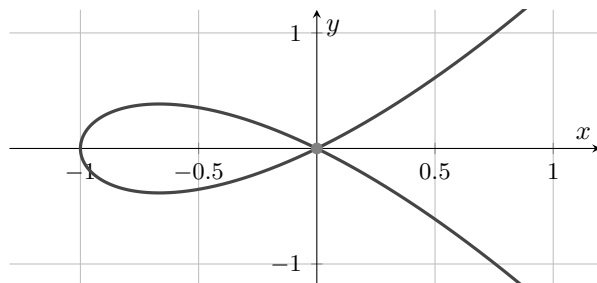


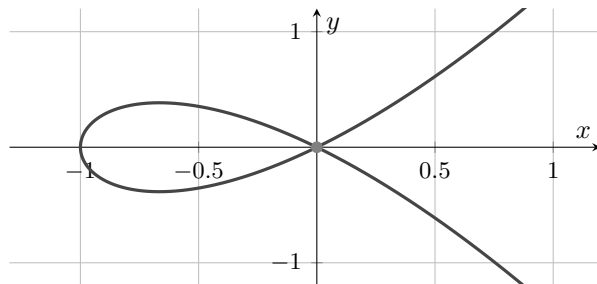
Question 7

Problem

The curve $x = t^2 - 1$, $y = t^3 - t$ crosses itself. Find the two parameter values at the crossing and describe the two tangent directions there.



Question 7 – Solution



Solution

1. Factor the coordinate equations:

$$x = t^2 - 1 = (t - 1)(t + 1), \quad y = t^3 - t = t(t - 1)(t + 1).$$

2. Both coordinates equal zero when $t = 1$ and when $t = -1$:

$$\mathbf{r}(1) = (0, 0), \quad \mathbf{r}(-1) = (0, 0).$$

Because two distinct parameter values produce the same point, the self-intersection is $(0, 0)$.

3. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1.$$

Since $dx/dt \neq 0$ at $t = \pm 1$, the tangent slope is

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}.$$

4. At $t = 1$,

$$\frac{dy}{dx} = \frac{3(1)^2 - 1}{2(1)} = 1,$$

so one tangent line is $y = x$.

5. At $t = -1$,

$$\frac{dy}{dx} = \frac{3(-1)^2 - 1}{2(-1)} = -1,$$

so the other tangent line is $y = -x$.

6. Therefore, the curve crosses itself at the origin with tangent directions of slopes

$$\boxed{1 \text{ and } -1}.$$