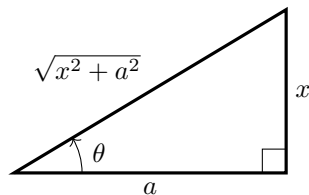


Question 9

For $a > 0$, derive the antiderivative

$$\int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

Question 9 – Solution



Let

$$x = a \tan \theta, \quad dx = a \sec^2 \theta d\theta.$$

Since $a > 0$, choose $-\pi/2 < \theta < \pi/2$. Then

$$\begin{aligned} (a^2 + x^2)^{3/2} &= (a^2 + a^2 \tan^2 \theta)^{3/2} \\ &= (a^2 \sec^2 \theta)^{3/2} = a^3 \sec^3 \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \int \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta \\ &= \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{1}{a^2} \sin \theta + C. \end{aligned}$$

From a reference triangle,

$$\tan \theta = \frac{x}{a}, \quad \sin \theta = \frac{x}{\sqrt{a^2 + x^2}}.$$

Thus,

$$I = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$$

Indeed,

$$\frac{d}{dx} \left(\frac{x}{a^2 \sqrt{a^2 + x^2}} \right) = \frac{1}{(a^2 + x^2)^{3/2}},$$

which verifies the result.