = 1 - a_n = 1/(n+1)$, which tends to zero.