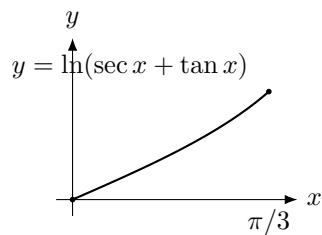


Question 4

Find the exact length of

$$y = \ln(\sec x + \tan x), \quad 0 \leq x \leq \frac{\pi}{3}.$$



Question 4 – Solution**Step 1: Differentiate.**

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Thus

$$L = \int_0^{\pi/3} \sqrt{1 + \sec^2 x} \, dx.$$

Step 2: Substitute $u = \tan x$. Then $du = \sec^2 x \, dx$ and the bounds become 0 and $\sqrt{3}$:

$$L = \int_0^{\sqrt{3}} \frac{\sqrt{u^2 + 2}}{1 + u^2} \, du.$$

Step 3: Substitute $u = \sqrt{2} \sinh t$. Then

$$du = \sqrt{2} \cosh t \, dt, \quad \sqrt{u^2 + 2} = \sqrt{2} \cosh t.$$

The upper bound is $T = \operatorname{arsinh} \sqrt{3/2}$, and

$$\begin{aligned} L &= \int_0^T \frac{2 \cosh^2 t}{1 + 2 \sinh^2 t} \, dt \\ &= \int_0^T (1 + \operatorname{sech}(2t)) \, dt \\ &= \left[t + \frac{1}{2} \arctan(\sinh 2t) \right]_0^T. \end{aligned}$$

Since $\sinh(2T) = \sqrt{15}$,

$$L = \operatorname{arsinh} \sqrt{\frac{3}{2}} + \frac{1}{2} \arctan \sqrt{15}$$