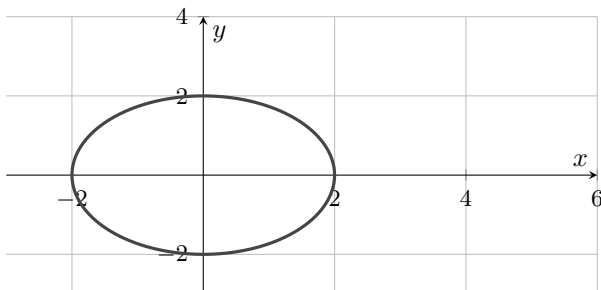


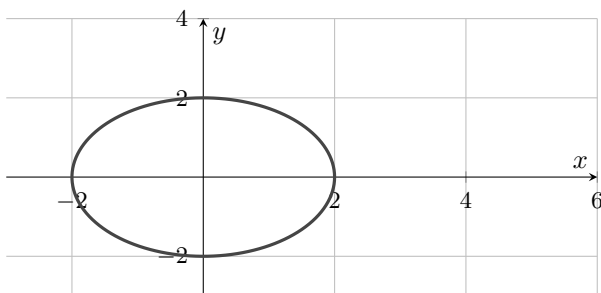
### Question 7

#### Problem

A curve is reparametrized from  $t$  to  $u = t^3$ . Explain why its length is unchanged when both parameters increase.



### Question 7 – Solution



### Solution

1. Let the original curve be  $\mathbf{r}(t)$  for  $a \leq t \leq b$ , and suppose  $u = t^3$  increases on this interval. Then

$$t = u^{1/3}, \quad \frac{dt}{du} = \frac{1}{3u^{2/3}}$$

wherever this derivative is defined.

2. By the chain rule, the velocity in the new parameter is

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \frac{dt}{du}.$$

Taking magnitudes gives

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \left\| \frac{d\mathbf{r}}{dt} \right\| \left| \frac{dt}{du} \right|.$$

3. Because both parameters increase,  $dt/du \geq 0$  and  $dt = (dt/du)du$ . Hence

$$\left\| \frac{d\mathbf{r}}{du} \right\| du = \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$

Integrating over corresponding endpoints yields the same value in either parameter.

4. Therefore, reparametrization changes the rate at which the curve is traced but not the geometric arc length:

$$\boxed{L_u = L_t}.$$

A possible derivative singularity at the single point  $u = 0$  does not change the value of the proper or corresponding improper integral.