

Question 10

For $n > 0$, evaluate the improper integral

$$J_n = \int_0^1 x^n \ln(x) \, dx.$$

Then describe the sign and limiting behavior of J_n as $n \rightarrow \infty$.

Question 10 – Solution

Because $\ln(x) \rightarrow -\infty$ as $x \rightarrow 0^+$, write the integral as

$$J_n = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^n \ln(x) dx.$$

For $\varepsilon > 0$, use integration by parts with

$$u = \ln(x), \quad dv = x^n dx, \quad du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}.$$

This choice is valid because $n > 0$, so in particular $n+1 \neq 0$. Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 x^n \ln(x) dx &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{1}{n+1} \int_{\varepsilon}^1 x^n dx \\ &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{x^{n+1}}{(n+1)^2} \Big|_{\varepsilon}^1. \end{aligned} \tag{1}$$

We must evaluate the boundary term at $x = 0$. Since $n+1 > 0$,

$$\lim_{x \rightarrow 0^+} x^{n+1} \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}}.$$

This has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}} &= \lim_{x \rightarrow 0^+} \frac{1/x}{-(n+1)x^{-(n+2)}} \\ &= \lim_{x \rightarrow 0^+} -\frac{x^{n+1}}{n+1} = 0. \end{aligned}$$

Also, $\lim_{x \rightarrow 0^+} x^{n+1} = 0$. At the upper endpoint, $\ln(1) = 0$. Therefore, taking $\varepsilon \rightarrow 0^+$ in equation (1) gives

$$\begin{aligned} J_n &= \frac{1^{n+1} \ln(1)}{n+1} - 0 - \frac{1^{n+1} - 0}{(n+1)^2} \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\boxed{J_n = -\frac{1}{(n+1)^2}}.$$

For every $n > 0$, $J_n < 0$, which agrees with $\ln(x) < 0$ on $0 < x < 1$. Moreover, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} -\frac{1}{(n+1)^2} = 0.$$

The values increase toward zero through negative values. In short,

$$\boxed{J_n \rightarrow 0^- \quad \text{as} \quad n \rightarrow \infty}.$$