

Question 7

Find an antiderivative F of

$$f(x) = \arcsin(x), \quad -1 < x < 1,$$

that satisfies the initial condition

$$F(0) = 2.$$

Question 7 – Solution

We begin by evaluating

$$\int \arcsin(x) dx.$$

Since the integrand can be viewed as $\arcsin(x) \cdot 1$, use integration by parts with

$$u = \arcsin(x), \quad dv = dx, \quad du = \frac{1}{\sqrt{1-x^2}} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\int \arcsin(x) dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx. \quad (1)$$

To evaluate the remaining integral, let

$$w = 1 - x^2.$$

Then

$$dw = -2x dx, \quad x dx = -\frac{1}{2} dw.$$

Therefore,

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int w^{-1/2} dw \\ &= -\frac{1}{2} (2w^{1/2}) \\ &= -\sqrt{w} \\ &= -\sqrt{1-x^2}. \end{aligned}$$

Substitute this result into equation (1):

$$\begin{aligned} \int \arcsin(x) dx &= x \arcsin(x) - (-\sqrt{1-x^2}) + C \\ &= x \arcsin(x) + \sqrt{1-x^2} + C. \end{aligned}$$

Thus, the general antiderivative is

$$F(x) = x \arcsin(x) + \sqrt{1-x^2} + C.$$

Now apply the condition $F(0) = 2$:

$$\begin{aligned} F(0) &= 0 \cdot \arcsin(0) + \sqrt{1-0^2} + C \\ &= 1 + C. \end{aligned}$$

Hence,

$$1 + C = 2 \quad \implies \quad C = 1.$$

Therefore, the required antiderivative is

$$\boxed{F(x) = x \arcsin(x) + \sqrt{1-x^2} + 1}.$$

To verify, differentiate:

$$\begin{aligned} F'(x) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x). \end{aligned}$$

Also, $F(0) = 0 + 1 + 1 = 2$, so both requirements are satisfied.