

Question 8

Evaluate

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx$$

using a substitution that produces reciprocal powers of the new variable.

Question 8 – Solution

Separate one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^3(x) = \sin(x)(1 - \cos^2(x)).$$

Then

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx = \int \frac{1 - \cos^2(x)}{\cos^4(x)} \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

The integral becomes

$$\begin{aligned} - \int \frac{1 - u^2}{u^4} du &= - \int (u^{-4} - u^{-2}) du \\ &= - \left(-\frac{1}{3}u^{-3} + u^{-1} \right) + C \\ &= \frac{1}{3}u^{-3} - u^{-1} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \frac{\sin^3(x)}{\cos^4(x)} dx = \frac{1}{3 \cos^3(x)} - \frac{1}{\cos(x)} + C}.$$

Equivalently, the answer is $\frac{1}{3} \sec^3(x) - \sec(x) + C$ on intervals where the original integrand is defined.