

Question 6

A student uses

$$\frac{1}{\sqrt{x^2 + 1}} < \frac{1}{x}$$

to claim convergence on $[1, \infty)$. Explain the error and classify the integral.

Question 6 – Solution

Step 1: Identify the error. The comparison integral

$$\int_1^{\infty} \frac{dx}{x}$$

diverges. Being smaller than a divergent function does not prove convergence; a smaller positive function may converge or diverge.

Step 2: Use limit comparison with $g(x) = 1/x$.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/\sqrt{x^2 + 1}}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and the harmonic integral diverges, the original integral also diverges.

diverges by limit comparison