

Question 7

Evaluate

$$\int \frac{dx}{x\sqrt[3]{x+1}}$$

using $u = \sqrt[3]{x+1}$.

Question 7 – Solution

Step 1: Substitute $u = \sqrt[3]{x+1}$. Then

$$u^3 = x + 1, \quad x = u^3 - 1, \quad dx = 3u^2 du.$$

Step 2: Rewrite and factor.

$$I = \int \frac{3u^2 du}{(u^3 - 1)u} = 3 \int \frac{u}{u^3 - 1} du,$$
$$u^3 - 1 = (u - 1)(u^2 + u + 1).$$

Step 3: Use partial fractions.

$$\frac{3u}{u^3 - 1} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + u + 1}.$$

After clearing denominators,

$$3u = A(u^2 + u + 1) + (Bu + C)(u - 1).$$

Matching coefficients gives

$$A + B = 0, \quad A - B + C = 3, \quad A - C = 0,$$

so $A = 1$, $B = -1$, and $C = 1$. Therefore,

$$\frac{3u}{u^3 - 1} = \frac{1}{u - 1} + \frac{-u + 1}{u^2 + u + 1}.$$

Step 4: Prepare the quadratic term.

$$-u + 1 = -\frac{1}{2}(2u + 1) + \frac{3}{2}.$$

Thus

$$I = \int \frac{du}{u - 1} - \frac{1}{2} \int \frac{2u + 1}{u^2 + u + 1} du + \frac{3}{2} \int \frac{du}{u^2 + u + 1}.$$

Complete the square:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Step 5: Integrate and return to x .

$$I = \ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C,$$

where $u = \sqrt[3]{x+1}$.

$$\ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$