

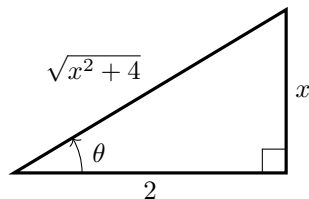
Question 6

Evaluate

$$\int \sqrt{x^2 + 4} \, dx$$

using trigonometric substitution, and verify the result by differentiation.

Question 6 – Solution



Let $x = 2 \tan \theta$, with $-\pi/2 < \theta < \pi/2$. Then

$$dx = 2 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 4} = 2 \sec \theta.$$

Therefore,

$$I = 4 \int \sec^3 \theta d\theta.$$

Using $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$,

$$I = 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C.$$

Now $\tan \theta = x/2$ and $\sec \theta = \sqrt{x^2 + 4}/2$. Hence,

$$\begin{aligned} I &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| + C \\ &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C, \end{aligned}$$

where $-2 \ln 2$ is absorbed into C . Thus,

$$\boxed{I = \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C}.$$

Differentiating and simplifying gives $I' = \sqrt{x^2 + 4}$, as required.