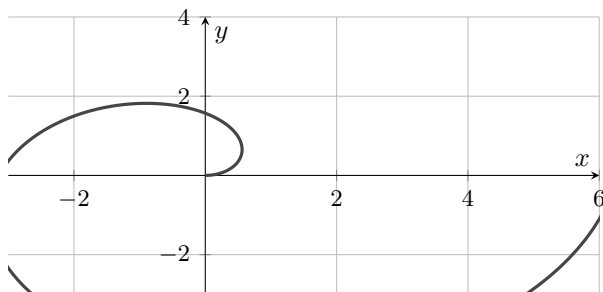


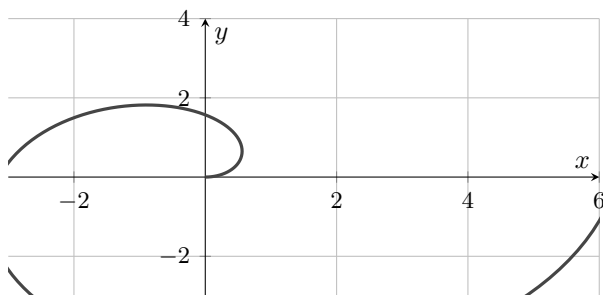
Question 3

Problem

A spiral-like path is $x = t \cos t$, $y = t \sin t$, $0 \leq t \leq 2\pi$. Set up its exact length and simplify the speed.



Question 3 – Solution



Solution

1. Apply the product rule to each coordinate:

$$x'(t) = \cos t - t \sin t, \quad y'(t) = \sin t + t \cos t.$$

2. Square and add:

$$\begin{aligned} x'^2 + y'^2 &= (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 \\ &= \cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) \\ &\quad - 2t \sin t \cos t + 2t \sin t \cos t \\ &= 1 + t^2. \end{aligned}$$

The cross terms cancel exactly.

3. Therefore, the speed is $\sqrt{1 + t^2}$, and the exact length is

$$L = \int_0^{2\pi} \sqrt{1 + t^2} dt.$$

4. If an evaluated form is desired, use

$$\int \sqrt{1 + t^2} dt = \frac{1}{2} \left(t\sqrt{1 + t^2} + \operatorname{arsinh} t \right),$$

giving

$$L = \pi\sqrt{1 + 4\pi^2} + \frac{1}{2} \operatorname{arsinh}(2\pi).$$