

Question 4

For which real a does the integral converge? Evaluate it when possible.

$$\int_0^{\infty} e^{-ax} dx$$

Question 4 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b e^{-ax} dx.$$

Step 2: Suppose $a \neq 0$ and integrate.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{a} e^{-ax} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \frac{1 - e^{-ab}}{a}. \end{aligned}$$

If $a > 0$, then $e^{-ab} \rightarrow 0$, so $I = 1/a$. If $a < 0$, then $e^{-ab} \rightarrow \infty$, so the integral diverges.

Step 3: Check $a = 0$. Then the integrand is 1, and

$$\int_0^\infty 1 dx$$

diverges.

$a > 0 : 1/a; \quad a \leq 0 : \text{diverges}$
