

Question 8

Find all real p for which the integral converges:

$$\int_1^{\infty} \frac{dx}{x^p + 1}.$$

Question 8 – Solution

Step 1: Consider $p > 0$. Compare with $g(x) = 1/x^p$:

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/(x^p + 1)}{1/x^p} \\ &= \lim_{x \rightarrow \infty} \frac{x^p}{x^p + 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + 1/x^p} = 1. \end{aligned}$$

Thus the original integral has the same behavior as the p -integral $\int_1^\infty x^{-p} dx$, which converges exactly when $p > 1$.

Step 2: Consider $p = 0$. The integrand is $1/2$, so the integral diverges.

Step 3: Consider $p < 0$. Then $x^p \rightarrow 0$, so

$$\frac{1}{x^p + 1} \rightarrow 1 \neq 0.$$

For $x \geq 1$ and $p < 0$, $x^p \leq 1$, so the integrand is at least $1/2$. Comparison with $\int_1^\infty (1/2) dx$ proves divergence.

$$\boxed{p > 1}$$