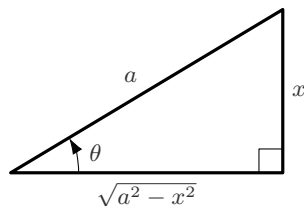


Question 1

Evaluate the integral both geometrically and analytically:

$$\int_0^a \sqrt{a^2 - x^2} \, dx, \quad a > 0.$$

Question 1 – Solution



Geometrically, $y = \sqrt{a^2 - x^2}$ is the upper semicircle $x^2 + y^2 = a^2$. On $0 \leq x \leq a$, the region is one quarter of a disk of radius a . Hence its area is

$$I = \frac{1}{4}\pi a^2.$$

For an analytic verification, let

$$x = a \sin \theta, \quad dx = a \cos \theta d\theta.$$

Since $0 \leq x \leq a$, the new bounds are $0 \leq \theta \leq \pi/2$. Also,

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a \cos \theta,\end{aligned}$$

because $\cos \theta \geq 0$ on this interval. Therefore,

$$\begin{aligned}I &= a^2 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi a^2}{4}.\end{aligned}$$

Thus both methods give

$$\boxed{\int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^2}{4}}.$$