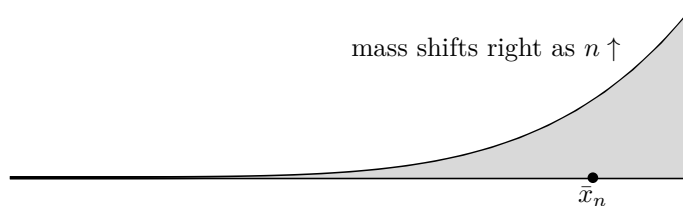
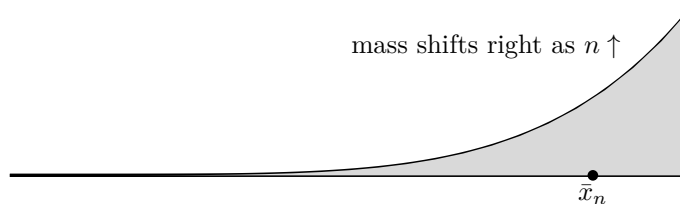


Question 8

A rod on $[0, 1]$ has density $\rho_n(x) = (n+1)x^n$, for integers $n \geq 0$. Find $\bar{x}_n$ and its limiting location as $n \rightarrow \infty$.



Question 8 – Solution



Step 1: Find the total mass. For $n \geq 0$,

$$\begin{aligned} m_n &= \int_0^1 \rho_n(x) dx = \int_0^1 (n+1)x^n dx \\ &= (n+1) \left[\frac{x^{n+1}}{n+1} \right]_0^1 = 1. \end{aligned}$$

Thus the density is normalized to have total mass 1.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_n &= \int_0^1 x \rho_n(x) dx = (n+1) \int_0^1 x^{n+1} dx \\ &= (n+1) \left[\frac{x^{n+2}}{n+2} \right]_0^1 = \frac{n+1}{n+2}. \end{aligned}$$

Step 3: Compute the center of mass.

$$\bar{x}_n = \frac{M_n}{m_n} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2}.$$

Step 4: Take the limit.

$$\lim_{n \rightarrow \infty} \bar{x}_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+2} = 1.$$

The density becomes increasingly concentrated near the right endpoint.

$\bar{x}_n = \frac{n+1}{n+2}, \quad \bar{x}_n \rightarrow 1$
--