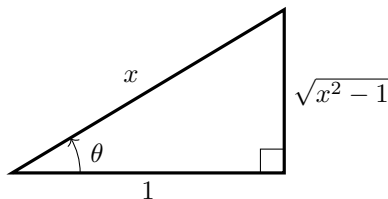


Question 10

Use a trigonometric or hyperbolic substitution to evaluate

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx, \quad x > 1.$$

Question 10 – Solution



Because $x > 1$, let

$$x = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Then

$$dx = \sec \theta \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \tan \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{\tan \theta}{\sec^2 \theta} (\sec \theta \tan \theta) \, d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} \, d\theta \\ &= \int \frac{\sec^2 \theta - 1}{\sec \theta} \, d\theta \\ &= \int (\sec \theta - \cos \theta) \, d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C. \end{aligned}$$

Back-substitute using

$$\sec \theta = x, \quad \tan \theta = \sqrt{x^2 - 1}, \quad \sin \theta = \frac{\sqrt{x^2 - 1}}{x}.$$

Since $x > 1$, the logarithm's argument is positive. Therefore,

$$I = \ln \left(x + \sqrt{x^2 - 1} \right) - \frac{\sqrt{x^2 - 1}}{x} + C.$$