

Question 7

Determine all integers $m, n \geq 1$ for which

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx \neq 0.$$

Justify your conclusion using a product-to-sum identity.

Question 7 – Solution

Use

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Setting $A = mx$ and $B = nx$ gives

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m + n)x) + \sin((m - n)x)]. \quad (1)$$

For every nonzero integer k ,

$$\begin{aligned} \int_0^{2\pi} \sin(kx) dx &= \left[-\frac{\cos(kx)}{k} \right]_0^{2\pi} \\ &= \frac{-\cos(2\pi k) + \cos(0)}{k} = 0, \end{aligned}$$

because $\cos(2\pi k) = 1$. Since $m + n \geq 2$, the first sine term in equation (1) always integrates to zero.

If $m \neq n$, then $m - n$ is a nonzero integer, so the second term also integrates to zero. If $m = n$, then

$$\sin((m - n)x) = \sin(0) = 0,$$

so it again contributes zero. Thus, in every case,

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0.$$

Therefore, there are no integers $m, n \geq 1$ for which the integral is nonzero:

There are no such integer pairs (m, n) .