

Question 6

Use integration by parts to prove that

$$\int_0^1 \ln(x) \, dx = -1.$$

Because $\ln(x)$ is unbounded as $x \rightarrow 0^+$, be sure to treat the lower endpoint correctly as an improper integral.

Question 6 – Solution

The function $\ln(x)$ is not defined at $x = 0$ and satisfies

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty.$$

Therefore, we must interpret the integral as

$$\int_0^1 \ln(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln(x) dx.$$

For $\varepsilon > 0$, evaluate the proper integral by integration by parts. Choose

$$u = \ln(x), \quad dv = dx, \quad du = \frac{1}{x} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 x \left(\frac{1}{x} \right) dx \\ &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 1 dx \\ &= [x \ln(x) - x]_{\varepsilon}^1. \end{aligned}$$

Now substitute the bounds:

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= (1 \ln(1) - 1) - (\varepsilon \ln(\varepsilon) - \varepsilon) \\ &= -1 - \varepsilon \ln(\varepsilon) + \varepsilon. \end{aligned}$$

It remains to evaluate the indeterminate endpoint term. Rewrite it as

$$\varepsilon \ln(\varepsilon) = \frac{\ln(\varepsilon)}{1/\varepsilon}.$$

As $\varepsilon \rightarrow 0^+$, this has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon \ln(\varepsilon) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} \\ &= \lim_{\varepsilon \rightarrow 0^+} (-\varepsilon) = 0. \end{aligned}$$

Also, $\lim_{\varepsilon \rightarrow 0^+} \varepsilon = 0$. Hence,

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln(\varepsilon) + \varepsilon) \\ &= -1. \end{aligned}$$

Therefore, the improper integral converges and

$$\boxed{\int_0^1 \ln(x) dx = -1}.$$

This sign is reasonable because $\ln(x) < 0$ for $0 < x < 1$.