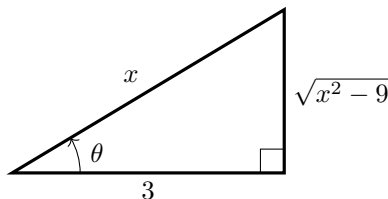


Question 8

Evaluate the definite integral exactly:

$$\int_3^5 \frac{\sqrt{x^2 - 9}}{x} dx.$$

Question 8 – Solution



Let

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta \, d\theta.$$

For $x \geq 3$, take $0 \leq \theta < \pi/2$. Then

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

Transform the bounds:

$$x = 3 \Rightarrow \sec \theta = 1 \Rightarrow \theta = 0, \quad x = 5 \Rightarrow \cos \theta = \frac{3}{5} \Rightarrow \theta = \arccos\left(\frac{3}{5}\right).$$

Thus,

$$\begin{aligned} I &= \int_0^{\arccos(3/5)} \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) \, d\theta \\ &= 3 \int_0^{\arccos(3/5)} \tan^2 \theta \, d\theta \\ &= 3 [\tan \theta - \theta]_0^{\arccos(3/5)}. \end{aligned}$$

At the upper bound, a 3-4-5 triangle gives $\tan \theta = 4/3$. Therefore,

$$\begin{aligned} I &= 3 \left(\frac{4}{3} - \arccos\left(\frac{3}{5}\right) \right) \\ &= 4 - 3 \arccos\left(\frac{3}{5}\right). \end{aligned}$$

Hence,

$$\boxed{I = 4 - 3 \arccos\left(\frac{3}{5}\right)}.$$