

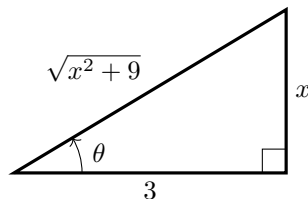
Question 7

For

$$\int \frac{dx}{\sqrt{x^2 + 9}},$$

two substitutions are proposed: $x = 3 \tan \theta$ and $x = 3 \sin \theta$. Select the efficient substitution, explain the identity it uses, and evaluate the integral.

Question 7 – Solution



Choose

$$x = 3 \tan \theta.$$

This substitution uses $1 + \tan^2 \theta = \sec^2 \theta$:

$$x^2 + 9 = 9 \tan^2 \theta + 9 = 9 \sec^2 \theta.$$

Taking $-\pi/2 < \theta < \pi/2$ gives $\sec \theta > 0$, so

$$dx = 3 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 9} = 3 \sec \theta.$$

Hence,

$$\begin{aligned} I &= \int \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Since $\tan \theta = x/3$ and $\sec \theta = \sqrt{x^2 + 9}/3$,

$$\begin{aligned} I &= \ln \left| \frac{x + \sqrt{x^2 + 9}}{3} \right| + C \\ &= \ln |x + \sqrt{x^2 + 9}| + C. \end{aligned}$$

The constant $-\ln 3$ is absorbed into C . Thus,

$$\int \frac{dx}{\sqrt{x^2 + 9}} = \ln |x + \sqrt{x^2 + 9}| + C.$$

The sine substitution does not turn $x^2 + 9$ into a single squared trigonometric function, so it does not simplify the radical.