

Question 10

1. Define and evaluate the Cauchy principal value:

$$\text{PV} \int_{-1}^1 \frac{dx}{x}$$

2. Does the ordinary improper integral converge? Check both sides of $x = 0$.

Question 10 – Solution**Part 1: Cauchy principal value.****Step 1: Approach zero by the same distance ε from both sides.**

$$\begin{aligned}\text{PV} \int_{-1}^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^1 \frac{dx}{x} \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} ([\ln |x|]_{-1}^{-\varepsilon} + [\ln |x|]_{\varepsilon}^1) \\ &= \lim_{\varepsilon \rightarrow 0^+} ((\ln \varepsilon - \ln 1) + (\ln 1 - \ln \varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0^+} 0 = 0.\end{aligned}$$

$$\boxed{\text{PV} \int_{-1}^1 \frac{dx}{x} = 0}.$$

Part 2: Ordinary improper integral.

The two one-sided integrals must converge separately. On the right,

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (0 - \ln \varepsilon) = +\infty.\end{aligned}$$

On the left,

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln |x|]_{-1}^{-\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - 0) = -\infty.\end{aligned}$$

Therefore, the ordinary improper integral diverges. The principal value is finite only because these opposite divergences cancel in a symmetric limit.