

Question 7

Use partial fractions to evaluate exactly:

$$\int_0^{1/2} \frac{dx}{1-x^2}.$$

Why is this integral proper on $[0, 1/2]$?

Question 7 – Solution

Step 1: Factor and set up partial fractions.

$$1 - x^2 = (1 - x)(1 + x),$$

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}.$$

Step 2: Clear denominators and find A and B .

$$1 = A(1 + x) + B(1 - x).$$

$$x = 1 : \quad 1 = 2A \implies A = \frac{1}{2},$$

$$x = -1 : \quad 1 = 2B \implies B = \frac{1}{2}.$$

Step 3: Integrate and evaluate both bounds.

$$\begin{aligned} I &= \frac{1}{2} \int_0^{1/2} \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) dx \\ &= \frac{1}{2} [-\ln(1 - x) + \ln(1 + x)]_0^{1/2} \\ &= \frac{1}{2} \left[-\ln \frac{1}{2} + \ln \frac{3}{2} - (-\ln 1 + \ln 1) \right] \\ &= \frac{1}{2} \ln \left(\frac{3/2}{1/2} \right) = \frac{1}{2} \ln 3. \end{aligned}$$

Step 4: Check whether the integral is proper. The singularities $x = \pm 1$ are outside $[0, 1/2]$, so the integrand is continuous on the whole interval and no improper limit is needed.

$$\boxed{\int_0^{1/2} \frac{dx}{1 - x^2} = \frac{1}{2} \ln 3.}$$