

Question 8

Evaluate

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

using a substitution that clears both radicals.

Question 8 – Solution

Step 1: Choose a power that clears both radicals. Since $\text{lcm}(2, 3) = 6$, let $x = u^6$. Then

$$\sqrt{x} = u^3, \quad \sqrt[3]{x} = u^2, \quad dx = 6u^5 du.$$

The bounds remain 0 and 1 because $u = x^{1/6}$. **Step 2: Rewrite and simplify.**

$$\begin{aligned} I &= 6 \int_0^1 \frac{u^5}{u^3 + u^2} du \\ &= 6 \int_0^1 \frac{u^5}{u^2(u+1)} du = 6 \int_0^1 \frac{u^3}{u+1} du. \end{aligned}$$

Step 3: Divide the polynomials.

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Step 4: Integrate and evaluate the bounds.

$$\begin{aligned} I &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1) \right]_0^1 \\ &= 6 \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2 \right) \\ &= 6 \left(\frac{5}{6} - \ln 2 \right) = 5 - 6 \ln 2. \end{aligned}$$

$$\boxed{I = 5 - 6 \ln 2}$$