

Question 10

For $a > 0$, evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2}.$$

Question 10 – Solution

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 1: Write the improper integral as a limit.**

$$I = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + a^2}.$$

Step 2: Find an antiderivative. Since $a > 0$,

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

Therefore,

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \frac{1}{a} \left[\arctan\left(\frac{x}{a}\right) \right]_{-R}^R \\ &= \frac{1}{a} \lim_{R \rightarrow \infty} \left[\arctan\left(\frac{R}{a}\right) - \arctan\left(\frac{-R}{a}\right) \right]. \end{aligned}$$

Step 3: Evaluate the limits.

$$\begin{aligned} I &= \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\ &= \frac{\pi}{a}. \end{aligned}$$

$$\boxed{I = \frac{\pi}{a}}$$