

Question 8

Reverse engineer an antiderivative by finding a quadratic polynomial $P(x)$ such that

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1).$$

Determine $P(x)$ and verify that it satisfies the equation.

Question 8 – Solution

We want to choose $P(x)$ so that differentiating $e^x P(x)$ produces the given expression. By the product rule,

$$\begin{aligned}\frac{d}{dx} [e^x P(x)] &= e^x P(x) + e^x P'(x) \\ &= e^x (P(x) + P'(x)).\end{aligned}$$

Thus, the required identity

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1)$$

will hold if

$$P(x) + P'(x) = x^2 + 1. \tag{1}$$

Because P is quadratic, write

$$P(x) = ax^2 + bx + c,$$

where a , b , and c are constants. Then

$$P'(x) = 2ax + b.$$

Adding $P(x)$ and $P'(x)$ gives

$$\begin{aligned}P(x) + P'(x) &= (ax^2 + bx + c) + (2ax + b) \\ &= ax^2 + (b + 2a)x + (c + b).\end{aligned}$$

Equation (1) requires this polynomial to equal

$$x^2 + 1 = 1x^2 + 0x + 1.$$

Matching coefficients of like powers of x yields

$$\begin{array}{rcl}a & = & 1, \quad \text{coefficient of } x^2, \\ b + 2a & = & 0, \quad \text{coefficient of } x, \\ c + b & = & 1, \quad \text{constant term.}\end{array}$$

Solve the equations in order. Since $a = 1$,

$$b + 2(1) = 0 \quad \implies \quad b = -2.$$

Then

$$c + (-2) = 1 \quad \implies \quad c = 3.$$

Therefore,

$$\boxed{P(x) = x^2 - 2x + 3}.$$

Finally, verify the result. Since $P'(x) = 2x - 2$,

$$\begin{aligned}\frac{d}{dx} [e^x (x^2 - 2x + 3)] &= e^x (x^2 - 2x + 3) + e^x (2x - 2) \\ &= e^x (x^2 - 2x + 3 + 2x - 2) \\ &= e^x (x^2 + 1),\end{aligned}$$

which is the required expression.