

Question 2

Use power-reduction identities to evaluate

$$\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx.$$

Question 2 – Solution

Begin with the double-angle identity

$$\sin(2x) = 2 \sin(x) \cos(x).$$

Squaring both sides gives

$$\sin^2(x) \cos^2(x) = \frac{1}{4} \sin^2(2x).$$

Now use $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ with $\theta = 2x$:

$$\sin^2(x) \cos^2(x) = \frac{1}{8} (1 - \cos(4x)).$$

Therefore,

$$\begin{aligned} \int_0^{\pi/2} \sin^2(x) \cos^2(x) dx &= \frac{1}{8} \int_0^{\pi/2} (1 - \cos(4x)) dx \\ &= \frac{1}{8} \left[x - \frac{1}{4} \sin(4x) \right]_0^{\pi/2}. \end{aligned}$$

Evaluate the endpoints:

$$\begin{aligned} \frac{1}{8} \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left(0 - \frac{1}{4} \sin(0) \right) \right] &= \frac{1}{8} \left(\frac{\pi}{2} \right) \\ &= \frac{\pi}{16}. \end{aligned}$$

Hence,

$$\boxed{\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx = \frac{\pi}{16}}.$$