

Question 4

Use partial fractions to evaluate:

$$\int \frac{x^2 + 1}{x(x^2 + 4)} dx.$$

Question 4 – Solution

Step 1: Set up the decomposition. Use a linear numerator over the irreducible quadratic:

$$\frac{x^2 + 1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Step 2: Clear denominators and collect like terms.

$$\begin{aligned}x^2 + 1 &= A(x^2 + 4) + x(Bx + C) \\&= (A + B)x^2 + Cx + 4A.\end{aligned}$$

Step 3: Match coefficients.

$$A + B = 1, \quad C = 0, \quad 4A = 1.$$

Therefore $A = 1/4$, $B = 1 - 1/4 = 3/4$, and $C = 0$.

Step 4: Integrate. For the second term, let $u = x^2 + 4$; then $du = 2x \, dx$.

$$\begin{aligned}I &= \frac{1}{4} \int \frac{dx}{x} + \frac{3}{4} \int \frac{x}{x^2 + 4} \, dx \\&= \frac{1}{4} \ln|x| + \frac{3}{4} \left(\frac{1}{2} \int \frac{du}{u} \right) \\&= \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C.\end{aligned}$$

Thus,

$$\boxed{I = \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C}.$$