

Question 4

Define

$$I_n = \int x^n e^x dx, \quad n \geq 1.$$

Use integration by parts to derive a recurrence relation connecting I_n and I_{n-1} . Then use the recurrence to evaluate I_4 .

Question 4 – Solution

Start with

$$I_n = \int x^n e^x dx.$$

To reduce the power of x , choose

$$u = x^n, \quad dv = e^x dx, \quad du = nx^{n-1} dx, \quad v = e^x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} I_n &= x^n e^x - \int e^x (nx^{n-1}) dx \\ &= x^n e^x - n \int x^{n-1} e^x dx. \end{aligned}$$

Since $I_{n-1} = \int x^{n-1} e^x dx$, the recurrence relation is

$$\boxed{I_n = x^n e^x - n I_{n-1}}.$$

Here, as usual for indefinite integrals, the constant of integration is added after the recurrence has been applied. The base case is

$$I_0 = \int e^x dx = e^x.$$

Now apply the recurrence repeatedly:

$$\begin{aligned} I_1 &= x e^x - I_0 = x e^x - e^x = e^x(x - 1), \\ I_2 &= x^2 e^x - 2I_1 \\ &= x^2 e^x - 2e^x(x - 1) = e^x(x^2 - 2x + 2), \\ I_3 &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3e^x(x^2 - 2x + 2) \\ &= e^x(x^3 - 3x^2 + 6x - 6), \\ I_4 &= x^4 e^x - 4I_3 \\ &= x^4 e^x - 4e^x(x^3 - 3x^2 + 6x - 6) \\ &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24). \end{aligned}$$

Therefore,

$$\boxed{I_4 = \int x^4 e^x dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C}$$

Indeed, differentiating the result gives

$$\begin{aligned} \frac{d}{dx} [e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)] &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) \\ &\quad + e^x(4x^3 - 12x^2 + 24x - 24) \\ &= x^4 e^x, \end{aligned}$$

which confirms the answer.