

Question 3

Consider the integral

$$\int x^2 e^x dx.$$

A student makes the integration-by-parts choice

$$u = e^x, \quad dv = x^2 dx.$$

Explain why this choice does not simplify the integral. Then evaluate the integral efficiently using a better choice of u and dv .

Question 3 – Solution

Recall the integration-by-parts formula

$$\int u \, dv = uv - \int v \, du.$$

With the student's choice $u = e^x$ and $dv = x^2 \, dx$, we have

$$du = e^x \, dx, \quad v = \frac{x^3}{3}.$$

Therefore,

$$\begin{aligned} \int x^2 e^x \, dx &= e^x \left(\frac{x^3}{3} \right) - \int \frac{x^3}{3} e^x \, dx \\ &= \frac{x^3 e^x}{3} - \frac{1}{3} \int x^3 e^x \, dx. \end{aligned}$$

The polynomial degree has increased from 2 to 3, so the remaining integral is more complicated. Repeating this choice would keep increasing the degree.

Instead, choose

$$u = x^2, \quad dv = e^x \, dx, \quad du = 2x \, dx, \quad v = e^x.$$

Applying integration by parts gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

For the remaining integral, use integration by parts again:

$$u = x, \quad dv = e^x \, dx, \quad du = dx, \quad v = e^x.$$

Thus,

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x. \end{aligned}$$

Substituting this result into the original calculation gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C. \end{aligned}$$

$$\boxed{\int x^2 e^x \, dx = e^x (x^2 - 2x + 2) + C}$$

Check by differentiating:

$$\begin{aligned} \frac{d}{dx} [e^x (x^2 - 2x + 2)] &= e^x (x^2 - 2x + 2) + e^x (2x - 2) \\ &= x^2 e^x, \end{aligned}$$

which is the original integrand.