

Question 2

Find the unique constant a such that

$$\int_0^1 x e^{ax} dx = 1.$$

Give the defining equation and a numerical value.

Question 2 – Solution

First consider the special case $a = 0$. In that case,

$$\int_0^1 x e^{0x} dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \neq 1.$$

Therefore, $a = 0$ is not a solution, so we may assume that $a \neq 0$.

To evaluate the integral, use integration by parts with

$$u = x, \quad dv = e^{ax} dx.$$

Then

$$du = dx, \quad v = \frac{e^{ax}}{a}.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_0^1 x e^{ax} dx &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{1}{a} \int_0^1 e^{ax} dx \\ &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{e^{ax}}{a^2} \Big|_0^1 \\ &= \frac{e^a}{a} - \frac{e^a - 1}{a^2} \\ &= \frac{a e^a - e^a + 1}{a^2} \\ &= \frac{e^a(a - 1) + 1}{a^2}. \end{aligned}$$

We want this integral to equal 1, so

$$\begin{aligned} \frac{e^a(a - 1) + 1}{a^2} &= 1, \\ e^a(a - 1) + 1 &= a^2. \end{aligned}$$

Thus, the defining equation is

$$\boxed{e^a(a - 1) + 1 = a^2}.$$

Substituting $a = 1$ verifies that it satisfies the equation:

$$e^1(1 - 1) + 1 = 1 = 1^2.$$

Hence, $a = 1$ is a solution.

To see that it is the only solution, define

$$F(a) = \int_0^1 x e^{ax} dx.$$

Differentiating with respect to a gives

$$F'(a) = \int_0^1 x^2 e^{ax} dx > 0$$

for every real a . Therefore, F is strictly increasing and can equal 1 for at most one value of a . Since $F(1) = 1$, the unique constant is

$$\boxed{a = 1.0000}.$$