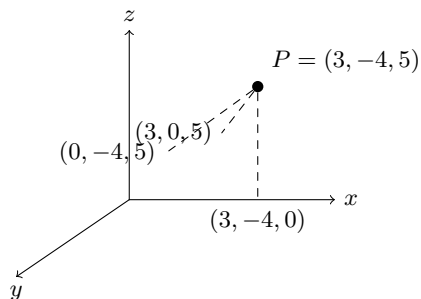


Question 1

A point is $P = (3, -4, 5)$. Find its perpendicular distances to each coordinate plane and to each coordinate axis. Identify the closest plane and the closest axis.

Question 1 – Solution

Strategy A coordinate plane is reached by changing one coordinate to zero. For an axis, two coordinates must become zero, so the distance comes from those two perpendicular components.



Coordinate planes The distances to the yz -, xz -, and xy -planes are $|3| = 3$, $|-4| = 4$, and $|5| = 5$, respectively. Thus the closest plane is the yz -plane.

Coordinate axes For the x -axis, $d = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$. Similarly,

$$d(P, y\text{-axis}) = \sqrt{3^2 + 5^2} = \sqrt{34}, \quad d(P, z\text{-axis}) = \sqrt{3^2 + (-4)^2} = 5.$$

Hence the closest axis is the z -axis, at distance $\boxed{5}$.

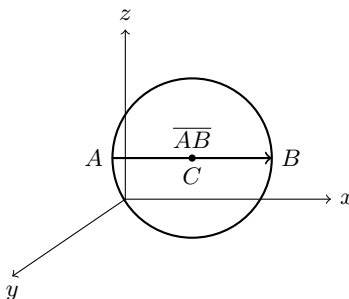
Verification Each plane distance uses exactly the omitted coordinate; each axis distance uses exactly the other two coordinates.

Question 2

The endpoints of a diameter of a sphere are $A = (-2, 1, 4)$ and $B = (6, -3, 0)$. Find the sphere's center, radius, and equation. Determine where the sphere intersects the z -axis.

Question 2 – Solution

Strategy The center is the midpoint of the diameter, and the radius is half its length. Points on the z -axis have the form $(0, 0, z)$.



Center and radius The midpoint is $C = (2, -1, 2)$. Also,

$$|AB| = \sqrt{8^2 + (-4)^2 + (-4)^2} = 4\sqrt{6}, \quad r = 2\sqrt{6}.$$

Therefore the sphere is

$$(x - 2)^2 + (y + 1)^2 + (z - 2)^2 = 24.$$

z -axis intersections Set $x = y = 0$: $4 + 1 + (z - 2)^2 = 24$, so $(z - 2)^2 = 19$. The intersection points are $(0, 0, 2 \pm \sqrt{19})$.

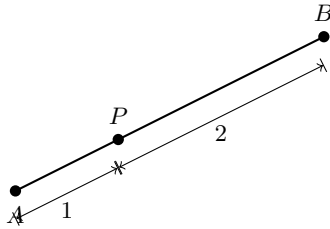
Verification Both A and B are $2\sqrt{6}$ units from C , and their midpoint is C .

Question 3

Point P divides the segment from $A = (-5, 2, 7)$ to $B = (4, -4, 1)$ internally in the ratio $AP : PB = 1 : 2$. Find P , then verify the ratio using distances.

Question 3 – Solution

Strategy Moving one-third of the way from A to B produces the required $1 : 2$ division.



Section formula Since $B - A = \langle 9, -6, -6 \rangle$,

$$P = A + \frac{1}{3}(B - A) = (-5, 2, 7) + (3, -2, -2) = \boxed{(-2, 0, 5)}.$$

Distance check We have $P - A = \langle 3, -2, -2 \rangle$ and $B - P = \langle 6, -4, -4 \rangle = 2(P - A)$. Therefore $AP = \sqrt{17}$ and $PB = 2\sqrt{17}$, giving $AP : PB = 1 : 2$.

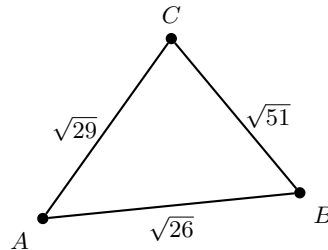
Key idea An internal ratio determines an affine weighted average; the distance check also confirms that P lies between the endpoints.

Question 4

The points $A = (1, 0, 2)$, $B = (4, 4, 1)$, and $C = (-1, 3, 6)$ form a triangle. Classify it as scalene, isosceles, or equilateral, and determine whether it is acute, right, or obtuse.

Question 4 – Solution

Strategy Compare squared side lengths; this avoids unnecessary square roots. The converse of the Pythagorean theorem classifies the largest angle.



Side lengths Direct calculation gives

$$AB^2 = 3^2 + 4^2 + (-1)^2 = 26, \quad AC^2 = (-2)^2 + 3^2 + 4^2 = 29, \quad BC^2 = (-5)^2 + (-1)^2 + 5^2 = 51.$$

The three lengths differ, so the triangle is scalene.

Angle classification The longest side is BC . Since $BC^2 = 51 < 26 + 29 = 55$, the angle opposite BC is acute. The other angles are necessarily smaller, so the triangle is scalene and acute.

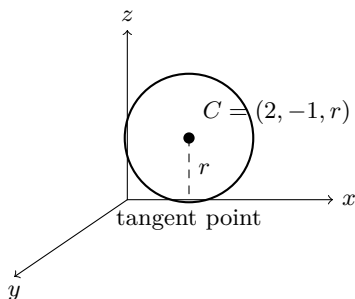
Verification The triangle inequality holds; for example, $\sqrt{26} + \sqrt{29} > \sqrt{51}$.

Question 5

A sphere has center $C = (2, -1, c)$, is tangent to the xy -plane, lies entirely above that plane, and passes through $P = (5, 3, 2)$. Find c and the sphere's equation.

Question 5 – Solution

Strategy Tangency to the xy -plane makes the radius equal to the center's height. The condition that the sphere lies above the plane forces $c > 0$.



Build the equation Here $r = c$. Because P is on the sphere,

$$(5 - 2)^2 + (3 + 1)^2 + (2 - c)^2 = c^2.$$

Thus $25 + (2 - c)^2 = c^2$, so $29 - 4c = 0$ and $c = 29/4$.

Result The radius is $29/4$, and

$$\boxed{(x - 2)^2 + (y + 1)^2 + \left(z - \frac{29}{4}\right)^2 = \left(\frac{29}{4}\right)^2}.$$

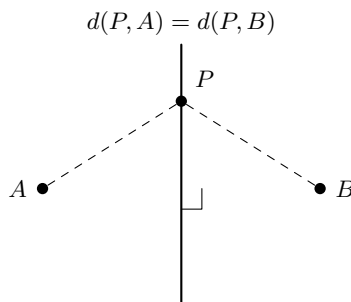
Verification The lowest z -coordinate is $c - r = 0$, so the sphere is tangent to the plane and otherwise lies above it.

Question 6

Describe geometrically the set of all points $P = (x, y, z)$ equidistant from $A = (2, -1, 3)$ and $B = (-4, 5, 1)$. Derive a Cartesian equation for the set and show that the midpoint of AB belongs to it.

Question 6 – Solution

Strategy Equate squared distances. Quadratic terms cancel, revealing the perpendicular-bisector surface.



Derivation Set

$$(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = (x + 4)^2 + (y - 5)^2 + (z - 1)^2.$$

After expanding and simplifying, $\boxed{3x - 3y + z + 7 = 0}$.

Interpretation This is the plane perpendicular to $\overrightarrow{AB} = \langle -6, 6, -2 \rangle$ through the midpoint $M = (-1, 2, 2)$. Indeed, $3(-1) - 3(2) + 2 + 7 = 0$.

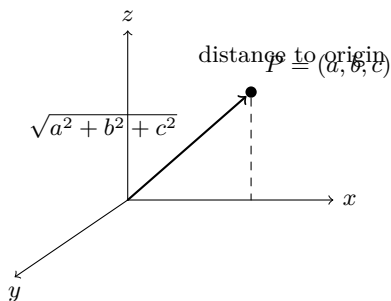
Verification Interchanging A and B does not change the locus, and every point in the plane has equal squared distance to the endpoints.

Question 7

A point $P = (a, b, c)$ is 7 units from the origin. Its distances to the yz - and xz -planes are 2 and 3, respectively, and P lies in the octant $x < 0, y > 0, z < 0$. Find P .

Question 7 – Solution

Strategy Distances to the yz - and xz -planes determine $|a|$ and $|b|$. The octant fixes signs, and the distance to the origin determines c .



Known coordinates We have $|a| = 2$ and $|b| = 3$, so $a = -2$ and $b = 3$.

Use the radial distance Since $a^2 + b^2 + c^2 = 49$,

$$4 + 9 + c^2 = 49 \implies c^2 = 36.$$

The octant condition $z < 0$ selects $c = -6$. Hence $\boxed{P = (-2, 3, -6)}$.

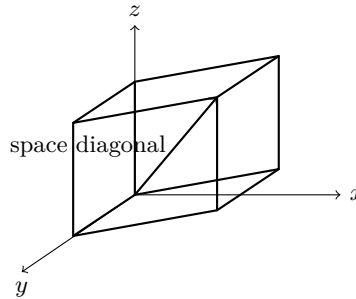
Verification Its distance from the origin is $\sqrt{4 + 9 + 36} = 7$, and its plane distances are 2 and 3 as required.

Question 8

A rectangular box has one vertex at the origin and the opposite vertex $P = (a, b, c)$ in the first octant. Its space diagonal has length 13, its xy -face diagonal has length 5, and the y -edge is longer than the x -edge. If the side lengths are positive integers, determine P .

Question 8 – Solution

Strategy Use the face diagonal to determine two integer side lengths, then use the space diagonal for the third.



Base dimensions The positive integer side lengths satisfying $a^2 + b^2 = 25$ are 3 and 4. Since the y -edge is longer than the x -edge, $a = 3$ and $b = 4$.

Third dimension The space diagonal gives

$$a^2 + b^2 + c^2 = 169 \implies 25 + c^2 = 169,$$

so $c = 12$. Therefore $\boxed{P = (3, 4, 12)}$.

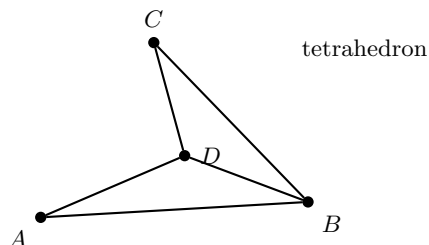
Verification The face diagonal is $\sqrt{3^2 + 4^2} = 5$ and the space diagonal is $\sqrt{3^2 + 4^2 + 12^2} = 13$.

Question 9

Four points are $A = (1, 1, 1)$, $B = (5, 1, 1)$, $C = (3, 1 + 2\sqrt{3}, 1)$, and $D = (3, 1 + 2/\sqrt{3}, 1 + 4\sqrt{2/3})$. Prove that they are the vertices of a regular tetrahedron and find its center.

Question 9 – Solution

Strategy A tetrahedron is regular if all six pairwise distances agree. Symmetry suggests that its center is the average of its four vertices.



Edge lengths Computation gives $AB = AC = BC = 4$. Also,

$$AD^2 = BD^2 = 4 + \frac{4}{3} + \frac{32}{3} = 16, \quad CD^2 = \frac{16}{3} + \frac{32}{3} = 16.$$

Thus all six edges have length 4, proving the tetrahedron is regular.

Center Averaging the vertices gives

$$G = \left(3, 1 + \frac{2\sqrt{3}}{3}, 1 + \sqrt{\frac{2}{3}} \right).$$

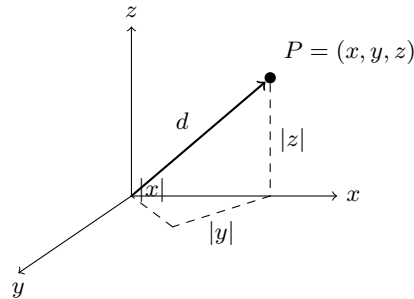
Verification By symmetry—or by direct substitution— G is equidistant from all four vertices.

Question 10

Derive the distance formula between $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ using the Pythagorean theorem twice. Then use it to find all values of t for which the distance from $(t, 2t, -1)$ to $(1, -2, 3)$ is 6.

Question 10 – Solution

Strategy First combine the x - and y -changes into a planar diagonal; then combine that diagonal with the z -change.



Derivation The planar displacement has length $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. A second right triangle therefore gives

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Application Set the squared distance equal to 36:

$$(t - 1)^2 + (2t + 2)^2 + (-4)^2 = 36.$$

This becomes $5t^2 + 6t - 15 = 0$, so

$$t = \frac{-3 \pm 2\sqrt{21}}{5}.$$

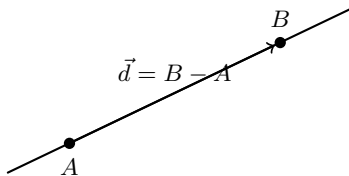
Verification Substitution into the quadratic recovers squared distance 36; both values are valid because distance is nonnegative.

Question 1

Find vector, parametric, and symmetric equations of the line through $A = (-2, 3, 1)$ and $B = (4, -1, 7)$. Determine the point on the line whose x -coordinate is 7.

Question 1 – Solution

Strategy Use $B - A$ as a direction vector and keep one common parameter in all coordinates.



Equations A direction is $\langle 6, -4, 6 \rangle = 2 \langle 3, -2, 3 \rangle$. Thus

$$\vec{r} = \langle -2, 3, 1 \rangle + t \langle 3, -2, 3 \rangle, \quad x = -2 + 3t, \quad y = 3 - 2t, \quad z = 1 + 3t,$$

and $\boxed{(x + 2)/3 = (y - 3)/(-2) = (z - 1)/3}$.

Requested point From $-2 + 3t = 7$, $t = 3$, giving $\boxed{(7, -3, 10)}$.

Verification $t = 0$ gives A and $t = 2$ gives B .

Question 2

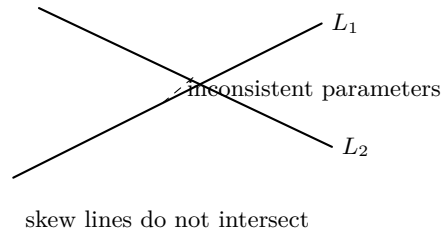
Determine whether the lines

$$L_1 : \vec{r} = \langle 1, 0, 2 \rangle + s \langle 2, -1, 3 \rangle, \quad L_2 : \vec{r} = \langle 3, -1, 1 \rangle + t \langle 1, 1, 2 \rangle$$

intersect, are parallel, or are skew. Justify the classification.

Question 2 – Solution

Strategy Equate coordinates. An intersection requires one pair (s, t) satisfying all three equations.



Coordinate equations The first two equations are $1 + 2s = 3 + t$ and $-s = -1 + t$. The second gives $t = 1 - s$; substitution into the first gives $s = 1$, hence $t = 0$.

Consistency check The third coordinates would be $2 + 3(1) = 5$ and $1 + 2(0) = 1$, so they disagree. The directions are not scalar multiples. Therefore the lines are skew.

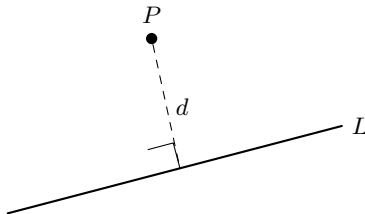
Key idea Solving only two coordinate equations can produce a false intersection; the remaining coordinate is essential.

Question 3

Find the distance from $P = (3, 1, -2)$ to the line $L : \vec{r} = \langle 1, -1, 2 \rangle + t \langle 2, 1, -2 \rangle$ and find the closest point on L .

Question 3 – Solution

Strategy At the closest point Q , the connector $P - Q$ is perpendicular to the line's direction.



Perpendicular condition Write $Q = (1 + 2t, -1 + t, 2 - 2t)$. Then

$$(P - Q) \cdot \langle 2, 1, -2 \rangle = 0.$$

Since $P - Q = \langle 2 - 2t, 2 - t, -4 + 2t \rangle$, the equation is $14 - 9t = 0$, so $t = 14/9$.

Closest point and distance Thus $Q = \boxed{(37/9, 5/9, -10/9)}$. Also $P - Q = \frac{1}{9} \langle -10, 4, -8 \rangle$, so

$$\boxed{d(P, L) = \frac{\sqrt{180}}{9} = \frac{2\sqrt{5}}{3}}.$$

Verification The dot product of $\langle -10, 4, -8 \rangle$ with $\langle 2, 1, -2 \rangle$ is zero.

Question 4

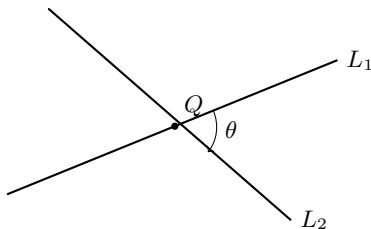
The lines

$$L_1 : \vec{r} = \langle 1, 2, -1 \rangle + s \langle 2, -1, 2 \rangle, \quad L_2 : \vec{r} = \langle 5, 0, 3 \rangle + t \langle 1, 2, -2 \rangle$$

intersect. Find their intersection and the acute angle between them.

Question 4 – Solution

Strategy Solve the coordinate equations for the intersection; use direction vectors for the angle.



Intersection Equating coordinates gives $1 + 2s = 5 + t$, $2 - s = 2t$, and $-1 + 2s = 3 - 2t$. The solution is $s = 2, t = 0$, so $Q = (5, 0, 3)$.

Angle With $\vec{d}_1 = \langle 2, -1, 2 \rangle$ and $\vec{d}_2 = \langle 1, 2, -2 \rangle$,

$$|\vec{d}_1 \cdot \vec{d}_2| = 4, \quad \|\vec{d}_1\| = 3, \quad \|\vec{d}_2\| = 3.$$

Hence $\theta = \cos^{-1}(4/9) \approx 63.6^\circ$.

Verification Substituting $s = 2$ and $t = 0$ gives the same point on both lines.

Question 5

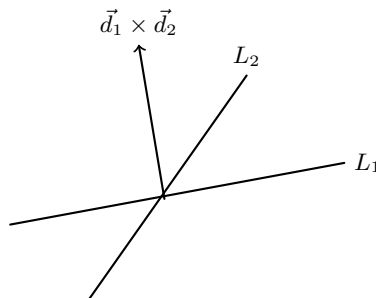
Find the line through $P = (2, -1, 4)$ whose direction is perpendicular to the directions of both lines

$$L_1 : \vec{r} = \langle 0, 1, 2 \rangle + s \langle 1, 2, -1 \rangle, \quad L_2 : \vec{r} = \langle 3, 0, -1 \rangle + t \langle 2, -1, 1 \rangle.$$

Explain why exactly one such direction exists.

Question 5 – Solution

Strategy A vector perpendicular to both directions is their cross product.



Direction Compute

$$\langle 1, 2, -1 \rangle \times \langle 2, -1, 1 \rangle = \langle 1, -3, -5 \rangle.$$

Therefore the requested line is

$$\boxed{\vec{r} = \langle 2, -1, 4 \rangle + u \langle 1, -3, -5 \rangle}.$$

Verification The new direction has dot product 0 with each original direction.

Uniqueness The original directions are not parallel, so their common perpendicular directions form the one-dimensional span of their nonzero cross product; reversing direction describes the same line.

Question 6

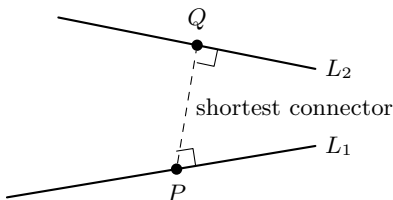
Find the shortest distance between the skew lines

$$L_1 : \vec{r} = \langle 0, 0, 0 \rangle + s \langle 1, 1, 0 \rangle, \quad L_2 : \vec{r} = \langle 0, 1, 1 \rangle + t \langle 1, -1, 0 \rangle,$$

and identify the pair of closest points.

Question 6 – Solution

Strategy The connector between closest points must be perpendicular to both line directions.



Closest points Let $P = (s, s, 0)$ and $Q = (t, 1 - t, 1)$. Requiring $(Q - P) \cdot \langle 1, 1, 0 \rangle = 0$ gives $1 - 2s = 0$, so $s = 1/2$. Requiring $(Q - P) \cdot \langle 1, -1, 0 \rangle = 0$ gives $2t - 1 = 0$, so $t = 1/2$.

Distance Thus $P = (1/2, 1/2, 0)$ and $Q = (1/2, 1/2, 1)$, with

$$\boxed{d(L_1, L_2) = |PQ| = 1}.$$

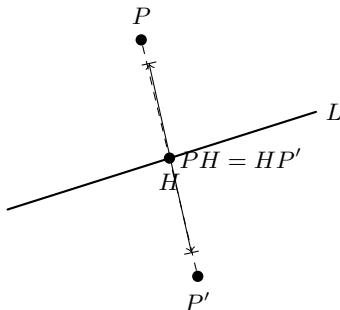
Verification $Q - P = \langle 0, 0, 1 \rangle$ is perpendicular to both direction vectors.

Question 7

Reflect the point $P = (4, 0, 1)$ across the line $L : \vec{r} = \langle 1, 1, 1 \rangle + t \langle 1, -1, 0 \rangle$. That is, find P' such that L perpendicularly bisects $\overline{PP'}$.

Question 7 – Solution

Strategy First find the perpendicular foot H on the line; then use $H = (P + P')/2$.



Projection point Write $H = (1 + t, 1 - t, 1)$. The condition $(P - H) \cdot \langle 1, -1, 0 \rangle = 0$ gives

$$\langle 3 - t, t - 1, 0 \rangle \cdot \langle 1, -1, 0 \rangle = 4 - 2t = 0,$$

so $t = 2$ and $H = (3, -1, 1)$.

Reflection Therefore $P' = 2H - P = \boxed{(2, -2, 1)}$.

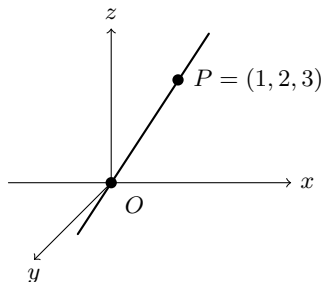
Verification The midpoint of P and P' is H , and $P - H = \langle 1, 1, 0 \rangle$ is perpendicular to the direction of L .

Question 8

Find the line that passes through $P = (1, 2, 3)$ and meets both the x -axis and the y -axis. Determine the two intercept points, or prove that no such line exists.

Question 8 – Solution

Strategy Distinct axis intercepts determine a line in the xy -plane, but both intercepts may be the origin.



Distinct intercepts A line through distinct points $X = (a, 0, 0)$ and $Y = (0, b, 0)$ lies in $z = 0$, so it cannot contain P .

Coincident intercepts The axes share the origin. The line through the origin and P is

$$\vec{r}(t) = t \langle 1, 2, 3 \rangle, \quad t \in \mathbb{R}.$$

It passes through P at $t = 1$ and meets each axis at $t = 0$. Thus both intercepts are $(0, 0, 0)$.

Key idea The distinct-point argument does not rule out coincident intercepts. This is the unique line satisfying the question.

Question 9

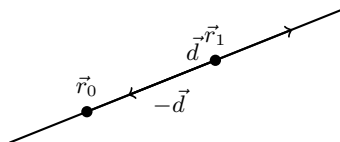
Show that

$$L_1 : \vec{r} = \langle 2, -1, 3 \rangle + t \langle 4, 2, -2 \rangle \quad \text{and} \quad L_2 : \vec{r} = \langle 8, 2, 0 \rangle + s \langle -2, -1, 1 \rangle$$

describe the same geometric line. Find the parameter conversion between t and s .

Question 9 – Solution

Strategy Check parallel directions and show that one base point lies on the other line.



different parameters, same point set

Same point set The second direction is $-\frac{1}{2}\langle 4, 2, -2 \rangle$. Also $\langle 8, 2, 0 \rangle = \langle 2, -1, 3 \rangle + \frac{3}{2}\langle 4, 2, -2 \rangle$, so the second base point lies on L_1 . Thus the lines coincide.

Parameter conversion Equating the representations gives

$$\vec{r} = \vec{r}_0 + t\vec{d} = \vec{r}_0 + \frac{3}{2}\vec{d} - \frac{s}{2}\vec{d},$$

so $\boxed{t = \frac{3}{2} - \frac{s}{2}}$, equivalently $s = 3 - 2t$.

Verification The conversion makes all three coordinates identical.

Question 10

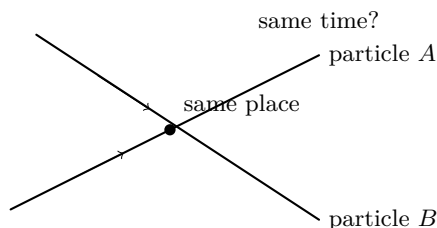
Two particles move according to

$$\vec{r}_A(t) = \langle 1, 0, 2 \rangle + t \langle 2, 1, -1 \rangle, \quad \vec{r}_B(t) = \langle 5, 2, 0 \rangle + t \langle -1, 0, 1 \rangle, \quad t \geq 0.$$

Do their paths intersect? Do the particles collide? Find their minimum separation for $t \geq 0$.

Question 10 – Solution

Strategy Path intersection permits different parameters; collision requires the same time. Minimum separation comes from minimizing the squared relative distance.



Paths Solving $\vec{r}_A(s) = \vec{r}_B(u)$ gives $s = 2, u = 0$, so the paths intersect at $(5, 2, 0)$. The particles do not collide there because they reach it at different times. Directly equating positions at the same t is inconsistent.

Minimum separation The relative position is

$$\vec{r}_A(t) - \vec{r}_B(t) = \langle -4 + 3t, -2 + t, 2 - 2t \rangle.$$

Thus $D^2(t) = 14t^2 - 36t + 24$, minimized at $t = 36/28 = 9/7$. The minimum squared distance is $6/7$, so

$$D_{\min} = \sqrt{6/7} = \frac{\sqrt{42}}{7}.$$

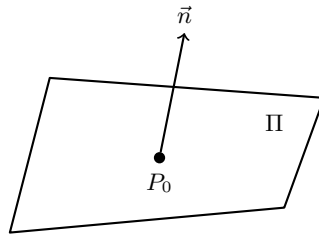
Verification The minimizing time is nonnegative, and the quadratic has positive leading coefficient.

Question 1

Find scalar and vector equations of the plane through $P_0 = (2, -1, 3)$ with normal vector $\vec{n} = \langle 4, 2, -1 \rangle$. Find its intercepts with the coordinate axes.

Question 1 – Solution

Strategy Use $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, then set two coordinates equal to zero for each intercept.



Equations The vector equation is $\langle 4, 2, -1 \rangle \cdot \langle x - 2, y + 1, z - 3 \rangle = 0$, which simplifies to $4x + 2y - z = 3$.

Intercepts The x -, y -, and z -intercepts are

$$(3/4, 0, 0), \quad (0, 3/2, 0), \quad (0, 0, -3).$$

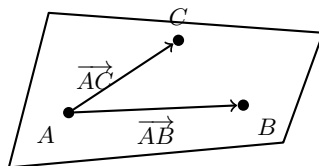
Verification Substituting P_0 gives $8 - 2 - 3 = 3$, and each intercept satisfies the scalar equation.

Question 2

Find an equation of the plane through $A = (1, 0, 2)$, $B = (3, -1, 4)$, and $C = (0, 2, 5)$. Determine whether the origin and $D = (2, 1, 3)$ lie on the same side of the plane.

Question 2 – Solution

Strategy Cross two independent displacement vectors to obtain a normal. Compare signs after substituting test points.



Plane Here $\overrightarrow{AB} = \langle 2, -1, 2 \rangle$ and $\overrightarrow{AC} = \langle -1, 2, 3 \rangle$, so $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -7, -8, 3 \rangle$. Thus

$$\boxed{7x + 8y - 3z = 1}.$$

Side test Write $F(x, y, z) = 7x + 8y - 3z - 1$. Then $F(0, 0, 0) = -1$, while $F(2, 1, 3) = 12$. The signs differ, so the points lie on opposite sides.

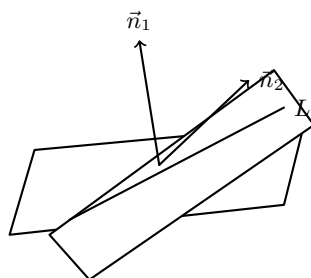
Verification Each of A, B, C satisfies the plane equation.

Question 3

Find the line of intersection of the planes $\Pi_1 : x + 2y - z = 3$ and $\Pi_2 : 2x - y + 2z = 1$. Also find the acute dihedral angle between the planes.

Question 3 – Solution

Strategy The intersection direction is perpendicular to both normals. Obtain one point by fixing a convenient coordinate.



Intersection line With $\vec{n}_1 = \langle 1, 2, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 2 \rangle$,

$$\vec{n}_1 \times \vec{n}_2 = \langle 3, -4, -5 \rangle.$$

Setting $z = 0$ gives $x + 2y = 3$, $2x - y = 1$, hence $(x, y) = (1, 1)$. Therefore

$$\vec{r} = \langle 1, 1, 0 \rangle + t \langle 3, -4, -5 \rangle.$$

Angle Since $\vec{n}_1 \cdot \vec{n}_2 = -2$, $\|\vec{n}_1\| = \sqrt{6}$, and $\|\vec{n}_2\| = 3$,

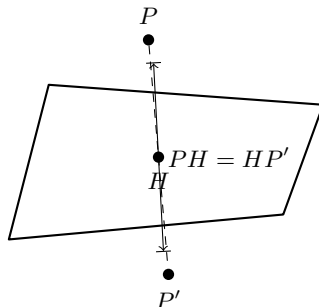
$$\theta = \cos^{-1} \left(\frac{2}{3\sqrt{6}} \right) \approx 74.2^\circ.$$

Question 4

Find the distance from $P = (4, -1, 5)$ to the plane $2x - y + 2z = 3$, the perpendicular foot H , and the reflection P' of P across the plane.

Question 4 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$.



Distance The signed numerator is $2(4) - (-1) + 2(5) - 3 = 16$, and $\|\vec{n}\| = 3$, so $d = 16/3$.

Foot For $H = P - \lambda\vec{n}$, impose the plane equation. Since $\lambda = 16/9$,

$$H = \left(4/9, 7/9, 13/9 \right).$$

Reflection Since H is the midpoint of PP' , $P' = 2H - P = \left(-28/9, 23/9, -19/9 \right)$.

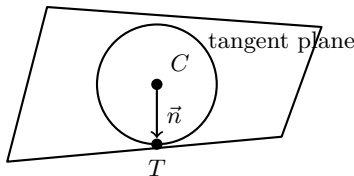
Verification H satisfies the plane equation, and $P - H$ is parallel to $\vec{n}$.

Question 5

The sphere $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 36$ contains the point $T = (3, 2, 7)$. Find the tangent plane at T and the line normal to the sphere there.

Question 5 – Solution

Strategy The radius from the center to T is normal to the tangent plane.



Normal direction The center is $C = (1, -2, 3)$ and $T - C = \langle 2, 4, 4 \rangle = 2\langle 1, 2, 2 \rangle$.

Tangent plane Using normal $\langle 1, 2, 2 \rangle$ through T ,

$$\boxed{(x - 3) + 2(y - 2) + 2(z - 7) = 0}, \quad \text{or} \quad \boxed{x + 2y + 2z = 21}.$$

Normal line $\boxed{\vec{r} = \langle 3, 2, 7 \rangle + t \langle 1, 2, 2 \rangle}$.

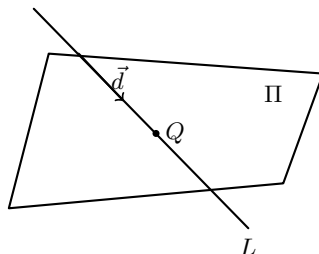
Verification $|T - C| = 6$, so T is indeed on the sphere.

Question 6

Classify the relationship between the line $L : \vec{r} = \langle 1, 2, 0 \rangle + t \langle 2, -1, 3 \rangle$ and the plane $\Pi : 3x + y - z = 5$. If they intersect, find the point and angle at which the line meets the plane.

Question 6 – Solution

Strategy Substitute the line into the plane. The line-plane angle is complementary to the angle between the direction and the normal.



Intersection Substitution gives $3(1 + 2t) + (2 - t) - 3t = 5$, so $2t = 0$ and $t = 0$. Thus $Q = \boxed{(1, 2, 0)}$.

Angle With $\vec{d} = \langle 2, -1, 3 \rangle$ and $\vec{n} = \langle 3, 1, -1 \rangle$,

$$\sin \alpha = \frac{|\vec{d} \cdot \vec{n}|}{\|\vec{d}\| \|\vec{n}\|} = \frac{2}{\sqrt{154}}.$$

Hence $\alpha = \sin^{-1}(2/\sqrt{154}) \approx 9.27^\circ$.

Verification Since $\vec{d} \cdot \vec{n} \neq 0$, the line is not parallel to the plane.

Question 7

Find the plane in the family containing the intersection line of $x + y + z = 2$ and $2x - y + 3z = 1$ that is perpendicular to the plane $x - 2y + z = 4$.

Question 7 – Solution

Strategy Apart from $2x - y + 3z = 1$ itself, every plane through the common line has equation $(x + y + z - 2) + \lambda(2x - y + 3z - 1) = 0$. The omitted plane has normal dot product $7 \neq 0$ with the given normal, so it cannot qualify. Choose λ so the family normal is perpendicular to the given normal.

pencil of planes



Normal condition The family normal is $\langle 1 + 2\lambda, 1 - \lambda, 1 + 3\lambda \rangle$. Dotting with $\langle 1, -2, 1 \rangle$ gives 7λ , so $\lambda = 0$.

Result The required plane is therefore simply

$$\boxed{x + y + z = 2}.$$

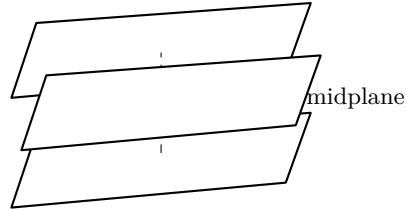
Verification Its normal $\langle 1, 1, 1 \rangle$ has dot product $1 - 2 + 1 = 0$ with $\langle 1, -2, 1 \rangle$, and it contains the stated intersection line.

Question 8

Find the distance between the parallel planes $2x - y + 2z = 3$ and $4x - 2y + 4z = 21$. Find the plane exactly halfway between them.

Question 8 – Solution

Strategy Normalize the equations to identical left sides before comparing constants.



Distance Divide the second equation by 2 to get $2x - y + 2z = 21/2$. Therefore

$$d = \frac{|21/2 - 3|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{15/2}{3} = \boxed{5/2}.$$

Midplane Average the normalized constants: $(3 + 21/2)/2 = 27/4$. Hence

$$\boxed{2x - y + 2z = 27/4}.$$

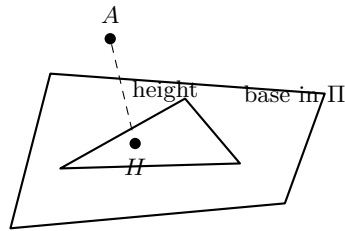
Verification Its constant differs from each neighboring constant by $15/4$, corresponding to distance $5/4$.

Question 9

A tetrahedron has base vertices $B = (0, 0, 0)$, $C = (4, 0, 0)$, $D = (1, 3, 0)$ and apex $A = (2, 1, 6)$. Find the base plane, the height, and the volume. Then find the plane parallel to the base that cuts the tetrahedron at half its height.

Question 9 – Solution

Strategy The base is horizontal. Volume is one third of base area times perpendicular height.



Base and height The base plane is $z = 0$. Its triangular area is $\frac{1}{2}(4)(3) = 6$, and the apex is 6 units from the plane.

Volume Thus $V = \frac{1}{3}(6)(6) = 12$.

Half-height plane The plane parallel to $z = 0$ halfway from the base to the apex is $z = 3$.

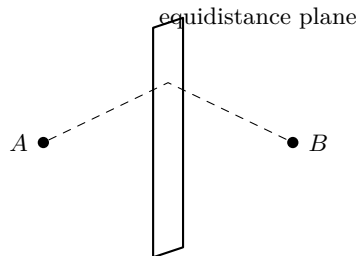
Verification Distance from $z = 3$ to both $z = 0$ and A 's level $z = 6$ is 3.

Question 10

Prove that the set of points equidistant from two distinct points A and B is a plane perpendicular to $\overline{AB}$ through its midpoint. Then find that plane for $A = (1, -2, 4)$ and $B = (5, 0, -2)$.

Question 10 – Solution

Strategy Expand $\|\vec{r} - \vec{a}\|^2 = \|\vec{r} - \vec{b}\|^2$ and cancel $\vec{r} \cdot \vec{r}$.



General proof Expansion gives

$$2(\vec{b} - \vec{a}) \cdot \vec{r} = \|\vec{b}\|^2 - \|\vec{a}\|^2.$$

This is a plane with normal $\vec{b} - \vec{a}$. Substituting $(\vec{a} + \vec{b})/2$ verifies that the midpoint lies on it.

Application Here $\vec{b} - \vec{a} = \langle 4, 2, -6 \rangle$ and the midpoint is $(3, -1, 1)$. Thus

$$4(x - 3) + 2(y + 1) - 6(z - 1) = 0,$$

or $\boxed{2x + y - 3z = 2}$.

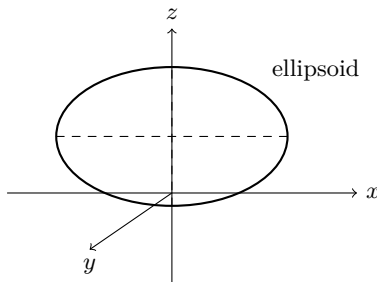
Verification Direct substitution of the midpoint gives $6 - 1 - 3 = 2$.

Question 1

Put $4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0$ in standard form. Classify the surface and give its center, semiaxes, and coordinate-direction extreme points.

Question 1 – Solution

Strategy Complete the square separately in x, y, z , then divide by the resulting constant.



Standard form Completing squares gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45,$$

so

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its axial extreme points are

$$(1 \pm 3\sqrt{5}/2, -3, 2), \quad (1, -3 \pm 3\sqrt{5}, 2), \quad (1, -3, 2 \pm \sqrt{5}).$$

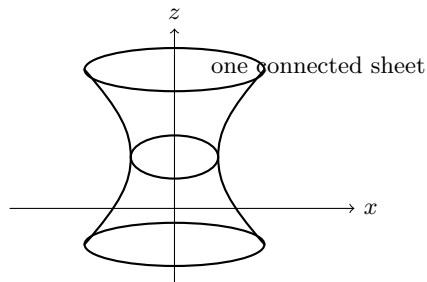
Verification Substitution of any listed endpoint makes exactly one normalized square equal to 1.

Question 2

Analyze $x^2 + 4y^2 - z^2 - 2x + 16y + 4z - 12 = 0$. Put it in standard form, classify it, identify its axis, and describe the traces $z = 2$ and $z = 2 + k$.

Question 2 – Solution

Strategy Complete squares and inspect the signs. Then substitute fixed z -levels.



Standard form Simplification gives

$$(x-1)^2 + 4(y+2)^2 - (z-2)^2 = 25, \quad \boxed{\frac{(x-1)^2}{25} + \frac{(y+2)^2}{25/4} - \frac{(z-2)^2}{25} = 1}.$$

It is a hyperboloid of one sheet centered at $(1, -2, 2)$ with axis parallel to z .

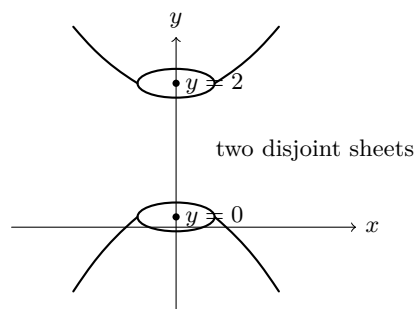
Traces At $z = 2$ the waist ellipse is $(x-1)^2 + 4(y+2)^2 = 25$. At $z = 2+k$ it becomes $(x-1)^2 + 4(y+2)^2 = 25 + k^2$, so ellipses expand as $|k|$ grows.

Question 3

For $9x^2 - 4y^2 + 9z^2 + 36x + 8y - 18z + 45 = 0$, find the standard form, classify the surface, and determine the vertices and which coordinate planes fail to intersect it.

Question 3 – Solution

Strategy Complete squares; a single negative term with right side -1 is more clearly read after multiplying by -1 .



Standard form Completing squares gives

$$9(x+2)^2 - 4(y-1)^2 + 9(z-1)^2 = -4,$$

hence

$$\frac{(y-1)^2}{1} - \frac{(x+2)^2}{4/9} - \frac{(z-1)^2}{4/9} = 1.$$

This is a hyperboloid of two sheets with axis parallel to y .

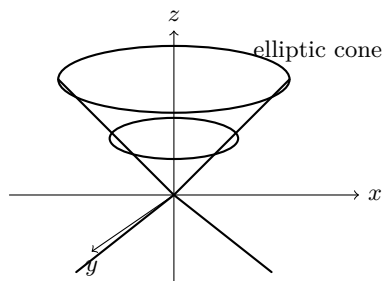
Vertices and coordinate traces The vertices are $(-2, 0, 1), (-2, 2, 1)$. All three coordinate planes intersect the surface: $y = 0$ contains the first vertex, while $x = 0$ and $z = 0$ give hyperbolas. The plane $y = 1$ does not intersect the surface, but it is not a coordinate plane.

Question 4

Classify the surface $4x^2 + y^2 - 9z^2 = 0$. Describe its traces in the planes $z = k$, $x = k$, and $y = k$, and find the lines obtained in the trace $y = 0$.

Question 4 – Solution

Strategy A homogeneous quadratic with two positive squares and one negative square describes a cone.



Classification The equation is $4x^2 + y^2 = 9z^2$, an elliptic cone with vertex at the origin and axis the z -axis.

Traces For $z = k \neq 0$, $4x^2 + y^2 = 9k^2$ is an ellipse; for $z = 0$ it is only the vertex. For fixed $x = k$ or $y = k$, the traces are hyperbolas when the fixed value is nonzero and intersecting lines when it is zero.

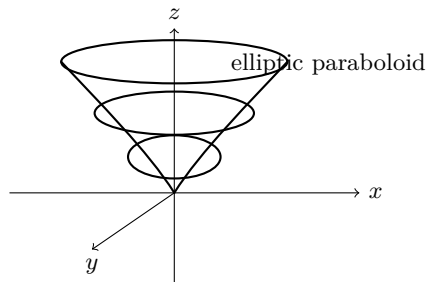
$y = 0$ Then $4x^2 = 9z^2$, so $z = \pm \frac{2}{3}x$.

Question 5

The surface $z = x^2 + 4y^2 - 2x + 8y + 7$ is cut by the plane $z = 10$. Identify the surface, locate its vertex, and find the semiaxes and area of the resulting planar region.

Question 5 – Solution

Strategy Complete squares in x, y ; a horizontal cut of an elliptic paraboloid is an ellipse.



Standard form We obtain

$$z = (x - 1)^2 + 4(y + 1)^2 + 2.$$

Thus the surface is an elliptic paraboloid opening upward with vertex $(1, -1, 2)$.

Cut at $z = 10$ The boundary satisfies $(x - 1)^2 + 4(y + 1)^2 = 8$, or

$$\frac{(x - 1)^2}{8} + \frac{(y + 1)^2}{2} = 1.$$

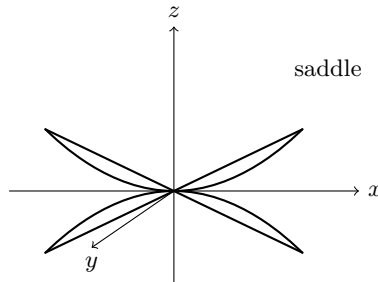
Its semiaxes are $2\sqrt{2}$ and $\sqrt{2}$, so the enclosed area is $\pi(2\sqrt{2})(\sqrt{2}) = 4\pi$.

Question 6

Analyze the hyperbolic paraboloid $z = x^2 - 4y^2$. Find the traces in $x = 0$, $y = 0$, $z = 0$, and $z = k \neq 0$. Explain how these traces establish the saddle shape.

Question 6 – Solution

Strategy Substitute each fixed coordinate and classify the resulting planar curve.



Vertical traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -4y^2$ opens downward. Their opposite concavities create the saddle.

Horizontal traces At $z = 0$, $x^2 - 4y^2 = 0$, so $x = \pm 2y$, two intersecting lines. For $k > 0$, $x^2 - 4y^2 = k$ is a hyperbola opening in the x -direction; for $k < 0$, it opens in the y -direction.

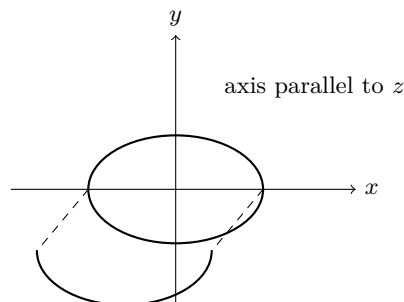
Conclusion The origin is neither a local maximum nor minimum because nearby traces rise in one direction and fall in the perpendicular direction.

Question 7

Classify $9x^2 + 4y^2 - 18x + 16y - 11 = 0$, describe its axis, and find the area of a cross-section perpendicular to that axis. Does the surface intersect the z -axis?

Question 7 – Solution

Strategy Because z is absent, any planar curve in xy is extruded parallel to the z -axis.



Standard form Completing squares yields

$$9(x-1)^2 + 4(y+2)^2 = 36, \quad \boxed{\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1}.$$

This is an elliptic cylinder parallel to the z -axis.

Cross-section Its semiaxes are 2 and 3, so each perpendicular cross-section has area $\boxed{6\pi}$.

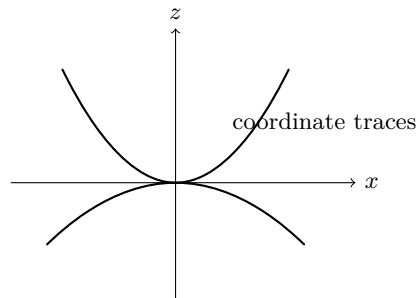
z -axis test At $x = y = 0$, the left side of the original equation is $-11 \neq 0$, so the cylinder does not meet the z -axis.

Question 8

A quadric has the traces $z = x^2$ in the plane $y = 0$ and $z = -y^2$ in the plane $x = 0$. Every horizontal trace $z = k \neq 0$ is a hyperbola. Determine a simplest equation for the surface and explain whether the trace information uniquely fixes all coefficients.

Question 8 – Solution

Strategy Combine the two vertical traces into a quadratic having the required restrictions.



Simplest model The equation

$$z = x^2 - y^2$$

has exactly the stated coordinate-plane traces and horizontal hyperbolas. It is a hyperbolic paraboloid.

Uniqueness issue The given coordinate traces force the coefficients of x^2 and y^2 in a diagonal model, but without an assumption excluding a mixed term, $z = x^2 - y^2 + cxy$ has the same traces in $x = 0$ and $y = 0$. Its horizontal traces remain hyperbolic when the quadratic form is indefinite.

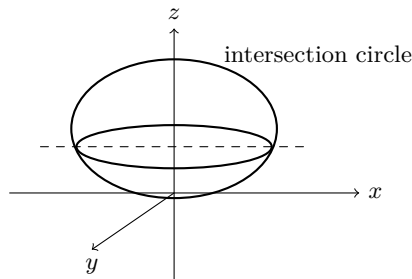
Conclusion The simplest axis-aligned equation is determined, but the data do not uniquely exclude rotated saddles.

Question 9

Find and classify the curve of intersection of the sphere $x^2 + y^2 + z^2 = 25$ with the plane $x + 2y + 2z = 9$. Find the curve's center and radius, and determine whether the plane is tangent, secant, or disjoint.

Question 9 – Solution

Strategy The circle center is the perpendicular projection of the sphere center onto the plane.



Plane distance The origin-to-plane distance is $d = 9/\sqrt{1+4+4} = 3$. Since $3 < 5$, the plane is secant.

Center and radius The foot from the origin lies along $\langle 1, 2, 2 \rangle$ and is

$$C = \frac{9}{9} \langle 1, 2, 2 \rangle = \boxed{(1, 2, 2)}.$$

By the right triangle from sphere center to circle edge, $r^2 = 5^2 - 3^2 = 16$, so $\boxed{r = 4}$.

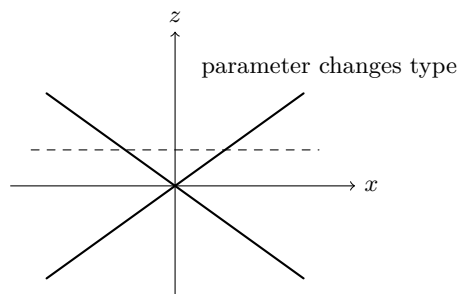
Classification The intersection is a circle in the given plane, centered at $(1, 2, 2)$ with radius 4.

Question 10

Classify the family $x^2 + y^2 - z^2 = \lambda$ for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Describe how the topology changes as λ passes through zero and give the vertices when they exist.

Question 10 – Solution

Strategy Divide by $|\lambda|$ and inspect which side has the single positive term.



$\lambda > 0$ The form $x^2/\lambda + y^2/\lambda - z^2/\lambda = 1$ is a hyperboloid of one sheet, connected, with axis z .

$\lambda = 0$ The equation $x^2 + y^2 = z^2$ is a double circular cone with vertex at the origin.

$\lambda < 0$ Writing $z^2/|\lambda| - x^2/|\lambda| - y^2/|\lambda| = 1$ gives a hyperboloid of two sheets. Its vertices are $\boxed{(0, 0, \pm\sqrt{|\lambda|})}$.

Transition At zero, the one-sheet surface pinches to a cone and separates into two components.

Question 1

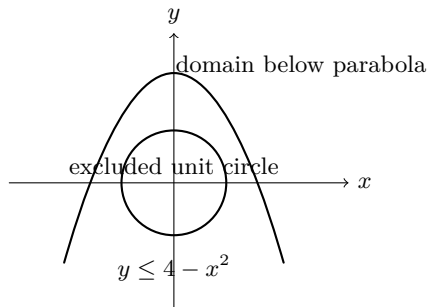
Find and sketch the domain of

$$f(x, y) = \frac{\sqrt{4 - x^2 - y^2}}{\ln(x^2 + y^2)}.$$

State which boundary pieces are included and whether the domain is bounded, open, closed, or neither.

Question 1 – Solution

Strategy Enforce the square-root, logarithm, and denominator restrictions simultaneously.



Restrictions We need $4 - x^2 - y \geq 0$, $x^2 + y^2 > 0$, and $\ln(x^2 + y^2) \neq 0$. Thus

$$D = \{(x, y) : y \leq 4 - x^2, x^2 + y^2 \neq 0, 1\}.$$

Boundary and topology The entire parabola $y = 4 - x^2$ is included; it does not meet the unit circle or the origin. The origin and entire unit circle are excluded. The region is unbounded downward. It is not open because the parabolic boundary is included, and not closed because excluded circle points are limit points.

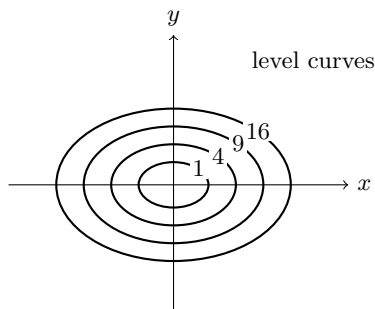
Check Every remaining point makes the radicand nonnegative and the denominator defined and nonzero.

Question 2

For $f(x, y) = 9 - 4(x - 1)^2 - (y + 2)^2$, find the range, global extrema, and level curves $f = c$. Describe how the curves change as c varies.

Question 2 – Solution

Strategy The subtracted squares are nonnegative; solve $f = c$ for the level sets.



Range and extrema The maximum is $\boxed{9}$ at $(1, -2)$. There is no minimum because $f \rightarrow -\infty$ as $\|(x, y)\| \rightarrow \infty$, so the range is $\boxed{(-\infty, 9]}$.

Level curves For $c < 9$,

$$4(x - 1)^2 + (y + 2)^2 = 9 - c,$$

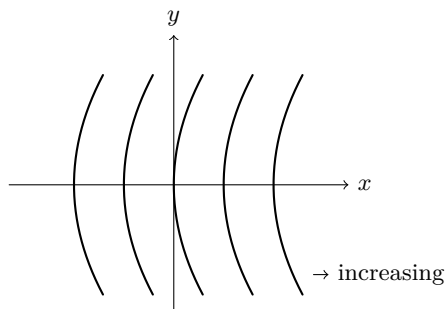
an ellipse centered at $(1, -2)$ with semiaxes $\sqrt{9 - c}/2$ and $\sqrt{9 - c}$. For $c = 9$ the level is one point; for $c > 9$ it is empty.

Question 3

A contour map consists of the curves $x - y^2 = c$ for all real c , labeled so values increase to the right. Reconstruct a simplest function producing the map. Find the level curve through $(3, -1)$ and decide where the function is positive.

Question 3 – Solution

Strategy Read the constant expression directly from the family of contours.



Function A simplest choice is $f(x, y) = x - y^2$. Any strictly increasing relabeling $g(f)$ would preserve the geometric contours but change their numerical labels.

Requested contour At $(3, -1)$, $c = 3 - 1 = 2$, so the curve is $x - y^2 = 2$.

Sign region The function is positive precisely when $x > y^2$, the region to the right of the parabola $x = y^2$.

Question 4

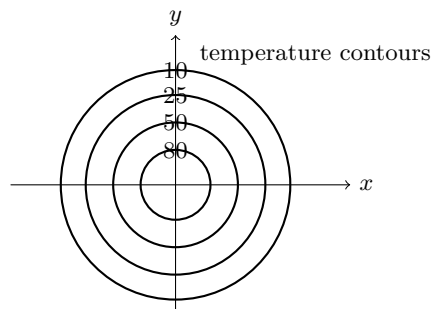
A thin plate has temperature

$$T(x, y) = 100e^{-(x^2+4y^2)/8} \quad (^\circ\text{C}).$$

Find the 50° isotherm, its semiaxes and enclosed area. Determine the warmest point and explain the contour spacing as temperature decreases.

Question 4 – Solution

Strategy Set T equal to a constant and take logarithms.



Isotherm Setting $T = 50$ gives $e^{-(x^2+4y^2)/8} = 1/2$, hence

$$x^2 + 4y^2 = 8 \ln 2.$$

The semiaxes are $\sqrt{8 \ln 2}$ and $\sqrt{2 \ln 2}$, so the area is $\boxed{4\pi \ln 2}$.

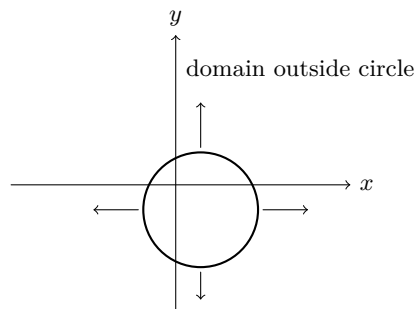
Maximum and spacing The maximum is 100° at $(0,0)$. For level $0 < c < 100$, $x^2 + 4y^2 = 8 \ln(100/c)$; lower temperatures correspond to larger ellipses, with nonlinear spacing caused by exponential decay.

Question 5

Let $f(u, v) = \sqrt{u - v}$ and $g(x, y) = (x^2 + y, x - y^2)$. Find $(f \circ g)(x, y)$ and its domain. Determine the boundary and whether the point $(1, 1)$ is interior, exterior, or on the boundary.

Question 5 – Solution

Strategy Substitute both components of g into f and require a nonnegative radicand.



Composition We obtain

$$(f \circ g)(x, y) = \sqrt{x^2 + y - x + y^2}.$$

Thus the domain is $x^2 - x + y^2 + y \geq 0$, or

$$\boxed{(x - 1/2)^2 + (y + 1/2)^2 \geq 1/2}.$$

It is the exterior of a disk, including its circular boundary.

Point test At $(1, 1)$ the original radicand is $2 > 0$, so the point is interior to the domain.

Question 6

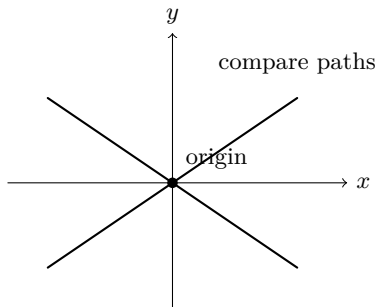
Investigate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Evaluate along at least three families of paths and explain why the evidence is decisive.

Question 6 – Solution

Strategy A multivariable limit must have the same value along every path.



Path tests Along $y = 0$, the expression equals 1; along $x = 0$, it equals -1 ; and along $y = mx$, it equals

$$\frac{1 - m^2}{1 + m^2},$$

which varies continuously with m .

Conclusion Since even two paths give different values, the limit does not exist.

Why decisive Path agreement can only suggest a limit, but one disagreement proves nonexistence because the definition requires uniform behavior over all approaches.

Question 7

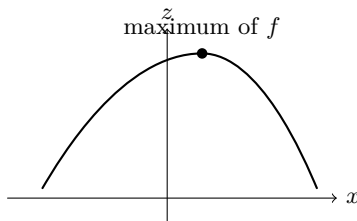
Without using derivatives, find the range and all global extrema of

$$f(x, y) = 7 - 2x^2 - 2xy - y^2 + 8x + 6y.$$

Hint: rewrite the quadratic as a sum of squares.

Question 7 – Solution

Strategy Complete squares in a way that respects the mixed term.



Rewrite Since $2x^2 + 2xy + y^2 = (x + y)^2 + x^2$,

$$f = 7 - (x + y)^2 - x^2 + 8x + 6y.$$

Set $u = x + y$. Then $y = u - x$ and $f = 7 - u^2 - x^2 + 2x + 6u = 17 - (x - 1)^2 - (u - 3)^2$.

Conclusion Therefore

$$\boxed{f(x, y) = 17 - (x - 1)^2 - (x + y - 3)^2}.$$

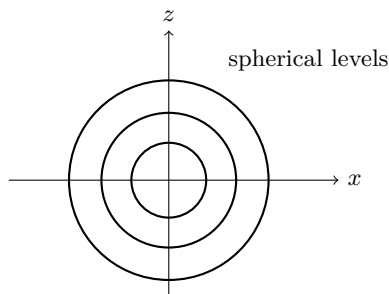
The global maximum is 17 at $x = 1$, $x + y = 3$, hence $(1, 2)$. There is no minimum, and the range is $(-\infty, 17]$.

Question 8

For $F(x, y, z) = \sqrt{(x-1)^2 + (y+2)^2 + (z-3)^2}$, interpret F geometrically. Describe every level surface $F = c$, determine the range, and find the level surface through $(4, 2, 3)$.

Question 8 – Solution

Strategy Recognize the three-dimensional distance formula.



Interpretation F is distance from $C = (1, -2, 3)$, so its range is $[0, \infty)$.

Levels For $c > 0$, $F = c$ is the sphere

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = c^2.$$

For $c = 0$ it is the single point C ; for $c < 0$ it is empty.

Requested surface The point $(4, 2, 3)$ is distance $\sqrt{3^2 + 4^2} = 5$ from C , so its level surface is the sphere of radius 5 .

Question 9

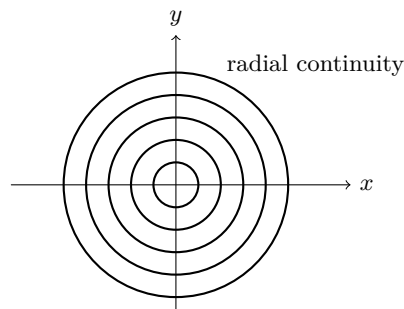
Define

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove directly that f is continuous at the origin, despite the direction-dependent factor in its polar form.

Question 9 – Solution

Strategy Bound the absolute value by a constant times the distance to the origin.



Estimate For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = |x| \frac{|x^2 - y^2|}{x^2 + y^2} \leq |x| \leq \sqrt{x^2 + y^2}.$$

Therefore $|f(x, y) - 0| \leq r$, where r is the distance to the origin.

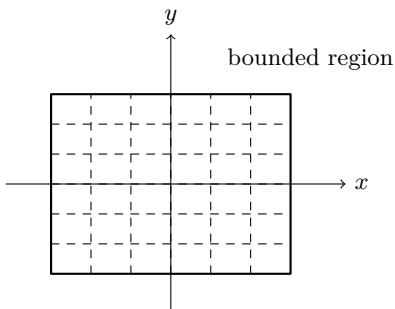
Epsilon argument Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $0 < r < \delta$, then $|f(x, y) - f(0, 0)| < \varepsilon$. Hence $\lim f = 0 = f(0, 0)$ and f is continuous.

Question 10

Let $f(x, y) = \frac{xy}{1 + x^2 + y^2}$ on the closed rectangle $R = [-2, 2] \times [-3, 3]$. Prove that $|f| < 1/2$ everywhere, and explain why f must attain a maximum and minimum on R even without finding them.

Question 10 – Solution

Strategy Compare $2|xy|$ with $x^2 + y^2$, then use continuity on a closed bounded domain.



Bound From $(|x| - |y|)^2 \geq 0$, $2|xy| \leq x^2 + y^2$. Hence

$$|f(x, y)| \leq \frac{x^2 + y^2}{2(1 + x^2 + y^2)} < \boxed{\frac{1}{2}}.$$

The strict inequality comes from the added 1 in the denominator.

Existence of extrema The denominator never vanishes, so f is continuous. The rectangle R is closed and bounded (compact). By the Extreme Value Theorem, f attains both an absolute maximum and absolute minimum on R .

Question 1

Find the domain of

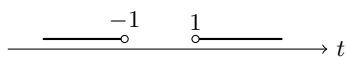
$$\vec{r}(t) = \left\langle \sqrt{t+2}, \ln(1-t), \frac{1}{t^2-1} \right\rangle.$$

Evaluate $\vec{r}(-1/2)$ and determine whether $\lim_{t \rightarrow -1^+} \vec{r}(t)$ exists in $\mathbb{R}^3$.

Question 1 – Solution

Strategy Intersect the domains of all three scalar components; vector limits exist exactly when every component limit is finite.

common component domain



Domain The restrictions are $t \geq -2$, $t < 1$, and $t \neq \pm 1$. Hence

$$\boxed{[-2, -1) \cup (-1, 1]}.$$

Also $\vec{r}(-1/2) = \langle \sqrt{3/2}, \ln(3/2), -4/3 \rangle$.

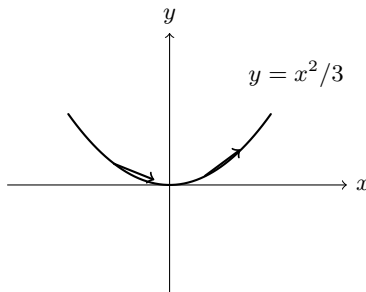
Limit As $t \rightarrow -1^+$, $1/(t^2 - 1) \rightarrow -\infty$, so no finite vector limit exists in $\mathbb{R}^3$.

Question 2

For $\vec{r}(t) = \langle 3t, t^2, 2 - t \rangle$, eliminate the parameter to describe the curve as the intersection of two surfaces. State its orientation as t increases and locate the point corresponding to $t = -1$.

Question 2 – Solution

Strategy Solve the simplest component for t and substitute into the others.



Elimination Since $x = 3t$, $t = x/3$. Therefore

$$\boxed{y = x^2/9, \quad z = 2 - x/3}.$$

The curve is the intersection of a parabolic cylinder and a plane.

Orientation and point As t increases, x increases and z decreases. At $t = -1$, the point is $\boxed{(-3, 1, 3)}$.

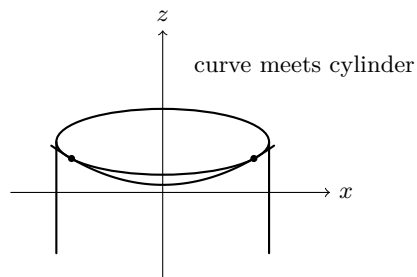
Verification Substituting these coordinates into both surface equations confirms membership.

Question 3

The curve $\vec{r}(t) = \langle t, t^2, 2t - 1 \rangle$ meets the cylinder $x^2 + y^2 = 2$. Find every intersection point and the corresponding parameter values.

Question 3 – Solution

Strategy Substitute the curve's coordinate functions into the surface equation.



Parameter equation We need $t^2 + t^4 = 2$. Let $u = t^2 \geq 0$; then $u^2 + u - 2 = 0$, so $(u - 1)(u + 2) = 0$. Thus $t^2 = 1$ and $t = \pm 1$.

Points At $t = 1$, $\vec{r} = (1, 1, 1)$; at $t = -1$, $\vec{r} = (-1, 1, -3)$. Hence the intersections are

$$\boxed{(1, 1, 1) \quad \text{and} \quad (-1, 1, -3)}.$$

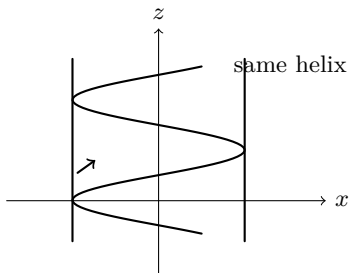
Verification Both points satisfy $x^2 + y^2 = 2$.

Question 4

Show that $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$ and $\vec{r}_2(u) = \langle \cos(2u), \sin(2u), 2u \rangle$ have the same image. Compare their orientations and speeds of traversal at matching points. Then give a reparameterization with the opposite orientation.

Question 4 – Solution

Strategy Relate parameters by matching all three components.



Same image Setting $t = 2u$ makes $\vec{r}_1(t) = \vec{r}_2(u)$, so both trace the same helix. As each parameter increases, z increases, so orientations agree.

Traversal rate A change Δu produces twice the angular and vertical change of the same Δt ; equivalently, $\vec{r}_2(u) = \vec{r}_1(2u)$ traverses the image twice as fast with respect to its parameter.

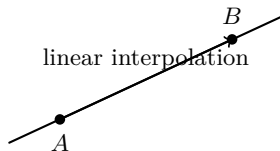
Reverse orientation For example, $\boxed{\vec{q}(s) = \langle \cos s, -\sin s, -s \rangle = \vec{r}_1(-s)}$.

Question 5

Construct a vector function that travels from $A = (-2, 1, 4)$ to $B = (3, -1, 0)$ during $0 \leq t \leq 1$, but does so nonuniformly: it starts and ends at rest while remaining on the line segment.

Question 5 – Solution

Strategy Use $\vec{r} = A + s(t)(B - A)$ with a scalar progress function satisfying $s(0) = 0$, $s(1) = 1$, and $s'(0) = s'(1) = 0$.



Progress function The cubic $s(t) = 3t^2 - 2t^3$ has the required endpoint values and zero endpoint derivatives. Since $0 \leq s(t) \leq 1$ on $[0, 1]$, the path stays on the segment.

Vector function Because $B - A = \langle 5, -2, -4 \rangle$,

$$\boxed{\vec{r}(t) = \langle -2, 1, 4 \rangle + (3t^2 - 2t^3) \langle 5, -2, -4 \rangle}, \quad 0 \leq t \leq 1.$$

Verification Substitution gives the correct endpoints, and $\vec{r}'(0) = \vec{r}'(1) = \vec{0}$.

Question 6

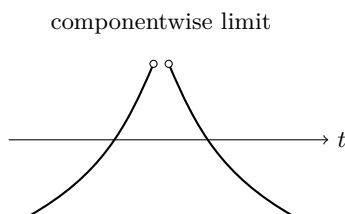
Find constants a, b, c so that

$$\lim_{t \rightarrow 0} \left\langle \frac{\sqrt{1+at}-1}{t}, \frac{\sin(bt)}{t}, \frac{e^{ct}-1}{t} \right\rangle = \langle 2, 3, -4 \rangle.$$

Justify the limit component by component.

Question 6 – Solution

Strategy A vector limit equals the target precisely when all component limits equal their corresponding target components.



First component Rationalizing gives

$$\frac{\sqrt{1+at}-1}{t} = \frac{a}{\sqrt{1+at}+1} \rightarrow \frac{a}{2},$$

so $a = 4$.

Other components Standard limits give $\sin(bt)/t \rightarrow b$ and $(e^{ct}-1)/t \rightarrow c$. Hence $a = 4, b = 3, c = -4$.

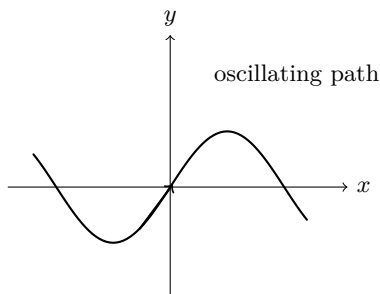
Verification The three component limits are respectively $2, 3, -4$.

Question 7

Analyze the planar vector function $\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ for $0 \leq t \leq 4\pi$. Find the points for $t = 0, \pi, 2\pi, 3\pi, 4\pi$, identify repeated heights, and describe the geometric motion.

Question 7 – Solution

Strategy Evaluate at multiples of π and use the periodicity of sine and cosine.



Key points The points are

$$(0, 0), (\pi, 2), (2\pi, 0), (3\pi, 2), (4\pi, 0).$$

The height oscillates between 0 and 2 while $x = t - \sin t$ increases from 0 to 4π .

Geometry The curve is two arches of a cycloid, moving left to right, with cusps at $(0, 0), (2\pi, 0), (4\pi, 0)$.

Verification Since $1 - \cos t \in [0, 2]$ and $x'(t) = 1 - \cos t \geq 0$, the description is consistent.

Question 8

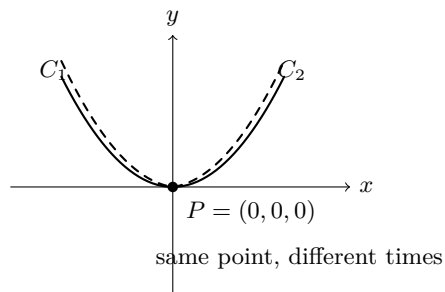
Curves C_1 and C_2 are given by

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \vec{r}_2(s) = \langle 1 - s, (1 - s)^2, s - 1 \rangle.$$

Determine all geometric intersection points. If particles use their displayed parameters as time, determine whether they collide.

Question 8 – Solution

Strategy Geometric intersection permits $t \neq s$; collision requires $t = s$ as well as equal positions.



Intersection From the first coordinate, set $u = 1 - s = t$. The second coordinates then agree automatically. The third coordinates require $t^3 = -u = -t$, so $t(t^2 + 1) = 0$. Thus $t = 0$ and $s = 1$, giving the sole intersection $\boxed{(0, 0, 0)}$.

Collision The required times differ, so particles using the displayed parameters do not collide.

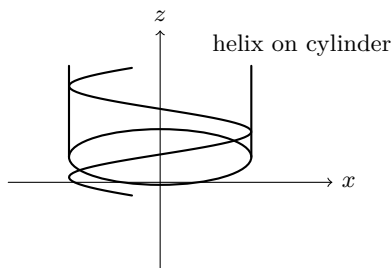
Check At $t = 0$, $\vec{r}_1 = \vec{0}$; at $s = 1$, $\vec{r}_2 = \vec{0}$.

Question 9

Construct a vector function for a helix that lies on the cylinder $(x - 2)^2 + (y + 1)^2 = 9$, begins at $(5, -1, 0)$ when $t = 0$, rises 4 units during each counterclockwise revolution, and makes two revolutions.

Question 9 – Solution

Strategy Use the cylinder center and radius for the circular components, then choose a linear height matching the pitch.



Construction Counterclockwise motion from the rightmost point is

$$x = 2 + 3 \cos t, \quad y = -1 + 3 \sin t.$$

One revolution is $\Delta t = 2\pi$ and should raise z by 4, so $z = (2/\pi)t$. Two revolutions require $0 \leq t \leq 4\pi$. Thus

$$\boxed{\vec{r}(t) = \langle 2 + 3 \cos t, -1 + 3 \sin t, 2t/\pi \rangle}, \quad 0 \leq t \leq 4\pi.$$

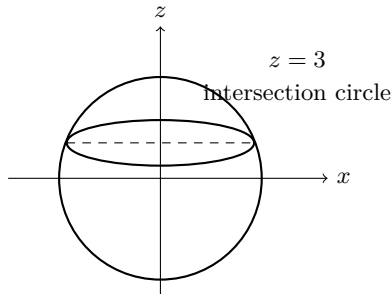
Verification The cylinder equation is identically 9, and the final height is 8.

Question 10

Find a parametrization of the curve where the sphere $x^2 + y^2 + z^2 = 25$ meets the plane $z = 3$. Give both orientations and explain why the parametrization covers the entire intersection exactly once on your chosen interval.

Question 10 – Solution

Strategy A horizontal plane cuts the sphere in a circle; determine its radius.



Circle Substituting $z = 3$ gives $x^2 + y^2 = 16$, a circle of radius 4 centered at $(0, 0, 3)$. A counterclockwise parametrization viewed from positive z is

$$\vec{r}(t) = \langle 4 \cos t, 4 \sin t, 3 \rangle, \quad 0 \leq t < 2\pi.$$

The opposite orientation is $\vec{q}(t) = \langle 4 \cos t, -4 \sin t, 3 \rangle$.

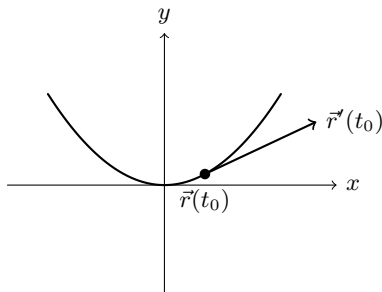
Coverage Sine and cosine generate every radius-4 circle point; the half-open interval prevents duplication of the initial point at 2π .

Question 1

For $\vec{r}(t) = \langle t^2 - 1, e^t, \sin(2t) \rangle$, find $\vec{r}'(t)$ and $\vec{r}''(t)$. Find the tangent line at $t = 0$ and a second-order approximation to $\vec{r}(0.1)$.

Question 1 – Solution

Strategy Differentiate componentwise; use the vector Taylor polynomial through second order.



Derivatives $\vec{r}'(t) = \langle 2t, e^t, 2\cos(2t) \rangle$ and $\vec{r}''(t) = \langle 2, e^t, -4\sin(2t) \rangle$. At $t = 0$, $\vec{r}(0) = \langle -1, 1, 0 \rangle$ and $\vec{r}'(0) = \langle 0, 1, 2 \rangle$.

Tangent line $\vec{L}(s) = \langle -1, 1, 0 \rangle + s \langle 0, 1, 2 \rangle$.

Approximation Since $\vec{r}''(0) = \langle 2, 1, 0 \rangle$,

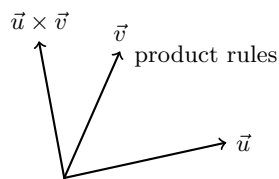
$$\vec{r}(0.1) \approx \vec{r}(0) + 0.1\vec{r}'(0) + \frac{0.1^2}{2}\vec{r}''(0) = \langle -0.99, 1.105, 0.2 \rangle.$$

Question 2

Let $\vec{u}(t) = \langle t, t^2, 1 \rangle$ and $\vec{v}(t) = \langle e^t, \cos t, t \rangle$. Compute $\frac{d}{dt}(\vec{u} \cdot \vec{v})$ and $\frac{d}{dt}(\vec{u} \times \vec{v})$ at $t = 0$ using product rules, then verify the dot-product result by expanding first.

Question 2 – Solution

Strategy Use $(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$ and $(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$.



Values at zero $\vec{u}(0) = \langle 0, 0, 1 \rangle$, $\vec{u}'(0) = \langle 1, 0, 0 \rangle$, $\vec{v}(0) = \langle 1, 1, 0 \rangle$, and $\vec{v}'(0) = \langle 1, 0, 1 \rangle$.

Results Thus

$$(\vec{u} \cdot \vec{v})'(0) = 1 + 1 = \boxed{2},$$

and

$$(\vec{u} \times \vec{v})'(0) = \langle 0, 0, 1 \rangle + \langle 0, 1, 0 \rangle = \boxed{\langle 0, 1, 1 \rangle}.$$

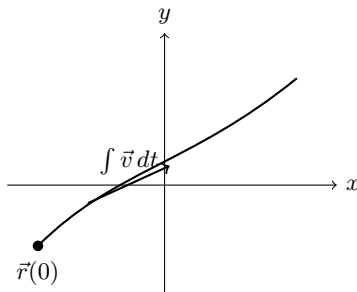
Expanding $\vec{u} \cdot \vec{v} = te^t + t^2 \cos t + t$ and differentiating also gives 2 at zero.

Question 3

A curve has velocity $\vec{r}'(t) = \langle 2t, 3t^2, e^{-t} \rangle$ and passes through $P = (1, -2, 4)$ at $t = 0$. Find $\vec{r}(t)$ and the net displacement from $t = 0$ to $t = 2$.

Question 3 – Solution

Strategy Integrate each velocity component and use the initial condition to determine the constant vector.



Antiderivative $\vec{r}(t) = \langle t^2, t^3, -e^{-t} \rangle + \vec{C}$. Since $\vec{r}(0) = \langle 0, 0, -1 \rangle + \vec{C} = P$, $\vec{C} = \langle 1, -2, 5 \rangle$. Therefore

$$\boxed{\vec{r}(t) = \langle t^2 + 1, t^3 - 2, 5 - e^{-t} \rangle}.$$

Displacement $\vec{r}(2) - \vec{r}(0) = \int_0^2 \vec{r}'(t) dt = \boxed{\langle 4, 8, 1 - e^{-2} \rangle}$.

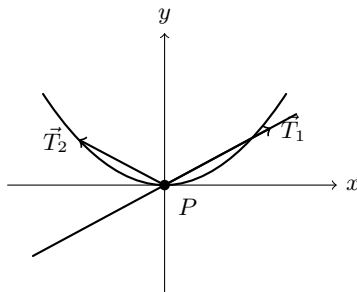
Verification Differentiation recovers the given velocity and substitution gives P .

Question 4

The curves $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(s) = \langle s^2, s, s^3 \rangle$ meet at the origin and at $(1, 1, 1)$. Find the smaller angle between their tangent lines at each intersection.

Question 4 – Solution

Strategy Differentiate each parametrization and evaluate using its own parameter at the shared point.



At the origin Both use parameter 0. The tangent vectors are $\vec{r}_1'(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}_2'(0) = \langle 0, 1, 0 \rangle$, so the angle is $\boxed{90^\circ}$.

At $(1, 1, 1)$ Both use parameter 1. The tangents are $\langle 1, 2, 3 \rangle$ and $\langle 2, 1, 3 \rangle$. Their dot product is 13 and both magnitudes are $\sqrt{14}$, so

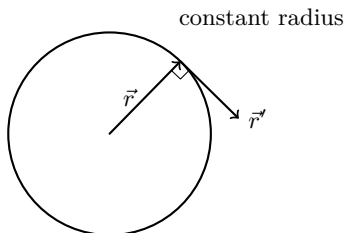
$$\boxed{\theta = \cos^{-1}(13/14) \approx 21.8^\circ}.$$

Question 5

Suppose a differentiable vector function satisfies $\|\vec{r}(t)\| = 5$ for every t . Prove that $\vec{r}(t) \perp \vec{r}'(t)$. Apply the result to $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle$ and verify directly.

Question 5 – Solution

Strategy Differentiate the squared magnitude rather than the square root.



Proof Since $\vec{r} \cdot \vec{r} = 25$, differentiation gives $2\vec{r} \cdot \vec{r}' = 0$. Thus $\boxed{\vec{r} \cdot \vec{r}' = 0}$, proving orthogonality wherever $\vec{r}'$ is nonzero.

Application Here $\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 0 \rangle$ and

$$\vec{r}' \cdot \vec{r}' = -9 \cos t \sin t + 9 \sin t \cos t = 0.$$

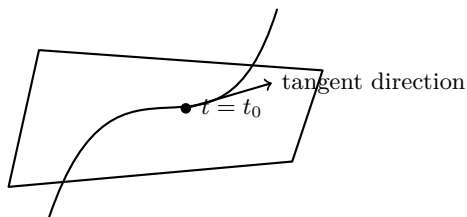
Also $\|\vec{r}'(t)\| = \sqrt{9 + 16} = 5$, as required.

Question 6

For $\vec{r}(t) = \langle t, t^2, t^3 - 3t \rangle$, find all points where the tangent line is parallel to the plane $2x - y + z = 4$. Determine whether any tangent line is perpendicular to that plane.

Question 6 – Solution

Strategy A tangent is parallel to a plane when its direction is orthogonal to the plane normal; it is perpendicular when its direction is parallel to the normal.



Parallel condition $\vec{r}'(t) = \langle 1, 2t, 3t^2 - 3 \rangle$ and $\vec{n} = \langle 2, -1, 1 \rangle$. Thus

$$\vec{r}'(t) \cdot \vec{n} = 3t^2 - 2t - 1 = (3t + 1)(t - 1).$$

So $t = 1$ or $t = -1/3$, giving points $\boxed{(1, 1, -2)}$ and $\boxed{(-1/3, 1/9, 26/27)}$.

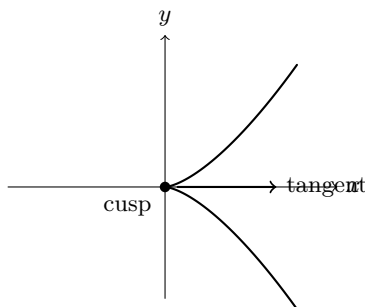
Perpendicular condition Parallelism of $\vec{r}'(t)$ with $\vec{n}$ would force scale $1/2$ from the first component, then $2t = -1/2$, but the third component at $t = -1/4$ is not $1/2$. Hence none exist.

Question 7

Analyze the smoothness and regularity of $\vec{r}(t) = \langle t^2, t^3 \rangle$ near $t = 0$. Does the curve have a tangent line at the origin even though $\vec{r}'(0) = \vec{0}$? Justify using a reparameterization or limiting secant directions.

Question 7 – Solution

Strategy A zero derivative makes the given parametrization nonregular, but the geometric curve may still have a tangent.



Regularity $\vec{r}'(t) = \langle 2t, 3t^2 \rangle$, so $\vec{r}'(0) = \vec{0}$ and this polynomial parametrization is smooth but not regular at 0. Eliminating t gives $y^2 = x^3$, a semicubical cusp.

Tangent direction For $t \neq 0$, the secant direction from the origin is proportional to $\langle t^2, t^3 \rangle = t^2 \langle 1, t \rangle$, which approaches $\langle 1, 0 \rangle$ from both sides. Thus the geometric tangent line exists and is $y = 0$.

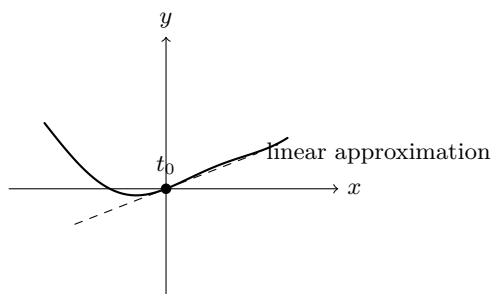
Distinction The curve has a tangent at the cusp, but no nonzero velocity there.

Question 8

For $\vec{r}(t) = \langle \cos t, \sin t, e^t \rangle$, form the second-order Taylor approximation about $t = 0$. Use it to approximate $\vec{r}(0.05)$ and state the order of the neglected error.

Question 8 – Solution

Strategy Use $\vec{r}(0) + t\vec{r}'(0) + \frac{1}{2}t^2\vec{r}''(0)$.



Data $\vec{r}(0) = \langle 1, 0, 1 \rangle$, $\vec{r}'(0) = \langle 0, 1, 1 \rangle$, and $\vec{r}''(0) = \langle -1, 0, 1 \rangle$. Hence

$$\vec{P}_2(t) = \langle 1 - t^2/2, t, 1 + t + t^2/2 \rangle.$$

At $t = 0.05$, $\vec{r}(0.05) \approx \langle 0.99875, 0.05, 1.05125 \rangle$.

Error Because the components have bounded third derivatives near zero, the remainder is $O(t^3)$, here on the scale of 0.05^3 .

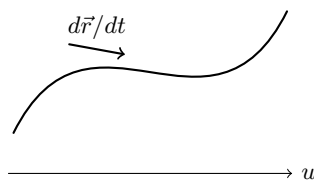
Question 9

Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ and reparameterize by $t = u^3 - u$. Find $d\vec{r}/du$. Identify every u where the reparameterized curve momentarily stops, and explain why the geometric helix itself remains smooth there.

Question 9 – Solution

Strategy Apply the vector chain rule and distinguish the image curve from the chosen parameter clock.

speed changes, image fixed



Chain rule With $t = g(u) = u^3 - u$,

$$\frac{d\vec{r}}{du} = \vec{r}'(g(u))g'(u) = (3u^2 - 1) \langle -\sin(u^3 - u), \cos(u^3 - u), 1 \rangle.$$

It vanishes when $3u^2 - 1 = 0$, so $\boxed{u = \pm 1/\sqrt{3}}$.

Interpretation The original derivative $\langle -\sin t, \cos t, 1 \rangle$ never vanishes. The stops occur because $dt/du = 0$; they reflect a nonregular reparameterization, not a cusp in the helix.

Question 10

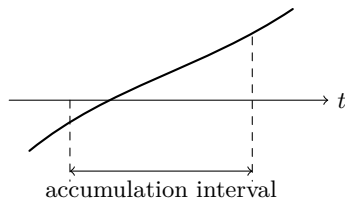
Evaluate

$$\int_0^{\pi} \langle t \sin t, t \cos t, \sin^2 t \rangle dt.$$

Then interpret the result as displacement if the integrand is velocity, and find the average velocity on $[0, \pi]$.

Question 10 – Solution

Strategy Integrate componentwise, using integration by parts for the first two components.



Integral We have

$$\int_0^\pi t \sin t \, dt = \pi, \quad \int_0^\pi t \cos t \, dt = -2, \quad \int_0^\pi \sin^2 t \, dt = \frac{\pi}{2}.$$

Therefore the displacement is $\boxed{\langle \pi, -2, \pi/2 \rangle}$.

Average velocity Divide displacement by the elapsed time π :

$$\boxed{\vec{v}_{\text{avg}} = \langle 1, -2/\pi, 1/2 \rangle}.$$

Verification Multiplying the average velocity by π recovers the net displacement.

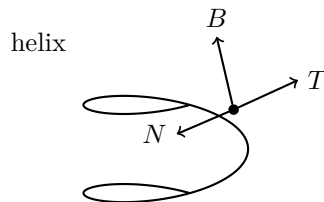
Question 1

For the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, find the unit tangent $\mathbf{T}$, principal unit normal $\mathbf{N}$, and binormal $\mathbf{B}$ for all t . Evaluate the frame at $t = 0$.

Calculus II Section 6-8: Tangent, Normal and Binormal Vectors

Question 1 – Solution

Strategy Normalize $\vec{r}'$, normalize $\mathbf{T}'$, then use $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.



Frame Since $\|\vec{r}'\| = \sqrt{2}$,

$$\mathbf{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle,$$

and

$$\mathbf{B} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle.$$

At $t = 0$ $\mathbf{T} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle, \mathbf{N} = \langle -1, 0, 0 \rangle, \mathbf{B} = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle.$

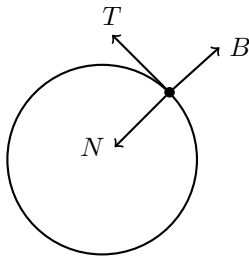
Verification The three vectors are unit, mutually orthogonal, and $\mathbf{T} \times \mathbf{N} = \mathbf{B}$.

Question 2

For the circle $\vec{r}(t) = \langle a \cos t, a \sin t, h \rangle$ with $a > 0$, compute $\mathbf{T}, \mathbf{N}, \mathbf{B}$. Explain why $\mathbf{N}$ is independent of a and why $\mathbf{B}$ is constant.

Question 2 – Solution

Strategy The parameter changes angular position while a changes only the circle's scale.



Computation $\vec{r}' = a \langle -\sin t, \cos t, 0 \rangle$, so

$$\mathbf{T} = \langle -\sin t, \cos t, 0 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

Interpretation The principal normal points radially inward regardless of radius. The curve lies in the fixed horizontal plane $z = h$, so the right-handed binormal is its constant upward normal.

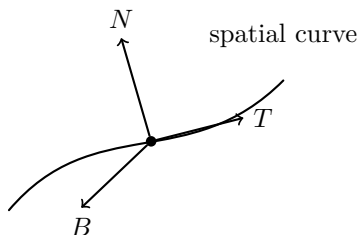
Question 3

Find the Frenet frame for $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Use the cross-product method for $\mathbf{B}$ and verify the orientation.

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Question 3 – Solution

Strategy Use $\mathbf{T} = \vec{r}' / \|\vec{r}'\|$, $\mathbf{B} = (\vec{r}' \times \vec{r}'') / \|\vec{r}' \times \vec{r}''\|$, and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.



Derivatives At $t = 1$, $\vec{r}' = \langle 1, 2, 3 \rangle$ and $\vec{r}'' = \langle 0, 2, 6 \rangle$. Thus

$$\mathbf{T} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle, \quad \mathbf{B} = \frac{1}{\sqrt{19}} \langle 3, -3, 1 \rangle.$$

Normal Therefore

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle.$$

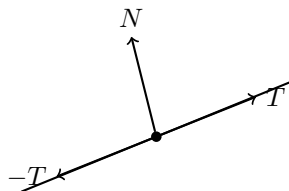
Verification Direct cross multiplication gives $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ and all norms equal 1.

Question 4

A regular curve $\vec{r}(t)$ with nonzero curvature is reparameterized with the opposite orientation by $\vec{q}(u) = \vec{r}(\phi(u))$, where $\phi'(u) < 0$. Determine how $\mathbf{T}, \mathbf{N}, \mathbf{B}$ change. Give a geometric explanation.

Question 4 – Solution

Strategy Apply the chain rule and track the sign introduced by the decreasing parameter.



Tangent Since $\vec{q}' = \vec{r}'(\phi)\phi'$, normalization gives $\mathbf{T}_q = -\mathbf{T}_r$.

Normal Differentiating $\mathbf{T}_q = -\mathbf{T}_r(\phi)$ with respect to u introduces another negative factor ϕ' , so its direction agrees with $\mathbf{N}_r$. Hence $\mathbf{N}_q = \mathbf{N}_r$.

Binormal $\mathbf{B}_q = \mathbf{T}_q \times \mathbf{N}_q = -\mathbf{B}_r$. Thus

$$(\mathbf{T}, \mathbf{N}, \mathbf{B}) \mapsto (-\mathbf{T}, \mathbf{N}, -\mathbf{B}).$$

The bending side is geometric, while forward and right-handed directions reverse.

Question 5

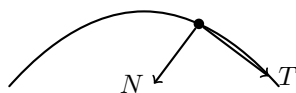
Let $\vec{r}(t) = \langle t, t^2, 0 \rangle$. Find $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for all t . Explain why the binormal is constant and identify the direction in which the curve bends.

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Question 5 – Solution

Strategy Normalize $\vec{r}'$, differentiate $\mathbf{T}$, and use the planar geometry as a check.

B perpendicular to curve's plane



Tangent and normal With $w = \sqrt{1 + 4t^2}$,

$$\mathbf{T} = \frac{1}{w} \langle 1, 2t, 0 \rangle, \quad \mathbf{N} = \frac{1}{w} \langle -2t, 1, 0 \rangle.$$

Consequently $\boxed{\mathbf{B} = \langle 0, 0, 1 \rangle}$.

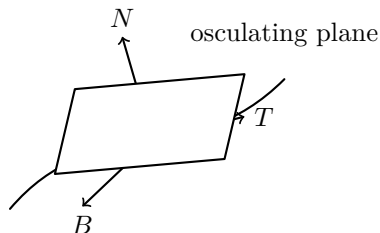
Interpretation The curve is contained in the xy -plane, whose oriented unit normal is constant. The normal has positive y -component for all t , so the upward-opening parabola bends toward its concave side.

Question 6

For $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at $t = 0$, find equations of the normal, osculating, and rectifying planes. State which Frenet vectors span each plane.

Question 6 – Solution

Strategy Use the frame from the helix and choose the Frenet vector perpendicular to each named plane.



Frame and point At $t = 0$, $P = (1, 0, 0)$, $\mathbf{T} \parallel \langle 0, 1, 1 \rangle$, $\mathbf{N} = \langle -1, 0, 0 \rangle$, and $\mathbf{B} \parallel \langle 0, -1, 1 \rangle$.

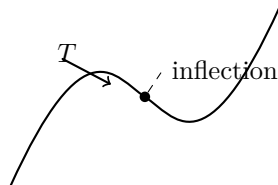
Planes The normal plane is perpendicular to $\mathbf{T}$: $y + z = 0$, and is spanned by $\mathbf{N}, \mathbf{B}$. The osculating plane is perpendicular to $\mathbf{B}$: $-y + z = 0$, spanned by $\mathbf{T}, \mathbf{N}$. The rectifying plane is perpendicular to $\mathbf{N}$: $x = 1$, spanned by $\mathbf{T}, \mathbf{B}$.

Question 7

For $\vec{r}(t) = \langle t, t^3, 0 \rangle$, determine where the Frenet principal normal is undefined. Does the curve still have a tangent there? Analyze the limiting normals on the two sides.

Question 7 – Solution

Strategy Compute $\mathbf{T}'$ and distinguish loss of curvature from loss of tangent direction.



Tangent $\vec{r}'(t) = \langle 1, 3t^2, 0 \rangle$ never vanishes, so the curve is regular and has tangent $\mathbf{T}(0) = \langle 1, 0, 0 \rangle$.

Normal failure Differentiating the normalized tangent shows $\mathbf{T}'(0) = \vec{0}$, so $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$ is undefined at 0. For $t > 0$, the normal tends to $\langle 0, 1, 0 \rangle$; for $t < 0$, it tends to $\langle 0, -1, 0 \rangle$.

Interpretation The origin is an inflection point: curvature vanishes and the bending side changes.

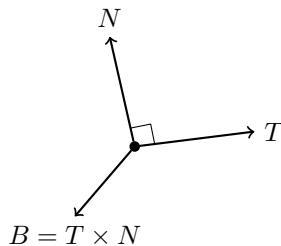
Question 8

Assume $\mathbf{T}$ is a differentiable unit tangent and $\mathbf{T}' \neq \vec{0}$. Prove that the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a right-handed orthonormal basis.

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Question 8 – Solution

Strategy Differentiate $\mathbf{T} \cdot \mathbf{T} = 1$, normalize $\mathbf{T}'$, and use cross-product identities.



Orthogonality Differentiation gives $2\mathbf{T} \cdot \mathbf{T}' = 0$. Since $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$, $\mathbf{T} \perp \mathbf{N}$, and both are unit.

Binormal Define $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The cross product of perpendicular unit vectors is unit and perpendicular to both.

Orientation By definition $\mathbf{T} \times \mathbf{N} = \mathbf{B}$, so the ordered frame is right-handed. Hence its Gram matrix is the identity and its orientation determinant is $+1$.

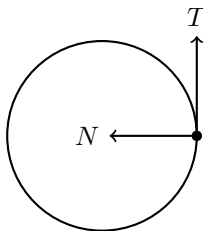
Question 9

The circle $x^2 + y^2 = 4$ is parametrized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3), 0 \rangle$ for $t > 0$. Find its Frenet frame and explain which parts depend on the varying speed.

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Question 9 – Solution

Strategy Let $\theta = t^3$ and separate angular position from the positive rate $d\theta/dt$.



same frame, nonuniform speed

Frame Because $3t^2 > 0$,

$$\mathbf{T} = \langle -\sin(t^3), \cos(t^3), 0 \rangle, \quad \mathbf{N} = \langle -\cos(t^3), -\sin(t^3), 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

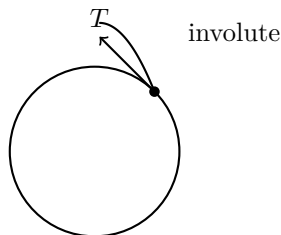
Dependence The frame at a geometric point depends on position and orientation, not on how fast that point is reached. The speed $\|\dot{\mathbf{r}}\| = 6t^2$ varies, but normalization removes it. At $t = 0$ the parametrization stops, so the usual Frenet construction from velocity is not regular there.

Question 10

For the unit circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$, define its involute by $\vec{q}(t) = \vec{r}(t) - t\mathbf{T}(t)$ for $t > 0$. Find $\mathbf{T}_q(t)$ and show that it is parallel to the circle's principal normal at the corresponding point.

Question 10 – Solution

Strategy Differentiate the involute and use the circle's simple Frenet derivatives.



Circle frame $\mathbf{T}(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{T}'(t) = \langle -\cos t, -\sin t \rangle = \mathbf{N}(t)$.

Involute derivative Differentiating,

$$\vec{q}' = \vec{r}' - \mathbf{T} - t\mathbf{T}' = \mathbf{T} - \mathbf{T} - t\mathbf{N} = -t\mathbf{N}.$$

Since $t > 0$, $\|\vec{q}'\| = t$, so $\boxed{\mathbf{T}_q = -\mathbf{N} = \langle \cos t, \sin t \rangle}$.

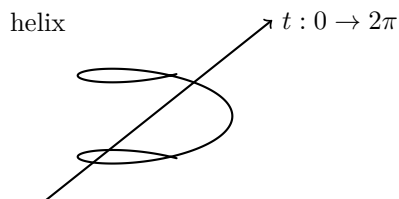
Interpretation The taut string leaves the circle tangentially, while the involute's tangent points along the outward radial direction.

Question 1

A particle follows the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ for $0 \leq t \leq 2\pi$. Find the exact distance traveled. Then explain why the vertical rise alone is not the distance.

Question 1 – Solution

Strategy Distance traveled is the integral of speed, not the net change in any one coordinate.



Speed

$$\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\vec{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Arc length Therefore

$$L = \int_0^{2\pi} 5 \, dt = 10\pi.$$

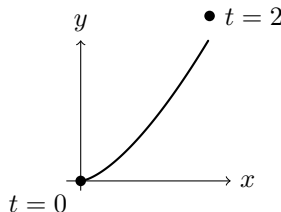
Interpretation The vertical rise is $4(2\pi) = 8\pi$, but the particle simultaneously travels around the cylinder. The velocity has perpendicular horizontal and vertical components of magnitudes 3 and 4, producing total speed 5.

Question 2

Find the exact arc length of $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3 \rangle$ on $0 \leq t \leq 2$. Show the algebraic step that makes the integral elementary.

Question 2 – Solution

Strategy Differentiate first, factor the speed, and use $u = 1 + t^2$.



Speed

$$\vec{r}'(t) = \langle 2t, 2t^2 \rangle, \quad \|\vec{r}'(t)\| = 2t\sqrt{1+t^2},$$

where $t \geq 0$ allows $\sqrt{t^2} = t$.

Integration

$$L = \int_0^2 2t\sqrt{1+t^2} dt = \frac{2}{3} \left[(1+t^2)^{3/2} \right]_0^2 = \boxed{\frac{2}{3}(5\sqrt{5} - 1)}.$$

Check Both coordinate functions increase. The straight-line distance between the endpoints is

$$\sqrt{4^2 + (16/3)^2} = \frac{20}{3}.$$

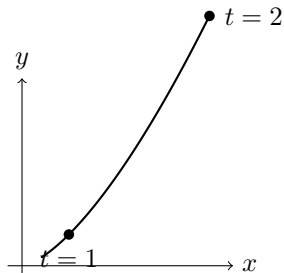
The computed curve length is larger, as it must be.

Question 3

For $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, 0 \rangle$, find the arc length from the point corresponding to $t = 1$ to the point corresponding to $t = 2$. Also find the chord length and compare the two quantities.

Question 3 – Solution

Strategy Use the same speed structure as in Question 2, but retain the nonzero lower limit.



Arc length

$$L = \int_1^2 2t\sqrt{1+t^2} dt = \boxed{\frac{2}{3}(5\sqrt{5} - 2\sqrt{2})}.$$

Chord The endpoints are $P = (1, 2/3, 0)$ and $Q = (4, 16/3, 0)$, so

$$\|Q - P\| = \sqrt{3^2 + \left(\frac{14}{3}\right)^2} = \boxed{\frac{\sqrt{277}}{3}}.$$

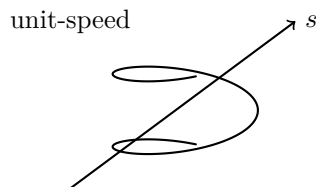
Comparison Numerically, $L \approx 5.568$ while the chord is about 5.548. The curve length is slightly larger, as required; equality would occur only for a straight segment traversed without reversal.

Question 4

Let $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ with starting point at $t = 0$. Find its arc-length function $s(t)$ and reparameterize the helix by arc length. Verify directly that the new parameterization has unit speed.

Question 4 – Solution

Strategy Integrate speed from the stated base point, solve for t , and substitute.



Arc-length coordinate From Question 1, $\|\vec{r}'(t)\| = 5$. Hence $s(t) = \int_0^t 5 \, du = 5t$, so $t = s/5$.

Reparameterization

$$\boxed{\vec{R}(s) = \langle 3 \cos(s/5), 3 \sin(s/5), 4s/5 \rangle}, \quad s \geq 0.$$

Verification

$$\vec{R}'(s) = \left\langle -\frac{3}{5} \sin(s/5), \frac{3}{5} \cos(s/5), \frac{4}{5} \right\rangle, \quad \|\vec{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

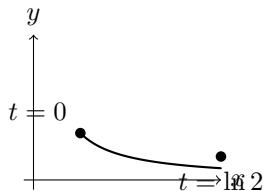
Thus an increase of one unit in s represents exactly one unit of distance along the helix.

Question 5

Evaluate exactly the length of $\vec{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$ for $0 \leq t \leq \ln 2$. The square root in the speed appears complicated; identify its hidden perfect square.

Question 5 – Solution

Strategy Expand the squared speed and recognize $(e^t + e^{-t})^2$.



Speed

$$\|\vec{r}'(t)\|^2 = e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2.$$

Both terms are positive, so $\|\vec{r}'(t)\| = e^t + e^{-t}$.

Length

$$L = \int_0^{\ln 2} (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^{\ln 2} = \boxed{\frac{3}{2}}.$$

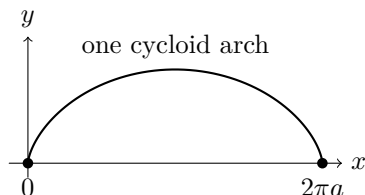
Insight Simplifying the norm before choosing a substitution is decisive here; treating the radical as generic would hide the elementary antiderivative.

Question 6

One arch of a cycloid is parameterized by $\vec{r}(t) = \langle a(t - \sin t), a(1 - \cos t) \rangle$, $0 \leq t \leq 2\pi$, where $a > 0$. Prove that its length is $8a$. Explain the absolute-value issue in simplifying the speed.

Question 6 – Solution

Strategy Use half-angle identities and determine the sign of $\sin(t/2)$ on the interval.



Speed

$$\vec{r}' = a \langle 1 - \cos t, \sin t \rangle, \quad \|\vec{r}'\| = a\sqrt{2 - 2\cos t} = 2a \left| \sin \frac{t}{2} \right|.$$

For $0 \leq t \leq 2\pi$, $\sin(t/2) \geq 0$, so the absolute value may be removed.

Length

$$L = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = \left[-4a \cos \frac{t}{2} \right]_0^{2\pi} = \boxed{8a}.$$

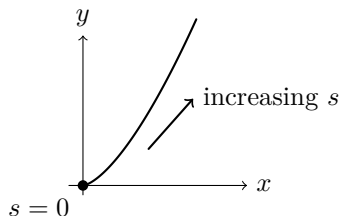
Endpoint note The velocity vanishes at the two cusps, but the speed is continuous and integrable, so the arch still has a well-defined finite length.

Question 7

For $t \geq 0$, let $\vec{r}(t) = \langle \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$. Construct the arc-length coordinate measured from $t = 0$, invert it explicitly, and give a unit-speed parameterization $\vec{R}(s)$.

Question 7 – Solution

Strategy The factor $t \geq 0$ makes the speed easy to integrate; then solve algebraically for t^2 .



Arc-length function

$$\|\vec{r}'(t)\| = \|\langle t, t^2 \rangle\| = t\sqrt{1+t^2}, \quad s(t) = \frac{1}{3}((1+t^2)^{3/2} - 1).$$

Inverse From $(1+t^2)^{3/2} = 3s+1$,

$$t(s) = \sqrt{(3s+1)^{2/3} - 1}.$$

Unit-speed form With $u = \sqrt{(3s+1)^{2/3} - 1}$,

$$\boxed{\vec{R}(s) = \left\langle \frac{1}{2}u^2, \frac{1}{3}u^3 \right\rangle}, \quad s \geq 0.$$

Because s was defined by integrating speed and is strictly increasing for $t > 0$, the chain rule gives $\|d\vec{R}/ds\| = 1$ there; the right-hand unit-speed limit also exists at $s = 0$.

Question 8

Suppose $\vec{q}(u) = \vec{r}(\phi(u))$ traces the same regular curve, where ϕ is continuously differentiable and strictly monotone. Prove that the arc length is unchanged by this reparameterization, including the case in which orientation is reversed.

Question 8 – Solution

Strategy Apply the chain rule to speed, then use substitution while carefully retaining the absolute value.

same trace, opposite orientations



Transformed speed

$$\vec{q}'(u) = \vec{r}'(\phi(u))\phi'(u), \quad \|\vec{q}'(u)\| = \|\vec{r}'(\phi(u))\| |\phi'(u)|.$$

Increasing case If $\phi' \geq 0$ and $\phi(\alpha) = a, \phi(\beta) = b$, substitution gives

$$\int_{\alpha}^{\beta} \|\vec{q}'(u)\| du = \int_a^b \|\vec{r}'(t)\| dt.$$

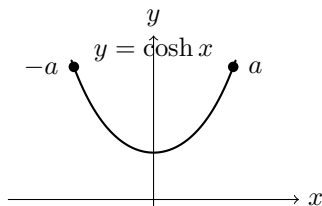
Decreasing case If $\phi' \leq 0$, then $\phi(\alpha) = b$ and $\phi(\beta) = a$. Since $|\phi'| = -\phi'$, reversing the substituted limits again produces $\int_a^b \|\vec{r}'(t)\| dt$. Thus length depends on the traced curve, not its orientation or timing.

Question 9

The catenary $\vec{r}(t) = \langle t, \cosh t \rangle$ is restricted to $-a \leq t \leq a$, where $a > 0$. Find its exact length in terms of a . Then determine the value of a that makes the length equal to 6.

Question 9 – Solution

Strategy Use the identity $1 + \sinh^2 t = \cosh^2 t$ and the positivity of $\cosh t$.



Length

$$\|\vec{r}'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t, \quad L = \int_{-a}^a \cosh t \, dt = 2 \sinh a.$$

Therefore $\boxed{L = 2 \sinh a}$.

Prescribed length Setting $2 \sinh a = 6$ gives $\sinh a = 3$, hence

$$\boxed{a = \operatorname{arsinh}(3) = \ln(3 + \sqrt{10})}.$$

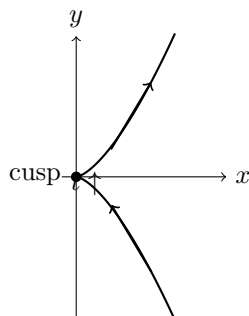
Symmetry The even speed makes the two halves have equal length; equivalently, $L = 2 \int_0^a \cosh t \, dt$.

Question 10

Consider $\vec{r}(t) = \langle t^2, t^3 \rangle$ on $-1 \leq t \leq 1$. The parameterization is not regular at $t = 0$. Determine the total arc length exactly and explain why one must split the integral or preserve an absolute value.

Question 10 – Solution

Strategy The speed contains $\sqrt{t^2} = |t|$; use symmetry across the cusp.



Speed

$$\vec{r}'(t) = \langle 2t, 3t^2 \rangle, \quad \|\vec{r}'(t)\| = |t|\sqrt{4 + 9t^2}.$$

Length

$$L = 2 \int_0^1 t\sqrt{4 + 9t^2} dt = \frac{2}{27} \left[(4 + 9t^2)^{3/2} \right]_0^1 = \boxed{\frac{2}{27}(13\sqrt{13} - 8)}.$$

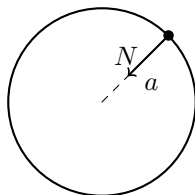
Why care about $|t|$? Replacing $|t|$ by t on the full symmetric interval would make the integrand odd and falsely give zero. Although velocity vanishes at the cusp, each half is smooth and their nonnegative lengths add.

Question 1

Starting from $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$, where $a > 0$, derive the curvature of a circle. Interpret the result in terms of turning sharpness and radius of curvature.

Question 1 – Solution

Strategy Use $\kappa = \|\vec{r}' \times \vec{r}''\| / \|\vec{r}'\|^3$, viewing planar vectors as three-dimensional vectors with zero third component.



Calculation $\vec{r}' = \langle -a \sin t, a \cos t, 0 \rangle$ and $\vec{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle$. Thus $\|\vec{r}'\| = a$ and $\|\vec{r}' \times \vec{r}''\| = a^2$. Therefore

$$\boxed{\kappa = \frac{a^2}{a^3} = \frac{1}{a}}, \quad \boxed{\rho = \frac{1}{\kappa} = a}.$$

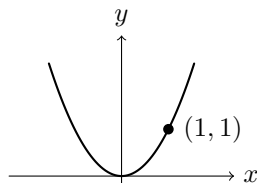
Interpretation Small circles turn more sharply and have larger curvature; the radius of curvature ρ is exactly the circle's geometric radius.

Question 2

Find the curvature of the parabola $\vec{r}(t) = \langle t, t^2 \rangle$ for general t , and evaluate it at the point $(1, 1)$. Where is the curvature largest?

Question 2 – Solution

Strategy Apply the planar curvature formula and then analyze its denominator.



Calculation With $x' = 1$, $y' = 2t$, $x'' = 0$, and $y'' = 2$,

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \boxed{\frac{2}{(1 + 4t^2)^{3/2}}}.$$

At $t = 1$, $\boxed{\kappa = 2/(5\sqrt{5})}$.

Maximum The numerator is constant and the denominator is minimized at $t = 0$. Hence the maximum curvature is $\kappa(0) = 2$, at the vertex.

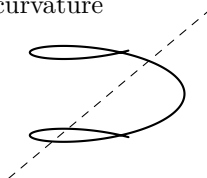
Question 3

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, compute the curvature for all t . Explain geometrically why a constant answer is reasonable.

Question 3 – Solution

Strategy Compute speed and the magnitude of $\vec{r}' \times \vec{r}''$ separately.

constant curvature



Calculation

$$\vec{r}' = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{r}'' = \langle -3 \cos t, -3 \sin t, 0 \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 12 \sin t, -12 \cos t, 9 \rangle.$$

Thus $\|\vec{r}'\| = 5$ and $\|\vec{r}' \times \vec{r}''\| = 15$, giving

$$\kappa = \frac{15}{5^3} = \frac{3}{25}.$$

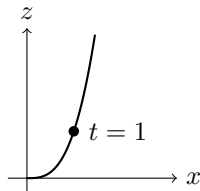
Geometry Every point of a circular helix has the same local shape: rotation about and translation along its axis carry one point to another, so its bending is uniform.

Question 4

Compute the curvature of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Show the cross product and simplify the exact value.

Question 4 – Solution

Strategy Use the parameter-independent cross-product formula.



Derivatives

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}'' = \langle 0, 2, 6t \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 6t^2, -6t, 2 \rangle.$$

At $t = 1$, its magnitude is $2\sqrt{19}$, while $\|\vec{r}'(1)\| = \sqrt{14}$. Therefore

$$\kappa(1) = \frac{2\sqrt{19}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}}.$$

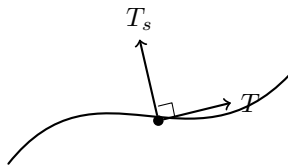
Check The cross product is nonzero, so the tangent is genuinely changing direction and the curvature is positive.

Question 5

Let $\vec{r}(s)$ be a twice-differentiable unit-speed curve. Prove that $\kappa(s) = \|\vec{r}''(s)\|$ and that the acceleration $\vec{r}''(s)$ is perpendicular to the tangent.

Question 5 – Solution

Strategy Differentiate the unit-speed identity $\vec{r}' \cdot \vec{r}' = 1$.



Orthogonality Because $\|\vec{r}'\| = 1$,

$$\vec{r}' \cdot \vec{r}' = 1 \implies 2\vec{r}' \cdot \vec{r}'' = 0.$$

Thus $\vec{r}'' \perp \vec{r}' = \mathbf{T}$.

Curvature For an arc-length parameter, $\mathbf{T} = d\vec{r}/ds$. By definition,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\|.$$

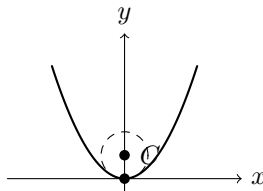
Interpretation Unit speed removes tangential acceleration. Any remaining second derivative measures only the rate at which the direction turns.

Question 6

Find the osculating circle of the parabola $y = x^2$ at its vertex. State its center, radius, and equation, and explain which side of the tangent contains the center.

Question 6 – Solution

Strategy Use curvature for the radius and the principal normal for the center direction.



Curvature and normal From Question 2, $\kappa(0) = 2$, so $\rho = 1/\kappa = 1/2$. At the vertex, the unit tangent points right and the parabola bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Center and equation

$$C = (0, 0) + \frac{1}{2} \langle 0, 1 \rangle = \left(0, \frac{1}{2} \right),$$

$$\boxed{x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4}}.$$

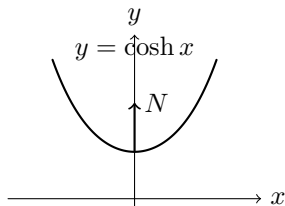
Geometry The center lies on the concave side of the curve, above the horizontal tangent. The circle has second-order contact with the parabola at the vertex.

Question 7

For the catenary $y = \cosh x$, find $\kappa(x)$ and the radius of curvature $\rho(x)$. Where is the curve most sharply bent, and what happens as $|x| \rightarrow \infty$?

Question 7 – Solution

Strategy Use the graph formula $\kappa = |y''|/(1 + y'^2)^{3/2}$ and a hyperbolic identity.



Calculation Since $y' = \sinh x$, $y'' = \cosh x$, and $1 + \sinh^2 x = \cosh^2 x$,

$$\kappa(x) = \frac{\cosh x}{\cosh^3 x} = \operatorname{sech}^2 x, \quad \rho(x) = \cosh^2 x.$$

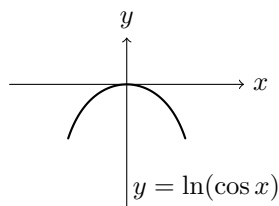
Analysis Curvature is largest at $x = 0$, where $\kappa = 1$. As $|x| \rightarrow \infty$, $\kappa \rightarrow 0$ and $\rho \rightarrow \infty$, so the arms become locally less sharply bent.

Question 8

On $-\frac{\pi}{2} < x < \frac{\pi}{2}$, determine the curvature of $y = \ln(\cos x)$. Explain why the stated domain makes the simplification unambiguous.

Question 8 – Solution

Strategy Differentiate twice and reduce the denominator using $1 + \tan^2 x = \sec^2 x$.



Calculation

$$y' = -\tan x, \quad y'' = -\sec^2 x.$$

Therefore

$$\kappa(x) = \frac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = \frac{\sec^2 x}{|\sec x|^3}.$$

On the stated interval, $\cos x > 0$ and $\sec x > 0$, so

$$\boxed{\kappa(x) = \cos x}.$$

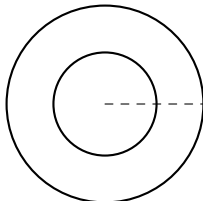
Behavior The maximum curvature is 1 at $x = 0$, while $\kappa \rightarrow 0$ as the graph approaches either vertical asymptote.

Question 9

A regular curve $\vec{r}(t)$ is enlarged by a factor $c > 0$ to form $\vec{q}(t) = c\vec{r}(t)$. Prove that $\kappa_q = \kappa_r/c$. What happens to the radius of curvature?

Question 9 – Solution

Strategy Track how first derivatives, second derivatives, cross products, and norms scale.



dilation by c lowers curvature by $1/c$

Scaling

$$\vec{q}' = c\vec{r}', \quad \vec{q}'' = c\vec{r}'', \quad \|\vec{q}' \times \vec{q}''\| = c^2 \|\vec{r}' \times \vec{r}''\|.$$

Also $\|\vec{q}'\|^3 = c^3 \|\vec{r}'\|^3$. Hence

$$\kappa_q = \frac{c^2}{c^3} \kappa_r = \frac{\kappa_r}{c}.$$

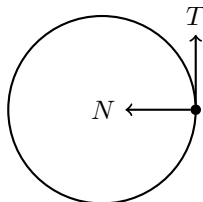
Radius Since $\rho = 1/\kappa$, $\boxed{\rho_q = c\rho_r}$. Enlargement makes every local osculating circle c times larger and the curve correspondingly less sharply bent.

Question 10

The circle of radius 2 is parameterized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3) \rangle$ for $t > 0$. Compute its curvature directly and explain why the changing speed does not affect the result.

Question 10 – Solution

Strategy Let $\theta = t^3$ and use the turning-angle relation $\kappa = \|d\mathbf{T}/dt\|/(ds/dt)$.



nonuniform parameter, same curvature

Tangent and rates Since $\theta' = 3t^2 > 0$,

$$\mathbf{T} = \langle -\sin \theta, \cos \theta \rangle, \quad \left\| \frac{d\mathbf{T}}{dt} \right\| = 3t^2.$$

The speed is $ds/dt = \|\vec{r}'\| = 2(3t^2) = 6t^2$.

Curvature

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{3t^2}{6t^2} = \frac{1}{2}.$$

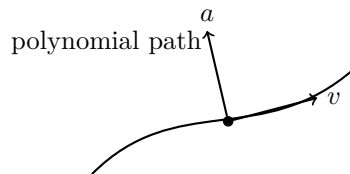
Interpretation Curvature measures direction change per unit distance, not per unit time. Both direction-change rate and speed acquire the same parameter factor, which cancels.

Question 1

A particle has position $\vec{r}(t) = \langle t^2 - 2t, t^3 - 3t, 2t \rangle$, $t \geq 0$. Find velocity, acceleration, and speed at $t = 2$. Determine whether it is speeding up or slowing down then.

Question 1 – Solution

Strategy Differentiate componentwise and use the sign of $\vec{v} \cdot \vec{a}$.



Kinematics

$$\vec{v} = \langle 2t - 2, 3t^2 - 3, 2 \rangle, \quad \vec{a} = \langle 2, 6t, 0 \rangle.$$

At $t = 2$, $\vec{v} = \langle 2, 9, 2 \rangle$, $\vec{a} = \langle 2, 12, 0 \rangle$, and

$\text{speed} = \|\vec{v}\| = \sqrt{89}.$

Speeding test $\vec{v} \cdot \vec{a} = 4 + 108 = 112 > 0$, so speed is increasing. The particle is

speeding up

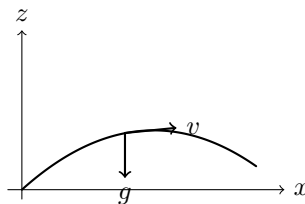
 at $t = 2$.

Question 2

A projectile is launched from $\langle 0, 0, 5 \rangle$ with initial velocity $\langle 12, 5, 20 \rangle$ ft/s. With gravity $\vec{a} = \langle 0, 0, -32 \rangle$ ft/s², find its position, time of impact with $z = 0$, and horizontal range vector.

Question 2 – Solution

Strategy Integrate constant acceleration and impose the ground condition on the vertical coordinate.



Position

$$\vec{r}(t) = \langle 12t, 5t, 5 + 20t - 16t^2 \rangle.$$

Impact Solve $5 + 20t - 16t^2 = 0$. The positive root is

$$t_* = \frac{5 + \sqrt{45}}{8} = \frac{5 + 3\sqrt{5}}{8}.$$

Horizontal range At impact,

$$\langle x(t_*), y(t_*) \rangle = \frac{5 + 3\sqrt{5}}{8} \langle 12, 5 \rangle.$$

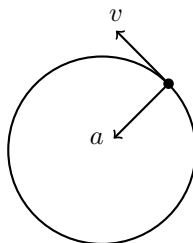
The horizontal velocity is constant because gravity has no horizontal component.

Question 3

A particle moves on the circle $\vec{r}(t) = \langle R \cos(\omega t), R \sin(\omega t) \rangle$, where $R, \omega > 0$. Find velocity, acceleration, speed, and show that acceleration points toward the center with magnitude v^2/R .

Question 3 – Solution

Strategy Differentiate twice and compare acceleration with the position vector.



Derivatives

$$\vec{v} = R\omega \langle -\sin \omega t, \cos \omega t \rangle, \quad \vec{a} = -R\omega^2 \langle \cos \omega t, \sin \omega t \rangle = -\omega^2 \vec{r}.$$

Thus $\boxed{v = R\omega}$ and acceleration is radially inward.

Magnitude

$$\|\vec{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \boxed{\frac{v^2}{R}}.$$

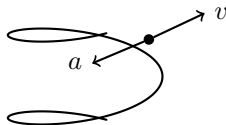
Velocity is tangent to the circle and perpendicular to the centripetal acceleration.

Question 4

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, find velocity and acceleration. Prove that its speed is constant and decide whether acceleration is entirely normal.

Question 4 – Solution

Strategy Compute $\vec{v} \cdot \vec{a}$ after differentiating.



Derivatives

$$\vec{v} = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{a} = \langle -3 \cos t, -3 \sin t, 0 \rangle.$$

The speed is $\|\vec{v}\| = 5$, independent of t .

Decomposition Since

$$\vec{v} \cdot \vec{a} = 9 \sin t \cos t - 9 \sin t \cos t = 0,$$

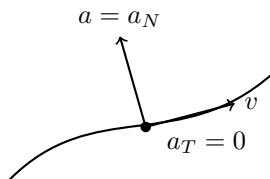
acceleration is perpendicular to velocity. Hence $a_T = 0$ and all acceleration is normal, with $a_N = \|\vec{a}\| = 3$.

Question 5

At an instant, a particle has velocity $\vec{v} = \langle 2, 1, 2 \rangle$ and acceleration $\vec{a} = \langle 1, 2, -2 \rangle$. Find the scalar tangential and normal components a_T, a_N . State what each says about the motion.

Question 5 – Solution

Strategy Use $a_T = (\vec{v} \cdot \vec{a})/\|\vec{v}\|$ and $a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$.



Components Here $\|\vec{v}\| = 3$, $\|\vec{a}\| = 3$, and $\vec{v} \cdot \vec{a} = 0$. Thus

$$\boxed{a_T = 0}, \quad \boxed{a_N = 3}.$$

Interpretation The instantaneous speed is not changing because the tangential component is zero. The full acceleration changes the direction of motion; it is perpendicular to the velocity.

Question 6

Prove that a particle with nonzero velocity is speeding up exactly when $\vec{v} \cdot \vec{a} > 0$, and slowing down exactly when $\vec{v} \cdot \vec{a} < 0$.

Question 6 – Solution

Strategy Differentiate speed through its square to avoid differentiating a square root prematurely.

$$v \cdot a > 0$$



Derivation Let $v = \|\vec{v}\| > 0$. Then

$$v^2 = \vec{v} \cdot \vec{v}.$$

Differentiating gives $2v \, dv/dt = 2\vec{v} \cdot \vec{a}$, so

$$\boxed{\frac{dv}{dt} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}}.$$

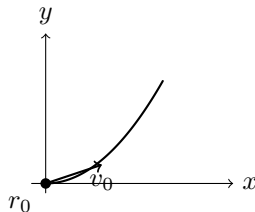
Conclusion The denominator is positive. Therefore the sign of the speed derivative is exactly the sign of $\vec{v} \cdot \vec{a}$: positive means speeding up, negative means slowing down, and zero means instantaneously constant speed.

Question 7

A particle has acceleration $\vec{a}(t) = \langle 2, 6t, e^t \rangle$, initial velocity $\vec{v}(0) = \langle 1, -1, 0 \rangle$, and initial position $\vec{r}(0) = \langle 2, 0, 3 \rangle$. Determine $\vec{v}(t)$ and $\vec{r}(t)$.

Question 7 – Solution

Strategy Integrate twice and use each vector initial condition to determine constants.



Velocity

$$\vec{v}(t) = \langle 2t + C_1, 3t^2 + C_2, e^t + C_3 \rangle.$$

Using $\vec{v}(0) = \langle 1, -1, 0 \rangle$ gives

$$\boxed{\vec{v}(t) = \langle 2t + 1, 3t^2 - 1, e^t - 1 \rangle}.$$

Position Integrating again and applying $\vec{r}(0) = \langle 2, 0, 3 \rangle$,

$$\boxed{\vec{r}(t) = \langle t^2 + t + 2, t^3 - t, e^t - t + 2 \rangle}.$$

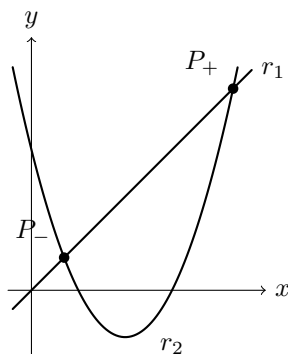
Verification Differentiating the final position recovers $\vec{v}$, and differentiating once more recovers the given acceleration.

Question 8

Particles follow $\vec{r}_1(t) = \langle t, t, 0 \rangle$ and $\vec{r}_2(t) = \langle 2 - t, t^2 - 1, 0 \rangle$. Do their paths intersect? Do the particles collide for $t \geq 0$? Clearly distinguish the two questions.

Question 8 – Solution

Strategy Path intersection may use different parameter values; collision requires equal positions at the same time.



Path intersection Write the first path as $y = x$. On the second, $x = 2 - u$ and $y = u^2 - 1$. Thus

$$u^2 + u - 3 = 0, \quad u = \frac{-1 \pm \sqrt{13}}{2}.$$

The full paths intersect at

$$\left(\frac{5 \mp \sqrt{13}}{2}, \frac{5 \mp \sqrt{13}}{2}, 0 \right).$$

If both paths are restricted to nonnegative times, only the point with $5 - \sqrt{13}$ remains.

Collision test At the same time t , equality of x -coordinates requires $t = 2 - t$, so $t = 1$. But

$$\vec{r}_1(1) = \langle 1, 1, 0 \rangle, \quad \vec{r}_2(1) = \langle 1, 0, 0 \rangle.$$

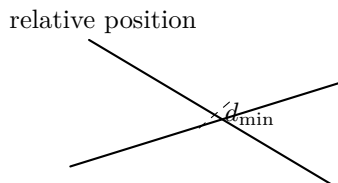
Therefore the particles do not collide. The geometric intersections occur at different parameter times.

Question 9

Two particles move according to $\vec{r}_1(t) = \langle t, 2t, 0 \rangle$ and $\vec{r}_2(t) = \langle 4 - t, 1, 2t \rangle$ for $t \geq 0$. Find the time and value of their minimum separation.

Question 9 – Solution

Strategy Minimize squared distance; it has the same minimizer as distance and avoids a square root.



Relative position

$$\vec{d}(t) = \vec{r}_1 - \vec{r}_2 = \langle 2t - 4, 2t - 1, -2t \rangle.$$

Thus

$$D^2(t) = (2t - 4)^2 + (2t - 1)^2 + (-2t)^2 = 12t^2 - 20t + 17.$$

Minimum The vertex occurs at $t = 20/24 = 5/6$. Then

$$D^2\left(\frac{5}{6}\right) = \frac{26}{3}, \quad \boxed{D_{\min} = \sqrt{\frac{26}{3}}}.$$

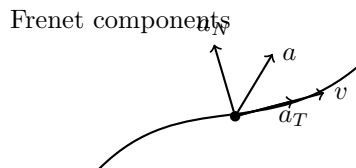
Because the quadratic opens upward and $5/6 \geq 0$, this is the required global minimum.

Question 10

For a regular motion with speed $v = ds/dt$, prove the Frenet acceleration formula $\vec{a} = (dv/dt)\mathbf{T} + \kappa v^2\mathbf{N}$. Identify the geometric role of each term.

Question 10 – Solution

Strategy Write velocity as speed times unit tangent and apply the product and chain rules.



Derivation

$$\vec{v} = v\mathbf{T}.$$

Therefore

$$\vec{a} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Since $d\mathbf{T}/ds = \kappa\mathbf{N}$ and $ds/dt = v$,

$$\boxed{\vec{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}}.$$

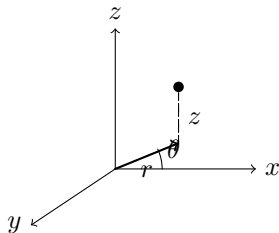
Meaning The tangential term changes speed. The normal term changes direction and points toward the local center of curvature; its magnitude grows quadratically with speed.

Question 1

Convert the rectangular point $(-3, 3\sqrt{3}, 5)$ to cylindrical coordinates using the convention $r \geq 0$ and $0 \leq \theta < 2\pi$. Verify by converting back.

Question 1 – Solution

Strategy Compute r first, then place the angle in the correct quadrant.



Conversion

$$r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = 6.$$

Because $x < 0$ and $y > 0$, the point lies in Quadrant II. With $\tan \theta = -\sqrt{3}$, $\theta = 2\pi/3$. Hence

$$(r, \theta, z) = \left(6, \frac{2\pi}{3}, 5\right).$$

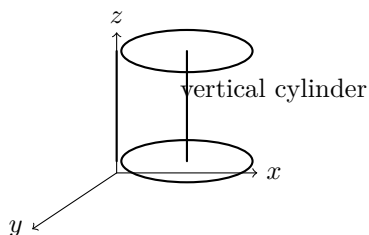
Verification $x = 6 \cos(2\pi/3) = -3$, $y = 6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 5$.

Question 2

Rewrite the surface $x^2 + y^2 = 4x$ in cylindrical coordinates. Identify the surface geometrically, including its axis and radius.

Question 2 – Solution

Strategy Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Cylindrical equation

$$r^2 = 4r \cos \theta.$$

The equation $r = 4 \cos \theta$ describes the non-axis points; $r = 0$ is already included, so

$$\boxed{r = 4 \cos \theta}.$$

Geometry In rectangular form,

$$(x - 2)^2 + y^2 = 4.$$

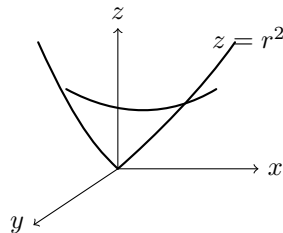
Thus the surface is a vertical circular cylinder of radius 2, whose axis is the line $\boxed{x = 2, y = 0}$ parallel to the z -axis.

Question 3

Convert the paraboloid $z = x^2 + y^2$ to cylindrical coordinates. Describe its traces for fixed z and for fixed θ , and explain why the new equation reveals its symmetry.

Question 3 – Solution

Strategy Use $x^2 + y^2 = r^2$ and interpret one coordinate at a time.



Equation

$$\boxed{z = r^2}, \quad r \geq 0.$$

Horizontal traces For $z = c \geq 0$, $r = \sqrt{c}$, a circle centered on the z -axis. There are no points for $c < 0$.

Vertical traces Fixing θ selects a vertical half-plane through the z -axis; within it, $z = r^2$ is one half of an upward-opening parabola (including its vertex).

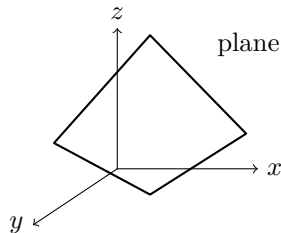
Symmetry The equation contains no θ , so rotating any point about the z -axis preserves the surface.

Question 4

Express the plane $x + y + z = 4$ in cylindrical coordinates. Solve for r where possible and state the angles at which that solved form breaks down.

Question 4 – Solution

Strategy Substitute $x = r \cos \theta$ and $y = r \sin \theta$.



Equation

$$r(\cos \theta + \sin \theta) + z = 4.$$

If $\cos \theta + \sin \theta \neq 0$, then

$$r = \frac{4 - z}{\cos \theta + \sin \theta}.$$

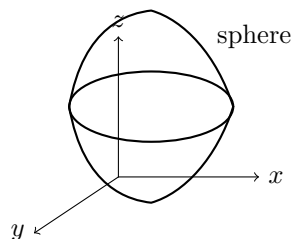
Exceptional angles $\cos \theta + \sin \theta = 0$ when $\theta = 3\pi/4$ or $7\pi/4$ modulo 2π . At those directions, the original cylindrical equation reduces to $z = 4$; dividing by the zero angular factor would incorrectly discard or invent points.

Question 5

Convert $x^2 + y^2 + z^2 = 6z$ to cylindrical coordinates and identify the surface. Describe the ranges of r and z on it.

Question 5 – Solution

Strategy Complete the square in z after replacing $x^2 + y^2$ by r^2 .



Equation

$$r^2 + z^2 = 6z \iff \boxed{r^2 + (z - 3)^2 = 9}.$$

Geometry This is a sphere of radius 3 centered at $(0, 0, 3)$. It is independent of θ , reflecting rotational symmetry about the z -axis.

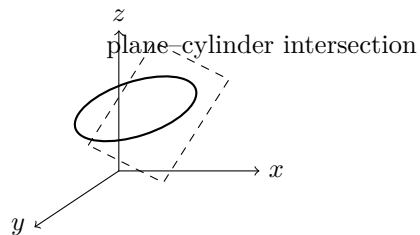
Ranges Since $r^2 \leq 9$, $\boxed{0 \leq r \leq 3}$. The vertical extent is $\boxed{0 \leq z \leq 6}$. At a fixed height z , the horizontal trace has radius $\sqrt{9 - (z - 3)^2}$.

Question 6

Find a one-parameter vector representation for the curve where the cylinder $x^2 + y^2 = 4$ meets the plane $z = x + y$. State an interval that traces the curve exactly once.

Question 6 – Solution

Strategy The cylinder fixes $r = 2$; use $\theta = t$ and substitute into the plane.



Coordinates

$$x = 2 \cos t, \quad y = 2 \sin t, \quad z = x + y = 2 \cos t + 2 \sin t.$$

Therefore

$$\boxed{\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 2 \cos t + 2 \sin t \rangle}, \quad 0 \leq t < 2\pi.$$

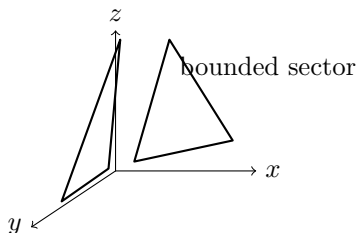
Verification The first two components satisfy $x^2 + y^2 = 4$, and the third is their sum. The half-open interval prevents duplicating the starting point.

Question 7

Describe in cylindrical inequalities the solid in the first octant that lies between the cylinders $r = 1$ and $r = 3$, above $z = 0$, and below the cone $z = 4 - r$. Explain each bound.

Question 7 – Solution

Strategy Translate the geometric restrictions directly into angular, radial, and vertical bounds.



Bounds First octant in the xy -plane gives $0 \leq \theta \leq \pi/2$. Between the cylinders gives $1 \leq r \leq 3$. At each (r, θ) , the solid lies from the xy -plane up to the cone:

$$\boxed{0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 3, \quad 0 \leq z \leq 4 - r.}$$

Consistency On $1 \leq r \leq 3$, the upper bound $4 - r$ remains positive. The cone is rotationally symmetric, while the first-octant restriction selects only a quarter-sector.

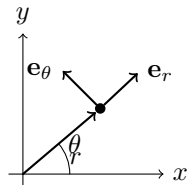
Question 8

A moving point has cylindrical coordinates $r(t), \theta(t), z(t)$. Derive its rectangular velocity and prove that its speed satisfies

$$\|\vec{v}\|^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

Question 8 – Solution

Strategy Differentiate $x = r \cos \theta$ and $y = r \sin \theta$, then simplify the sum of squares.



Velocity

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta, \quad \dot{z} = \dot{z}.$$

Speed identity Expanding $\dot{x}^2 + \dot{y}^2$ makes the cross terms cancel, while $\sin^2 \theta + \cos^2 \theta = 1$. Therefore

$$\|\vec{v}\|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

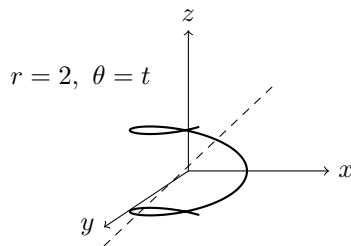
The radial, angular, and vertical velocity directions are mutually perpendicular.

Question 9

A curve is given in cylindrical form by $r = 2$, $\theta = t$, and $z = 3t$. Convert it to rectangular vector form, identify the curve, and find the vertical rise per complete revolution.

Question 9 – Solution

Strategy Substitute into $x = r \cos \theta$, $y = r \sin \theta$ and examine the change when t increases by 2π .



Rectangular form

$$\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle.$$

It is a circular helix of radius 2 around the z -axis.

Pitch One complete revolution corresponds to $\Delta t = 2\pi$. Therefore

$$\Delta z = 3(2\pi) = 6\pi.$$

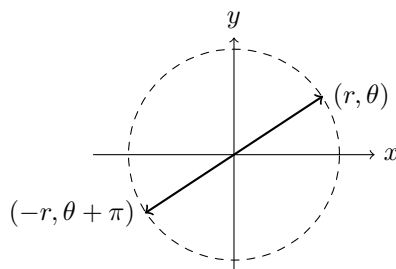
The constant r keeps the curve on a cylinder, while the linear z -coordinate produces uniform vertical advance.

Question 10

Explain why cylindrical coordinates are not unique if negative values of r are allowed. Prove the identity relating (r, θ, z) and $(-r, \theta + \pi, z)$, then state a standard uniqueness convention and its remaining exception.

Question 10 – Solution

Strategy Compare the rectangular coordinates produced by the two triples.



Identity

$$(-r) \cos(\theta + \pi) = (-r)(-\cos \theta) = r \cos \theta,$$

$$(-r) \sin(\theta + \pi) = (-r)(-\sin \theta) = r \sin \theta.$$

The z -coordinate is unchanged, so

$$\boxed{(r, \theta, z) \sim (-r, \theta + \pi, z)}.$$

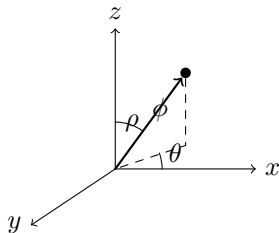
Convention Requiring $r \geq 0$ and $0 \leq \theta < 2\pi$ makes coordinates unique whenever $r > 0$. On the z -axis, $r = 0$ and every angle represents the same point, so θ remains intrinsically undetermined there.

Question 1

Convert the point $(1, \sqrt{3}, 2\sqrt{3})$ to spherical coordinates, using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. Verify the result.

Question 1 – Solution

Strategy Find distance, azimuth, and polar angle in that order.



Coordinates

$$\rho = \sqrt{1 + 3 + 12} = 4, \quad \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Also $\cos \phi = z/\rho = \sqrt{3}/2$, so $\phi = \pi/6$. Thus

$$\boxed{(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)}.$$

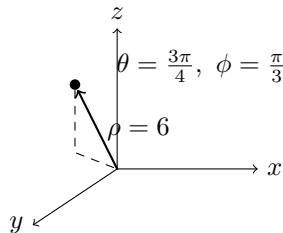
Verification Using $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ gives $1, \sqrt{3}, 2\sqrt{3}$, respectively.

Question 2

Convert the spherical point $(\rho, \theta, \phi) = (6, 3\pi/4, \pi/3)$ to rectangular and cylindrical coordinates. Keep all values exact.

Question 2 – Solution

Strategy First compute the cylindrical radius $r = \rho \sin \phi$ and height $z = \rho \cos \phi$.



Cylindrical form

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}, \quad z = 6 \cos \frac{\pi}{3} = 3.$$

Hence $(r, \theta, z) = (3\sqrt{3}, 3\pi/4, 3)$.

Rectangular form

$$x = 3\sqrt{3} \cos \frac{3\pi}{4} = -\frac{3\sqrt{6}}{2}, \quad y = 3\sqrt{3} \sin \frac{3\pi}{4} = \frac{3\sqrt{6}}{2}.$$

Therefore

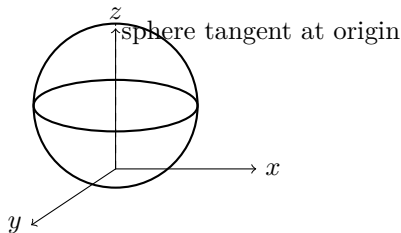
$$(x, y, z) = \left(-\frac{3\sqrt{6}}{2}, \frac{3\sqrt{6}}{2}, 3 \right).$$

Question 3

Convert the spherical equation $\rho = 4 \cos \phi$ to rectangular coordinates and identify the surface, including its center and radius. Explain why the origin belongs to it.

Question 3 – Solution

Strategy Multiply by ρ and use $\rho^2 = x^2 + y^2 + z^2$ and $\rho \cos \phi = z$.



Conversion

$$\rho^2 = 4\rho \cos \phi \implies x^2 + y^2 + z^2 = 4z.$$

Completing the square,

$$\boxed{x^2 + y^2 + (z - 2)^2 = 4}.$$

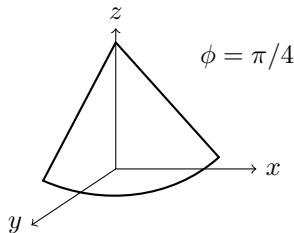
Geometry The surface is a sphere of radius 2 centered at $(0,0,2)$. It contains the origin because the distance from the origin to the center is exactly the radius. In the original spherical equation, $\rho = 0$ and $\phi = \pi/2$ represent the origin and satisfy $0 = 4 \cos(\pi/2)$.

Question 4

Identify and convert the surface $\phi = \pi/4$ to rectangular and cylindrical equations. Explain why it is only one nappe of a cone.

Question 4 – Solution

Strategy Relate the polar angle to cylindrical radius and height: $r = \rho \sin \phi$, $z = \rho \cos \phi$.



Conversion At $\phi = \pi/4$,

$$\frac{r}{z} = \tan \phi = 1,$$

so $\boxed{z = r}$ in cylindrical coordinates. Therefore

$$\boxed{z = \sqrt{x^2 + y^2}}.$$

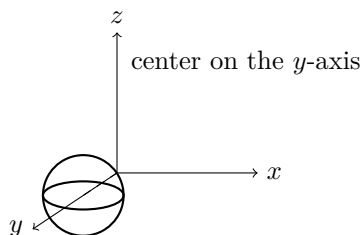
Geometry Because $\phi = \pi/4 < \pi/2$, $z = \rho \cos \phi \geq 0$. Thus this is the upper nappe only. Squaring to $z^2 = x^2 + y^2$ without retaining $z \geq 0$ would incorrectly add the lower nappe.

Question 5

Convert $\rho = 2 \sin \phi \sin \theta$ to rectangular coordinates and identify the surface. Which coordinate axis contains its center?

Question 5 – Solution

Strategy Recognize $\rho \sin \phi \sin \theta = y$, multiply the equation by ρ , and complete a square.



Conversion

$$\rho^2 = 2\rho \sin \phi \sin \theta = 2y.$$

Thus

$$x^2 + y^2 + z^2 = 2y \quad \Longleftrightarrow \quad \boxed{x^2 + (y - 1)^2 + z^2 = 1}.$$

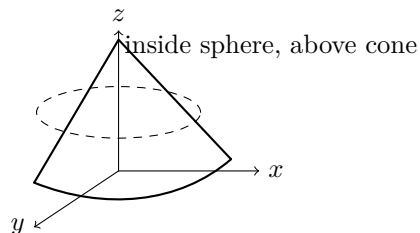
Geometry This is a sphere of radius 1 centered at $\boxed{(0, 1, 0)}$, so its center lies on the positive y -axis. The sphere is tangent to the origin.

Question 6

Describe with spherical inequalities the solid inside the sphere $\rho = 4$ and above the cone $\phi = \pi/3$. Include the full azimuthal range and explain the meaning of “above.”

Question 6 – Solution

Strategy A smaller polar angle lies closer to the positive z -axis.



Bounds Inside the sphere gives $0 \leq \rho \leq 4$. The full rotational sweep is $0 \leq \theta < 2\pi$. Above the cone means between the positive z -axis and the conical surface, so $0 \leq \phi \leq \pi/3$. Hence

$$0 \leq \rho \leq 4, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{3}.$$

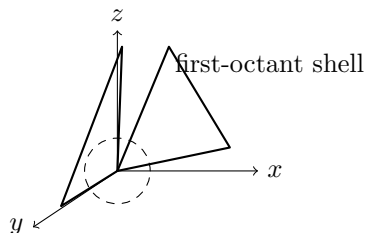
Boundary note At $\rho = 0$, angles are not unique, but including all angle values does not create additional geometric points.

Question 7

Write spherical-coordinate bounds for the first-octant portion of the shell between the spheres of radii 1 and 3. Justify the angular bounds.

Question 7 – Solution

Strategy Separate radial shell bounds from the two angular restrictions defining the first octant.



Bounds Between the spheres,

$$1 \leq \rho \leq 3.$$

The first quadrant in the xy -plane requires $0 \leq \theta \leq \pi/2$. Being above the xy -plane requires $0 \leq \phi \leq \pi/2$. Therefore

$$\boxed{1 \leq \rho \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}}.$$

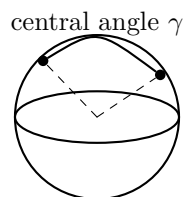
These bounds impose $x, y, z \geq 0$ while retaining every point of the radial shell in that octant.

Question 8

Two points have spherical coordinates $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$. Derive a formula for the square of their distance using their central angle.

Question 8 – Solution

Strategy Write the spherical unit direction vectors and use the dot-product form of the law of cosines.



Direction dot product If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Distance With position vectors $\vec{p}_1, \vec{p}_2$,

$$d^2 = \|\vec{p}_1 - \vec{p}_2\|^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma.$$

Therefore

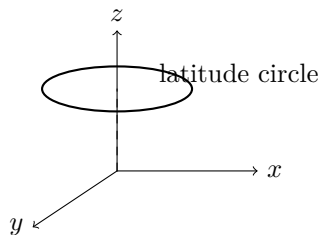
$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 [\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2)].$$

Question 9

A curve on the sphere $\rho = R$ has constant polar angle $\phi = \alpha$, while $\theta = t$ varies from 0 to 2π . Find a rectangular parameterization and the exact length of the curve.

Question 9 – Solution

Strategy Convert the fixed spherical coordinates and recognize a horizontal circle.



Parameterization

$$\vec{r}(t) = \langle R \sin \alpha \cos t, R \sin \alpha \sin t, R \cos \alpha \rangle, \quad 0 \leq t \leq 2\pi.$$

Length Differentiation gives

$$\|\vec{r}'(t)\| = R \sin \alpha$$

for $0 \leq \alpha \leq \pi$. Hence

$$L = 2\pi R \sin \alpha.$$

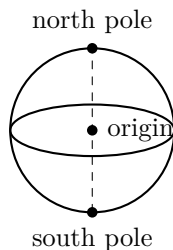
Geometry The latitude circle has Euclidean radius $R \sin \alpha$ and lies in the horizontal plane $z = R \cos \alpha$. At either pole, this radius and length are zero.

Question 10

Under the standard ranges $\rho \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$, discuss exactly where spherical coordinates fail to be unique and why.

Question 10 – Solution

Strategy Examine the conversion formulas at the origin and on the polar axis.



Origin If $\rho = 0$, then $x = y = z = 0$ for every θ and ϕ . Thus both angles are arbitrary at the origin.

Polar axis At $\phi = 0$ or $\phi = \pi$, $\sin \phi = 0$, so $x = y = 0$ regardless of θ . Therefore azimuth is arbitrary at the north and south polar rays.

Elsewhere For $\rho > 0$ and $0 < \phi < \pi$, the height uniquely determines $\phi = \arccos(z/\rho)$, and the nonzero projection (x, y) uniquely determines θ in $[0, 2\pi)$. Thus coordinates are unique away from the origin and z -axis.