

Question 9

Evaluate

$$\int_{-1}^1 x \arctan(x) dx.$$

Before using integration by parts, exploit the symmetry of the integrand to simplify the calculation.

Question 9 – Solution

Let

$$f(x) = x \arctan(x).$$

Both x and $\arctan(x)$ are odd functions. Therefore, their product is even:

$$\begin{aligned} f(-x) &= (-x) \arctan(-x) \\ &= (-x)(-\arctan(x)) \\ &= x \arctan(x) = f(x). \end{aligned}$$

Consequently,

$$\int_{-1}^1 x \arctan(x) dx = 2 \int_0^1 x \arctan(x) dx. \quad (1)$$

Now use integration by parts on the integral over $[0, 1]$. Choose

$$u = \arctan(x), \quad dv = x dx, \quad du = \frac{1}{1+x^2} dx, \quad v = \frac{x^2}{2}.$$

Then

$$\int_0^1 x \arctan(x) dx = \frac{x^2}{2} \arctan(x) \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx.$$

Multiplying by 2 as in equation (1) gives

$$\int_{-1}^1 x \arctan(x) dx = x^2 \arctan(x) \Big|_0^1 - \int_0^1 \frac{x^2}{1+x^2} dx. \quad (2)$$

For the remaining integral, rewrite the integrand:

$$\frac{x^2}{1+x^2} = \frac{(1+x^2) - 1}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^2} dx &= \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \\ &= [x - \arctan(x)]_0^1 \\ &= 1 - \frac{\pi}{4}. \end{aligned}$$

The boundary term in equation (2) is

$$x^2 \arctan(x) \Big|_0^1 = 1^2 \arctan(1) - 0^2 \arctan(0) = \frac{\pi}{4}.$$

Substituting both results into equation (2),

$$\begin{aligned} \int_{-1}^1 x \arctan(x) dx &= \frac{\pi}{4} - \left(1 - \frac{\pi}{4} \right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Thus,

$$\boxed{\int_{-1}^1 x \arctan(x) dx = \frac{\pi}{2} - 1}.$$

The answer is positive, as expected, because the even integrand is nonnegative on $[-1, 1]$.