

Question 5

Evaluate the cyclic integral

$$\int e^{2x} \cos(3x) dx$$

using integration by parts. Do not guess the form of the antiderivative.

Question 5 – Solution

Let

$$I = \int e^{2x} \cos(3x) dx.$$

For the first integration by parts, choose

$$u = \cos(3x), \quad dv = e^{2x} dx, \quad du = -3 \sin(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Using $\int u dv = uv - \int v du$, we get

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) - \int \frac{1}{2} e^{2x} (-3 \sin(3x)) dx \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) dx. \end{aligned} \tag{1}$$

Now evaluate the remaining integral. Let

$$J = \int e^{2x} \sin(3x) dx.$$

Use integration by parts again with

$$u = \sin(3x), \quad dv = e^{2x} dx, \quad du = 3 \cos(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Then

$$\begin{aligned} J &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I. \end{aligned} \tag{2}$$

Substitute equation (2) into equation (1):

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I \right) \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) - \frac{9}{4} I. \end{aligned}$$

Move the I -term to the left and solve:

$$\begin{aligned} I + \frac{9}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ \frac{13}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ I &= \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)). \end{aligned}$$

Therefore,

$$\boxed{\int e^{2x} \cos(3x) dx = \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) + C}$$

To check, differentiate the antiderivative:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) \right] &= \frac{e^{2x}}{13} [2(2 \cos(3x) + 3 \sin(3x)) \\ &\quad - 6 \sin(3x) + 9 \cos(3x)] \\ &= e^{2x} \cos(3x), \end{aligned}$$

which is the original integrand.