

Question 1

Let $a_n = \frac{3n-1}{n+2}$ for $n \geq 1$.

- (a) Find $L = \lim_{n \rightarrow \infty} a_n$.
- (b) Solve $|a_n - L| < 0.01$ exactly and determine the smallest integer N such that the inequality holds for every $n \geq N$.

Question 1 – Solution**Step 1: Find the candidate limit.**

The numerator and denominator have the same degree. Divide both by n :

$$a_n = \frac{3 - 1/n}{1 + 2/n} \rightarrow \frac{3}{1} = 3.$$

Since $1/n \rightarrow 0$ and $2/n \rightarrow 0$, the quotient law gives $L = 3$.

Step 2: Express the error exactly.

Rather than approximate numerically, subtract the limit and simplify:

$$|a_n - 3| = \left| \frac{3n - 1}{n + 2} - 3 \right| = \frac{7}{n + 2}.$$

Step 3: Solve the strict accuracy inequality.

Therefore

$$\frac{7}{n + 2} < 0.01 = \frac{1}{100} \iff n > 698.$$

Thus every integer $n \geq 699$ satisfies the requirement.

Step 4: Verify minimality.

The smallest possible integer is $N = 699$. At the preceding index, $n = 698$, the error is $7/700 = 0.01$, which equals the tolerance and is not strictly less than it. Hence no smaller N works.

Question 2

Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ for $n \geq 1$.

- (a) Show that (a_n) is bounded.
- (b) Find the limits of the even and odd subsequences (a_{2k}) and (a_{2k-1}) .
- (c) Use those limits to decide whether (a_n) converges and to evaluate the claim, “Every bounded sequence converges.”

Question 2 – Solution**Step 1: Prove boundedness.**

For $n \geq 1$, $1 < 1 + 1/n \leq 2$. Since $|(-1)^n| = 1$,

$$|a_n| = 1 + \frac{1}{n} \leq 2.$$

Thus $-2 \leq a_n \leq 2$, and the sequence is bounded.

Step 2: Analyze the even subsequence.

For even indices,

$$a_{2k} = 1 + \frac{1}{2k} \rightarrow 1.$$

Step 3: Analyze the odd subsequence.

For odd indices,

$$a_{2k-1} = -1 - \frac{1}{2k-1} \rightarrow -1.$$

Step 4: Apply the subsequence criterion.

If (a_n) converged to L , every subsequence would also converge to L . The even and odd subsequences instead approach 1 and -1 , respectively. Since these limits differ, (a_n) diverges.

Step 5: Evaluate the student's claim.

This bounded divergent sequence is a counterexample to the claim that every bounded sequence converges. Boundedness together with monotonicity, however, would guarantee convergence by the Monotone Convergence Theorem.

Question 3

Let $a_n = \frac{n}{n+1}$ for $n \geq 1$.

- (a) Prove algebraically (without derivatives) that (a_n) is strictly increasing.
- (b) Find an upper bound and use the Monotone Convergence Theorem to explain why a limit exists.
- (c) Find the limit and express the error $|a_n - L|$ exactly.

Question 3 – Solution**Step 1: Prove monotonicity algebraically.**

Compare consecutive terms by subtraction:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

The denominator is positive for $n \geq 1$, so $a_{n+1} - a_n > 0$ and (a_n) is strictly increasing.**Step 2: Find an upper bound.**

Rewrite the term as

$$a_n = 1 - \frac{1}{n+1} < 1,$$

Thus $a_n < 1$ for every n , so the sequence is bounded above by 1.**Step 3: Establish convergence and calculate the limit.**

An increasing sequence bounded above converges by the Monotone Convergence Theorem. Using the rewritten formula,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Step 4: Express the error exactly.Since $a_n < 1$, the exact error is $|a_n - 1| = 1 - a_n = 1/(n+1)$, which tends to zero.

Question 4

Let $a_n = \left(1 + \frac{2}{n}\right)^n$.

- (a) Interpret a_n as the balance produced by compounding a principal of 1 at a total rate of 2 over one time unit, using n periods.
- (b) Evaluate $\lim_{n \rightarrow \infty} a_n$ using the standard exponential limit.
- (c) Give a decimal approximation of the limiting balance.

Question 4 – Solution**Step 1: Interpret the compounding model.**

The principal is $P = 1$. Dividing the total rate 2 into n equal periods gives rate $2/n$ per period; applying that growth factor n times produces $1(1 + 2/n)^n$.

Step 2: Rewrite in the standard exponential form.

Put $m = n/2$. Then $m \rightarrow \infty$ and

$$a_n = \left[\left(1 + \frac{1}{m} \right)^m \right]^2.$$

Step 3: Apply the standard limit.

Using $\lim_{m \rightarrow \infty} (1 + 1/m)^m = e$ and continuity of the squaring function gives

$$\lim_{n \rightarrow \infty} a_n = e^2 \approx 7.38906.$$

Step 4: Interpret and verify.

This agrees with the continuous-compounding formula $Pe^{rt} = e^2$ for $P = 1$, $r = 2$, and $t = 1$. The standard exponential limit supplies the proof.

Question 5

Let $a_n = \sin(n\pi/2)$ for $n \geq 1$.

- (a) Make a table giving a_n for each residue class of n modulo 4, and list the first eight terms.
- (b) Determine whether the sequence converges. Justify your answer using subsequences.
- (c) Explain whether changing or rearranging finitely many initial terms can change convergence.

Question 5 – Solution**Step 1: Use periodicity.**

Since adding 2π does not change sine and increasing n by 4 adds 2π to $n\pi/2$, the values repeat every four indices:

$n \pmod{4}$	1	2	3	0
a_n	1	0	-1	0

Thus the first eight terms are $1, 0, -1, 0, 1, 0, -1, 0$.

Step 2: Select decisive subsequences.

In particular,

$$a_{4k+1} = 1 \quad \text{and} \quad a_{4k+3} = -1$$

for all applicable k . The first subsequence converges to 1, while the second converges to -1 .

Step 3: Conclude divergence.

If (a_n) converged, all its subsequences would share the same limit. These constant subsequences have different limits, so (a_n) diverges.

Step 4: Analyze finite modifications.

Changing, deleting, or rearranging only finitely many initial terms cannot change convergence because the definition concerns all terms beyond some index. No finite modification removes the infinitely recurring values 1 and -1 . Even an arbitrary permutation of all these terms still contains infinitely many occurrences of both values, so it cannot converge.

Question 6

Let $a_n = n^{1/n}$ for $n \geq 1$.

- (a) Use logarithms to find $\lim_{n \rightarrow \infty} a_n$.
- (b) For which integers n is $a_n > 1$?
- (c) Use the continuous extension $f(x) = x^{1/x}$ to explain why the terms first rise and then decrease, even though their limit is 1.

Question 6 – Solution**Step 1: Transform the variable exponent.**

Because $a_n = n^{1/n} > 0$, logarithms are valid. Let $b_n = \ln a_n$. Then

$$b_n = \ln(n^{1/n}) = \frac{\ln n}{n}.$$

Step 2: Evaluate the logarithmic limit.

Extend the quotient to $g(x) = \ln x/x$. It has the indeterminate form ∞/∞ , and L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Restricting this continuous limit to integer values gives $b_n \rightarrow 0$. By continuity of the exponential function, $a_n = e^{b_n} \rightarrow e^0 = 1$.

Step 3: Determine when the terms exceed the limit.

For every integer $n > 1$, the base $n > 1$ and exponent $1/n > 0$, so $a_n > 1$; also $a_1 = 1$. A sequence may approach a limit entirely from one side, so convergence to 1 does not require later terms to equal or cross 1.

Step 4: Explain the rise and fall.

For the continuous extension $f(x) = x^{1/x}$, logarithmic differentiation gives

$$\frac{f'(x)}{f(x)} = \frac{1 - \ln x}{x^2}.$$

Because $f(x) > 0$, the sign of $f'(x)$ is the sign of $1 - \ln x$. Thus f increases for $x < e$ and decreases for $x > e$. Among integer indices, the maximum occurs at $n = 3$, after which the sequence decreases toward 1 while remaining above it.

Question 7

Let $a_n = \frac{\ln n}{\sqrt{n}}$ for $n \geq 1$.

- (a) Use a continuous extension and L'Hopital's Rule to find the limit.
- (b) Explain why taking logarithms of a_n is not the most direct method.
- (c) Use the derivative of $f(x) = \ln x / \sqrt{x}$ to explain the initial increase of the sequence and identify where the eventual decrease begins.

Question 7 – Solution

Step 1: Choose a continuous extension.

Let $f(x) = \ln x / \sqrt{x}$ for $x \geq 1$. Both numerator and denominator tend to infinity, so the quotient has the indeterminate form ∞/∞ and L'Hopital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Restricting the continuous limit to integer inputs gives $a_n \rightarrow 0$.

Step 2: Compare possible methods.

Taking $\ln(a_n) = \ln(\ln n) - \frac{1}{2} \ln n$ would introduce an $\infty - \infty$ expression and still require comparing logarithmic growth. Applying L'Hopital's Rule directly to the original quotient is cleaner.

Step 3: Determine where the sequence increases and decreases.

Differentiate the extension:

$$f'(x) = x^{-3/2} \left(1 - \frac{1}{2} \ln x \right).$$

Since $x^{-3/2} > 0$, $f'(x) > 0$ for $x < e^2$, equals 0 at $e^2 \approx 7.39$, and is negative afterward. The integer terms rise through $n = 7$ and then decrease very slowly. This initial increase is consistent with the eventual limit of zero.

Question 8

Define $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

- (a) Prove by induction that $0 \leq a_n < 2$ for every n .
- (b) Prove that (a_n) is increasing, and conclude that it converges.
- (c) Find its limit by solving the appropriate fixed-point equation, and explain why the other algebraic root is inadmissible.

Question 8 – Solution**Step 1: Prove an invariant bound.**

We prove $0 \leq a_n < 2$ by induction. The base case holds because $a_1 = 0$. Assume $0 \leq a_n < 2$. Then

$$0 < 2 + a_n < 4 \implies 0 < a_{n+1} = \sqrt{2 + a_n} < 2.$$

Thus the bound holds for a_{n+1} , completing the induction.

Step 2: Prove monotonicity.

Use induction again. Since $a_2 = \sqrt{2} > a_1$, the base case holds. If $a_n > a_{n-1}$, the square-root function is strictly increasing, so

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{2 + a_{n-1}} = a_n.$$

Thus (a_n) is increasing. Together with the upper bound 2, the Monotone Convergence Theorem guarantees $a_n \rightarrow L$ for some $0 \leq L \leq 2$.

Step 3: Solve the fixed-point equation.

Only after proving convergence may we pass to the limit in the recurrence. Continuity of the square root gives

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0.$$

The algebraic candidates are $L = 2$ and $L = -1$. Because every term and hence its limit are non-negative, $L = -1$ is inadmissible. Therefore $L = 2$. Solving this equation alone would identify only possible limits; boundedness and monotonicity are what prove a limit actually exists.

Question 9

Let $a_n = \frac{(n!)^{1/n}}{n}$ for $n \geq 1$.

- (a) Apply Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ to rewrite a_n as a dominant constant times a factor tending to 1.
- (b) Find $\lim_{n \rightarrow \infty} a_n$ and give its decimal value.
- (c) Explain why the first dozen terms alone do not reveal the limit accurately.

Question 9 – Solution**Step 1: Insert Stirling's approximation.**

Stirling's formula states

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Step 2: Apply the n th root and normalize.All quantities are positive, so taking n th roots and dividing by n gives

$$a_n = \frac{1}{n} (2\pi n)^{1/(2n)} \frac{n}{e} (1 + o(1))^{1/n} = \frac{1}{e} (2\pi n)^{1/(2n)} (1 + o(1))^{1/n}.$$

Step 3: Evaluate the correction factors.

The extra factors tend to 1. For the first factor, take logarithms:

$$\ln \left((2\pi n)^{1/(2n)} \right) = \frac{\ln(2\pi n)}{2n} \rightarrow 0,$$

so exponentiating gives $(2\pi n)^{1/(2n)} \rightarrow 1$. Also, if $u_n = o(1)$, then $(1 + u_n)^{1/n} \rightarrow 1$ because its logarithm is $\ln(1 + u_n)/n \rightarrow 0$. Consequently,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{e} \approx 0.367879.$$

Step 4: Interpret the numerical evidence.

At $n = 12$, $a_{12} \approx 0.4407$, still about 0.0729 above $1/e$. The slowly decaying correction factor $(2\pi n)^{1/(2n)}$ explains why a short numerical table is suggestive but cannot identify the hidden constant reliably.

Question 10

Let $a_n = \cos(1/n)$, where angles are measured in radians.

- (a) Find $\lim_{n \rightarrow \infty} a_n$ using continuity.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain an explicit upper bound for the error $|a_n - L|$.
- (c) Find a simple integer N that guarantees $|a_n - L| < 10^{-4}$ for every $n \geq N$.

Question 10 – Solution

Step 1: Find the limit by continuity.

Since $1/n \rightarrow 0$ and cosine is continuous at 0,

$$L = \lim_{n \rightarrow \infty} \cos(1/n) = \cos 0 = 1.$$

Step 2: Express and bound the error.

Because $\cos(1/n) \leq 1$, the absolute error simplifies to

$$|a_n - L| = 1 - \cos(1/n).$$

Apply $0 \leq 1 - \cos x \leq x^2/2$ with $x = 1/n$:

$$0 \leq |a_n - 1| \leq \frac{1}{2n^2}.$$

Step 3: Solve for a sufficient index.

It is enough to make the upper bound smaller than the desired tolerance:

$$\frac{1}{2n^2} < 10^{-4} \iff n^2 > 5000 \iff n > \sqrt{5000} \approx 70.71.$$

Step 4: State the guarantee precisely.

Thus $N = 71$ guarantees $|a_n - 1| < 10^{-4}$ for every $n \geq 71$, because $1/(2n^2)$ decreases with n . This is a sufficient value obtained from an upper bound; it need not be the smallest possible N for the exact cosine error.

Question 1

Define $a_1 = 2$ and $a_{n+1} = \frac{1}{2} \left(a_n + \frac{3}{a_n} \right)$.

- (a) Explain why this is Newton's method applied to $x^2 - 3 = 0$.
- (b) Prove that $a_n \geq \sqrt{3}$ for every n , and prove that (a_n) is decreasing.
- (c) Conclude that the sequence converges and determine its limit.

Question 1 – Solution**Step 1: Identify the algorithm.**

For $f(x) = x^2 - 3$, Newton's method uses $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Since $f'(x) = 2x$, the update becomes

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{1}{2} \left(x + \frac{3}{x} \right),$$

which is exactly the given recurrence.

Step 2: Establish positivity and a lower bound.

All terms are positive because $a_1 = 2 > 0$ and, whenever $a_n > 0$, both a_n and $3/a_n$ are positive. More precisely,

$$a_{n+1} - \sqrt{3} = \frac{(a_n - \sqrt{3})^2}{2a_n} \geq 0,$$

The right side is nonnegative, so $a_{n+1} \geq \sqrt{3}$. Since $a_1 \geq \sqrt{3}$, induction proves $a_n \geq \sqrt{3}$ for every n .

Step 3: Prove monotonicity.

Using the lower bound,

$$a_{n+1} - a_n = \frac{3 - a_n^2}{2a_n} \leq 0,$$

Thus $a_{n+1} \leq a_n$, so (a_n) is decreasing.

Step 4: Prove convergence before solving for the limit.

The sequence is decreasing and bounded below by $\sqrt{3}$. The Monotone Convergence Theorem therefore guarantees $a_n \rightarrow L$ for some $L \geq \sqrt{3}$. Continuity and $L > 0$ allow passage to the limit:

$$L = \frac{1}{2} \left(L + \frac{3}{L} \right), \quad L^2 = 3.$$

The equation gives $L = \pm\sqrt{3}$, and the established positive lower bound selects $L = \sqrt{3}$.

Question 2

Let (b_n) be a bounded real sequence and define $a_n = \sup\{b_k : k \geq n\}$.

- (a) Explain why every a_n exists.
- (b) Prove that (a_n) is decreasing and bounded, and hence convergent.
- (c) State what its limit represents, and compute it when $b_n = (-1)^n + 1/n$.

Question 2 – Solution

Step 1: Verify that each supremum exists.

Let $T_n = \{b_k : k \geq n\}$. Each tail is nonempty, and because (b_n) is bounded, T_n is bounded above. The least-upper-bound property of $\mathbb{R}$ therefore guarantees that $a_n = \sup T_n$ exists.

Step 2: Prove monotonicity.

Removing the first element of a tail cannot increase its set of possible values. Formally, $T_{n+1} \subseteq T_n$, so

$$a_{n+1} = \sup T_{n+1} \leq \sup T_n = a_n.$$

Thus (a_n) is decreasing.

Step 3: Establish a lower bound and convergence.

If m is any lower bound for (b_n) , then $m \leq b_k \leq a_n$ for every $k \geq n$, so $m \leq a_n$. Hence (a_n) is bounded below. The Monotone Convergence Theorem now gives convergence, and its limit is called the limit superior:

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n.$$

Step 4: Apply the construction to the example.

For $b_n = (-1)^n + 1/n$, even terms equal $1 + 1/(2j)$ and odd terms are negative. Among the even terms in a tail, the earliest one is largest. If e_n is the first even integer satisfying $e_n \geq n$, then

$$a_n = 1 + \frac{1}{e_n} \longrightarrow 1.$$

Thus $\limsup b_n = 1$. The tail-supremum process has turned oscillating bounded data into a decreasing upper envelope.

Question 3

Let $a_n = \{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$.

- (a) Prove that $0 \leq a_n < 1$, so the sequence is bounded.
- (b) Using the fact that the fractional parts of multiples of an irrational number are dense in $[0, 1]$, find two different subsequential limits.
- (c) Decide whether (a_n) converges and explain why boundedness is insufficient.

Question 3 – Solution**Step 1: Establish boundedness.**

For every real x , its fractional part satisfies $0 \leq \{x\} < 1$. Therefore $0 \leq a_n < 1$ for all n , and the sequence is bounded.

Step 2: Use irrationality and density.

Because $\sqrt{2}$ is irrational, the density theorem for irrational rotations says that the set $\{\{n\sqrt{2}\} : n \geq 1\}$ is dense in $[0, 1]$. In particular, every open subinterval contains infinitely many terms, allowing the selection of increasing subsequence indices.

Step 3: Construct two subsequences.

For each positive integer j , choose increasing indices n_j and m_j such that

$$0 < a_{n_j} < 1/j, \quad 1/2 - 1/j < a_{m_j} < 1/2 + 1/j.$$

The first inequality and Squeeze Theorem give $a_{n_j} \rightarrow 0$; the second gives $a_{m_j} \rightarrow 1/2$.

Step 4: Apply the subsequence criterion.

Every subsequence of a convergent sequence must approach the same limit as the original sequence. Since these two subsequences have different limits, (a_n) diverges. Boundedness alone guarantees neither monotonicity nor convergence.

Question 4

Let $a_n = \sum_{k=1}^n 2^{-k}$.

- (a) Derive a closed formula for a_n from the finite geometric-sum formula.
- (b) Prove that (a_n) converges and find its limit.
- (c) Find the exact remaining gap $L - a_n$ and the smallest n for which this gap is below 10^{-3} .

Question 4 – Solution

Step 1: Identify and sum the finite geometric series.

The terms form a geometric sequence with first term $a = 1/2$ and common ratio $r = 1/2$. Therefore

$$a_n = \frac{(1/2)(1 - (1/2)^n)}{1 - 1/2} = 1 - 2^{-n}.$$

Step 2: Determine convergence and the limit.

Since $2^{-n} \rightarrow 0$, the closed formula gives $a_n \rightarrow L = 1$. Also $a_{n+1} - a_n = 2^{-(n+1)} > 0$ and $a_n < 1$, so the sequence is increasing and bounded above, independently confirming convergence by the Monotone Convergence Theorem.

Step 3: Compute the exact remaining gap.

The exact error after n terms is

$$L - a_n = 2^{-n}.$$

Step 4: Solve the accuracy inequality.

We need $2^{-n} < 10^{-3}$, equivalently $2^n > 1000$. Since $2^9 = 512 \leq 1000$ but $2^{10} = 1024 > 1000$, the smallest qualifying index is $n = 10$.

Question 5

Let $a_n = \frac{n^2}{2^n}$ for $n \geq 1$.

- (a) Use a_{n+1}/a_n to determine exactly when $a_{n+1} < a_n$.
- (b) Identify the largest term and the first index from which the sequence is strictly decreasing.
- (c) Prove that $a_n \rightarrow 0$.

Question 5 – Solution**Step 1: Compare consecutive positive terms.**

Because $a_n > 0$, the inequality $a_{n+1} < a_n$ is equivalent to $a_{n+1}/a_n < 1$. Compute

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2}.$$

Step 2: Solve the ratio inequality exactly.

Thus $a_{n+1} < a_n$ exactly when

$$(n+1)^2 < 2n^2 \iff n^2 - 2n - 1 > 0 \iff n > 1 + \sqrt{2}.$$

For integer $n \geq 1$, this is equivalent to $n \geq 3$. Since $a_1 = 1/2 < a_2 = 1 < a_3 = 9/8$ and $a_{n+1} < a_n$ for every $n \geq 3$, a_3 is the unique largest term and the sequence is strictly decreasing from index 3 onward.

Step 3: Prove the decreasing tail approaches zero.

A decreasing positive sequence has a limit, but the ratio bound identifies it directly. For $n \geq 7$,

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \leq \frac{1}{2} \left(\frac{8}{7}\right)^2 = \frac{32}{49} < 1.$$

Iterating this inequality gives $0 < a_n \leq a_7(32/49)^{n-7}$ for $n \geq 7$. The right side is geometric with ratio less than 1, so it tends to zero. The Squeeze Theorem yields $a_n \rightarrow 0$.

Question 6

Let $a_n = \frac{(-1)^n}{n}$.

- (a) Prove that $a_n \rightarrow 0$ using an absolute-value estimate.
- (b) Show that the full sequence is not monotone and is not eventually monotone.
- (c) Describe the monotonic behavior of its even and odd subsequences.

Question 6 – Solution**Step 1: Prove convergence.**

Use absolute values to remove the alternating sign:

$$|a_n - 0| = \frac{1}{n} \longrightarrow 0,$$

Since $1/n \rightarrow 0$, the definition of convergence gives $a_n \rightarrow 0$. Equivalently, $-1/n \leq a_n \leq 1/n$ and the Squeeze Theorem applies.

Step 2: Test eventual monotonicity.

For every positive integer k ,

$$a_{2k} = \frac{1}{2k} > 0, \quad a_{2k+1} = -\frac{1}{2k+1} < 0.$$

Thus $a_{2k} > a_{2k+1}$ but $a_{2k+1} < a_{2k+2}$, producing infinitely many decreases and increases. Because this occurs for arbitrarily large k , no tail can be monotone.

Step 3: Analyze the two subsequences.

The even subsequence $a_{2k} = 1/(2k)$ decreases to 0, while the odd subsequence $a_{2k-1} = -1/(2k-1)$ increases to 0. Both close in on the same limit from opposite sides. This demonstrates that convergence does not require eventual monotonicity.

Question 7

Let $a_n = \sqrt{n+1} - \sqrt{n}$.

- (a) Rationalize the expression and use the result to find the limit.
- (b) Prove that (a_n) is positive and decreasing.
- (c) Show that $a_n \sim 1/(2\sqrt{n})$ and interpret this as the spacing between neighboring square roots.

Question 7 – Solution**Step 1: Rationalize the difference.**

Multiplying numerator and denominator by the conjugate avoids the indeterminate form $\infty - \infty$:

$$a_n = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Step 2: Find the limit and monotonicity.

The denominator tends to infinity, so $a_n \rightarrow 0$. It is also positive. Furthermore, $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$; taking reciprocals of positive quantities reverses the inequality and gives $a_{n+1} < a_n$. Thus the sequence decreases to zero.

Step 3: Determine the asymptotic gap size.

Compare a_n with $1/(2\sqrt{n})$ by taking their ratio:

$$\frac{a_n}{1/(2\sqrt{n})} = \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{1+1/n} + 1} \rightarrow 1.$$

Because the ratio tends to 1, $a_n \sim 1/(2\sqrt{n})$. Therefore neighboring square roots get closer, and their gap is approximately $1/(2\sqrt{n})$ for large n .

Question 8

Let $a_n = \int_0^1 x^n dx$ for $n \geq 1$.

- (a) Evaluate the integral and find $\lim_{n \rightarrow \infty} a_n$.
- (b) Prove directly from the integrands that (a_n) is decreasing.
- (c) Give a geometric explanation for why the areas approach zero, despite $x^n = 1$ at $x = 1$.

Question 8 – Solution

Step 1: Evaluate the area exactly.

By the power rule,

$$a_n = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1} \rightarrow 0.$$

Step 2: Prove monotonicity directly from the integrands.

For $0 \leq x \leq 1$, $0 \leq x^{n+1} \leq x^n$, with strict inequality on $(0, 1)$. Integrating over $[0, 1]$ gives $0 < a_{n+1} < a_n$, so the areas decrease.

Step 3: Explain the geometry rigorously.

For every fixed $x \in [0, 1)$, $x^n \rightarrow 0$; only the single endpoint $x = 1$ remains at height 1, and one point has zero width and contributes no area. For a quantitative version, fix $0 < c < 1$ and split the interval:

$$\int_0^1 x^n dx = \int_0^c x^n dx + \int_c^1 x^n dx \leq c^n + (1 - c).$$

Given $\varepsilon > 0$, first choose c so that $1 - c < \varepsilon/2$. Then choose n so large that $c^n < \varepsilon/2$. The displayed bound gives $a_n < \varepsilon$, proving geometrically that the areas vanish.

Question 9

For $n \geq 2$, let $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$.

- (a) Take logarithms and use $\ln(1 - x) \leq -x$ for $0 < x < 1$ to obtain a useful bound.
- (b) Prove that $a_n \rightarrow 0$.
- (c) Use $\ln(1 - x) = -x - x^2/2 + O(x^3)$ to describe the rate of decay more precisely.

Question 9 – Solution

Step 1: Take logarithms.

For $n \geq 2$, $0 < 1 - 1/n < 1$, so $a_n > 0$ and logarithms are valid:

$$\ln a_n = n^2 \ln \left(1 - \frac{1}{n} \right).$$

Step 2: Obtain a decisive upper bound.

For $0 < x < 1$, $\ln(1-x) \leq -x$. With $x = 1/n$, this gives $\ln a_n \leq -n$. Because the exponential function is increasing,

$$0 < a_n \leq e^{-n} \longrightarrow 0.$$

The Squeeze Theorem proves $a_n \rightarrow 0$.

Step 3: Refine the decay rate.

For a sharper description, substitute $x = 1/n$ into $\ln(1-x) = -x - x^2/2 + O(x^3)$:

$$\ln a_n = n^2 \left(-\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n - \frac{1}{2} + O(n^{-1}).$$

Exponentiating both sides gives

$$a_n = e^{-n-1/2+O(1/n)} \sim e^{-n-1/2}.$$

The notation $u_n \sim v_n$ means $u_n/v_n \rightarrow 1$. Thus the quadratic exponent repeats a base close to 1 often enough to produce decay essentially like e^{-n} , with the additional constant factor $e^{-1/2}$.

Question 10

Define $a_1 = 0$ and $a_{n+1} = \frac{1}{2 + a_n}$.

- (a) Show that $0 \leq a_n \leq 1/2$ for all n .
- (b) Find the unique fixed point L in this interval and prove that consecutive terms alternate around it.
- (c) Establish an error contraction and use it to prove $a_n \rightarrow L$.

Question 10 – Solution**Step 1: Establish an invariant interval.**

The bound follows by induction. The base case $a_1 = 0$ lies in $[0, 1/2]$. If $0 \leq a_n \leq 1/2$, then $2 + a_n \geq 2$, so $0 < a_{n+1} = 1/(2 + a_n) \leq 1/2$. Hence every term remains in $[0, 1/2]$.

Step 2: Find the admissible fixed point.

A fixed point satisfies

$$L = \frac{1}{2 + L} \iff L^2 + 2L - 1 = 0,$$

so $L = -1 \pm \sqrt{2}$. Only $L = \sqrt{2} - 1$ lies in the invariant interval $[0, 1/2]$.

Step 3: Explain the alternating behavior.

Let $f(x) = 1/(2 + x)$. Because $f'(x) = -1/(2 + x)^2 < 0$, f is strictly decreasing. Since $a_1 < L$, $a_2 = f(a_1) > f(L) = L$. Applying the decreasing map reverses the side of L at every iteration, so successive terms alternate across the fixed point.

Step 4: Prove convergence with an error contraction.

$$|a_{n+1} - L| = \left| \frac{1}{2 + a_n} - \frac{1}{2 + L} \right| = \frac{|a_n - L|}{(2 + a_n)(2 + L)} \leq \frac{1}{4} |a_n - L|.$$

Both denominator factors are at least 2, which justifies the factor $1/4$. Iterating gives

$$|a_n - L| \leq \left(\frac{1}{4}\right)^{n-1} |a_1 - L| \longrightarrow 0.$$

Therefore $a_n \rightarrow \sqrt{2} - 1$; the even and odd subsequences approach the same value from opposite sides.

Question 1

Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

- (a) Identify the first term and common ratio, then evaluate the series using the infinite geometric-series formula.
- (b) Derive a formula for the N th partial sum s_N and evaluate $\lim_{N \rightarrow \infty} s_N$.
- (c) Find the exact remainder $R_N = S - s_N$ and determine the smallest N for which $R_N < 10^{-4}$.

Question 1 – Solution**Step 1: Identify the geometric structure.**

Writing the first terms as $3/4 + 3/16 + 3/64 + \cdots$ shows that the first term is $a = 3/4$ and the common ratio is $r = 1/4$. Since $|r| < 1$, the infinite geometric-series formula applies:

$$S = \frac{a}{1-r} = \frac{3/4}{1-1/4} = 1.$$

Step 2: Confirm the result from partial sums.

For a geometric series, the finite sum is $s_N = a(1 - r^N)/(1 - r)$. Substitution gives

$$s_N = \frac{3}{4} \frac{1 - (1/4)^N}{1 - 1/4} = 1 - 4^{-N},$$

Since $4^{-N} \rightarrow 0$, $s_N \rightarrow 1$, confirming the result from the definition of an infinite series.

Step 3: Find and use the exact remainder.

The exact remainder after N terms is

$$R_N = S - s_N = 4^{-N}.$$

We require $4^{-N} < 10^{-4}$, equivalently $4^N > 10^4$. Since $4^6 = 4096 \leq 10^4$ but $4^7 = 16384 > 10^4$, the smallest choice is $N = 7$. The strict inequality is important.

Question 2

Consider $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

- (a) Write out the first four terms and display the cancellation.
- (b) Find a formula for the N th partial sum, identifying the surviving boundary terms.
- (c) Decide whether the series converges and, if so, find its sum and exact remainder.

Question 2 – Solution**Step 1: Form a finite partial sum.**

Never telescope directly to infinity. First write the N th partial sum:

$$\begin{aligned}s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}.\end{aligned}$$

Step 2: Identify the surviving boundary terms.

Every interior fraction appears once positively and once negatively, so it cancels. Only the initial 1 and final $-1/(N+1)$ survive.

Step 3: Take the partial-sum limit.

By definition, the series converges exactly when (s_N) has a finite limit. Here

$$S = \lim_{N \rightarrow \infty} s_N = 1.$$

Step 4: Compute and verify the remainder.

The exact tail after N terms is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As an algebraic check, $s_N - s_{N-1} = 1/N - 1/(N+1)$, which reproduces the original N th summand exactly.

Question 3

For a real parameter x , consider $\sum_{n=0}^{\infty} x^n$.

- (a) For $x \neq 1$, derive the finite identity $s_N = (1 - x^{N+1})/(1 - x)$.
- (b) Determine every x for which the infinite series converges and find its sum there.
- (c) Analyze $x = 1$, $x = -1$, and $|x| > 1$ separately.

Question 3 – Solution

Step 1: Derive the finite identity.

Let $s_N = 1 + x + \cdots + x^N$. Then $xs_N = x + x^2 + \cdots + x^{N+1}$. Subtracting the second equation from the first gives

$$(1 - x)s_N = 1 - x^{N+1},$$

so for $x \neq 1$ division by $1 - x$ yields

$$s_N = \frac{1 - x^{N+1}}{1 - x}.$$

Step 2: Determine the convergent parameter region.

If $|x| < 1$, then $x^{N+1} \rightarrow 0$. Therefore the partial sums converge and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Step 3: Check the excluded cases.

At $x = 1$, $s_N = N + 1 \rightarrow \infty$. At $x = -1$, the partial sums alternate between 1 and 0 and have no limit. If $|x| > 1$, the terms x^n do not approach zero, so the nth-term test proves divergence. Thus convergence occurs exactly for $|x| < 1$; checking both endpoints separately is essential.

Question 4

Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

- (a) Rewrite the N th partial sum as the logarithm of a product.
- (b) Simplify that product and determine whether the series converges.
- (c) Explain how the term test and the growth of the partial sums illustrate different facts.

Question 4 – Solution**Step 1: Work with a finite partial sum.**

For positive factors, $\ln u + \ln v = \ln(uv)$. Therefore

$$s_N = \ln \prod_{n=1}^N \frac{n+1}{n} = \ln \left(\frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{N+1}{N} \right) = \ln(N+1).$$

Step 2: Simplify the product.

Every interior factor cancels: the numerator $2 \cdot 3 \cdots N$ cancels the same factors in the denominator, leaving $N+1$. This is multiplicative telescoping.

Step 3: Classify the series.

Since $s_N = \ln(N+1) \rightarrow \infty$, the sequence of partial sums is unbounded and the series diverges to $+\infty$.

Step 4: Interpret the nth-term condition.

The individual terms do satisfy

$$\ln \left(\frac{n+1}{n} \right) = \ln(1 + 1/n) \rightarrow 0,$$

which is necessary but not sufficient for convergence. The logarithm grows slowly, explaining why a short numerical table can look nearly stable even though the partial sums are unbounded.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find a partial-fraction decomposition that produces telescoping cancellation.
- (b) Derive the exact N th partial sum and verify it for $N = 1, 2, 3$.
- (c) Find the sum and the exact remainder after N terms.

Question 5 – Solution**Step 1: Find the partial fractions.**

Write $1/[n(n+2)] = A/n + B/(n+2)$. Multiplying by $n(n+2)$ gives $1 = A(n+2) + Bn$. Matching coefficients yields $A = 1/2$ and $B = -1/2$, so

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite partial sum.

Hence

$$s_N = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{3}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}.$$

The surviving terms are 1, $1/2$, $-1/(N+1)$, and $-1/(N+2)$. For $N = 1, 2, 3$, the formula gives $1/3$, $11/24$, and $21/40$, matching direct addition.

Step 3: Take the limit.

Since both remaining reciprocal terms approach zero,

$$S = \frac{3}{4}.$$

Step 4: State the exact error.

Subtracting the partial sum from S gives the exact positive tail

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

Question 6

Consider $\sum_{n=1}^{\infty} (-1)^n$.

- (a) List the first six terms and partial sums.
- (b) Prove that the sequence of partial sums is bounded but divergent.
- (c) Apply the nth-term test and explain why bounded partial sums alone do not imply convergence.

Question 6 – Solution**Step 1: Distinguish terms from partial sums.**

The series terms are $a_n = (-1)^n$, so the first six are $-1, 1, -1, 1, -1, 1$. Adding successively gives the partial sums

$$-1, 0, -1, 0, -1, 0, \dots$$

More precisely,

$$s_{2k} = 0, \quad s_{2k-1} = -1.$$

Step 2: Test convergence of the partial sums.

The sequence (s_N) is bounded between -1 and 0 , but $s_{2k} = 0 \rightarrow 0$ while $s_{2k-1} = -1 \rightarrow -1$. Since these subsequences have different limits, (s_N) does not converge. By definition, the series therefore diverges.

Step 3: Apply the nth-term test as a check.

The terms $(-1)^n$ do not approach zero, so the nth-term test also proves divergence immediately. Indeed, if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. This example shows that bounded partial sums are not enough; the partial sums must approach one number.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric-series identity to derive a formula for $\sum_{n=1}^{\infty} nx^{n-1}$ when $|x| < 1$.
- (b) Use that formula at $x = 1/2$ to evaluate the series.
- (c) Derive the finite partial sum and exact remainder after N terms.

Question 7 – Solution

Step 1: Start from the geometric series.

For $|x| < 1$, the geometric identity is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Step 2: Differentiate within the interval of convergence.

Power series may be differentiated term by term at every interior point of their interval of convergence. Thus

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x aligns the power with the desired form:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Step 3: Substitute the required value.

Since $|1/2| < 1$, substitution is valid and gives $(1/2)/(1/2)^2 = 2$.

Step 4: Derive the finite sum and remainder.

Differentiating the finite identity $1 + x + \cdots + x^N = (1 - x^{N+1})/(1 - x)$ and multiplying by x yields

$$\sum_{n=1}^N nx^n = \frac{x - (N+1)x^{N+1} + Nx^{N+2}}{(1-x)^2}.$$

At $x = 1/2$, this simplifies to

$$s_N = 2 - \frac{N+2}{2^N}, \quad R_N = 2 - s_N = \frac{N+2}{2^N}.$$

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$.

- (a) Reindex the series using $k = n + 1$.
- (b) Compare it with the Maclaurin series for e and identify every omitted term.
- (c) Find the sum and give a simple upper bound for the remainder after N terms.

Question 8 – Solution**Step 1: Reindex carefully.**

Let $k = n + 1$. When $n = 1$, the new index is $k = 2$, and as $n \rightarrow \infty$, so does k . Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{k=2}^{\infty} \frac{1}{k!}.$$

Step 2: Compare with the exponential series.

The Maclaurin series for $e = e^1$ is

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{\infty} \frac{1}{k!}.$$

The desired tail begins at $k = 2$, so the omitted terms are exactly $1/0! = 1$ and $1/1! = 1$. Therefore

$$S = e - 2.$$

Step 3: Bound the remainder.

After the original terms through $n = N$, the tail starts at $k = N + 2$. For $j \geq 0$, each new factorial factor is at least $N + 3$, so

$$(N + 2 + j)! \geq (N + 2)!(N + 3)^j.$$

Taking reciprocals and summing gives

$$0 < R_N \leq \frac{1}{(N + 2)!} \sum_{j=0}^{\infty} \frac{1}{(N + 3)^j} = \frac{N + 3}{(N + 2)(N + 2)!}.$$

This bound is explicit, positive, and tends to zero, confirming the convergence of the partial sums to $e - 2$.

Question 9

Consider $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

- (a) Write the first four terms and expose the telescoping cancellation.
- (b) Derive the N th partial sum and determine whether the series converges.
- (c) Reconcile the divergence with the fact that the individual terms approach zero.

Question 9 – Solution**Step 1: Form the finite partial sum.**

Write the N th partial sum before passing to infinity:

$$\begin{aligned}s_N &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{N+1} - \sqrt{N}) \\ &= \sqrt{N+1} - 1.\end{aligned}$$

Step 2: Analyze the partial-sum limit.

All interior square-root terms cancel, leaving only the first negative boundary term and final positive boundary term. Because $\sqrt{N+1} - 1 \rightarrow \infty$, the partial sums are unbounded and the series diverges to $+\infty$.

Step 3: Check the n th-term condition.

The summand does approach zero. Rationalization gives

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Step 4: Reconcile the two facts.

There is no contradiction: the n th-term condition is necessary, not sufficient. Here every increment is positive and shrinks to zero, but the accumulated total still grows without bound. Since the denominator above is asymptotic to $2\sqrt{n}$, each increment is asymptotic to $1/(2\sqrt{n})$, consistent with a divergent p -series scale.

Question 10

For a real parameter r , consider $\sum_{n=1}^{\infty} r^{2n}$.

- (a) Identify the first term and common ratio as functions of r .
- (b) Determine all r for which the series converges and find its sum.
- (c) Analyze the boundary values $r = \pm 1$ and the region $|r| > 1$ using the n th-term test.

Question 10 – Solution**Step 1: Identify the geometric data.**

Expanding the first terms gives

$$r^2 + r^4 + r^6 + \dots,$$

Thus the first term is $a = r^2$ and the common ratio is $q = r^2$. A geometric series converges precisely when $|q| < 1$, so

$$|r^2| < 1 \iff |r| < 1.$$

Step 2: Find the sum on the convergence interval.For $|r| < 1$, the geometric formula gives

$$S(r) = \frac{r^2}{1 - r^2}.$$

The finite partial sum provides a direct verification:

$$s_N = \frac{r^2(1 - r^{2N})}{1 - r^2} \quad (r^2 \neq 1),$$

When $|r| < 1$, $r^{2N} \rightarrow 0$, so $s_N \rightarrow r^2/(1 - r^2)$.**Step 3: Check the endpoints and exterior region.**

At both $r = 1$ and $r = -1$, every term r^{2n} equals 1, so $s_N = N \rightarrow \infty$. If $|r| > 1$, then r^{2n} grows rather than approaches zero; the nth-term test proves divergence. Thus the convergence set is exactly $(-1, 1)$.

Question 1

Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

- (a) Apply the nth-term test and explain why its result is inconclusive.
- (b) Group terms between consecutive perfect squares and find a lower bound for each block.
- (c) Use the block estimates to classify the series.

Question 1 – Solution**Step 1: Check the necessary condition.**

Let $a_n = 1/\sqrt{n}$. Since $\sqrt{n} \rightarrow \infty$, $a_n \rightarrow 0$. This necessary condition is satisfied, so the n th-term test is inconclusive; it can prove divergence when the limit is nonzero, but it can never prove convergence.

Step 2: Choose a grouping strategy.

The terms decrease too slowly for a geometric comparison. Grouping between consecutive squares makes their sizes and the number of terms in each block easy to control. For $k \geq 1$, consider $k^2 \leq n < (k+1)^2$. There are

$$(k+1)^2 - k^2 = 2k + 1$$

terms, and every one satisfies $\sqrt{n} < k+1$, hence $1/\sqrt{n} > 1/(k+1)$. Therefore the k th block has sum greater than

$$\frac{2k+1}{k+1} > 1.$$

Step 3: Convert the block estimate into divergence.

The partial sum through $(m+1)^2 - 1$ contains m such blocks and consequently exceeds m . Since m can be arbitrarily large, the increasing sequence of partial sums is unbounded.

Step 4: State and check the conclusion.

Therefore

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.}}$$

This agrees with the p -series rule: $p = 1/2 \leq 1$.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

- (a) Verify that the terms approach zero and explain why that does not settle convergence.
- (b) Group the indices into dyadic blocks $2^k \leq n < 2^{k+1}$.
- (c) Find a lower bound for each block and use it to classify the series.

Question 2 – Solution**Step 1: Check the terms.**

For $n \geq 2$, $a_n = 1/(n \ln n) > 0$, and $n \ln n \rightarrow \infty$. Thus $a_n \rightarrow 0$. The n th-term test is inconclusive, so a tail estimate is required.

Step 2: Form dyadic blocks.

Use the block $2^k \leq n < 2^{k+1}$. It contains exactly 2^k integers. Within this block,

$$n < 2^{k+1}, \quad \ln n < (k+1) \ln 2,$$

and hence

$$\frac{1}{n \ln n} > \frac{1}{2^{k+1}(k+1) \ln 2}.$$

Step 3: Estimate one complete block.

Because the block contains 2^k terms, its sum is greater than

$$2^k \frac{1}{2^{k+1}(k+1) \ln 2} = \frac{1}{2(k+1) \ln 2}.$$

Step 4: Compare the block series.

Summing these lower bounds over k produces the constant multiple $(2 \ln 2)^{-1} \sum 1/(k+1)$ of the divergent harmonic series. Hence the original partial sums are unbounded and

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Step 5: Interpret the slow growth.

The hidden continuous substitution is $u = \ln x$, since $du = dx/x$ and $\int dx/(x \ln x) = \ln(\ln x) + C$. This explains the exceptionally slow divergence; a large numerical cutoff can misleadingly suggest stabilization.

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ converges or diverges (angles in radians).

- (a) Explain why $0 \leq \sin^2 n/n \leq 1/n$ does not determine convergence.
- (b) Use $\sin^2 n = (1 - \cos 2n)/2$ to split the partial sums.
- (c) Show that $\sum \cos(2n)/n$ converges by the Dirichlet Test, then classify the original series.

Question 3 – Solution

Step 1: Audit the obvious comparison.

The terms satisfy $0 \leq \sin^2 n/n \leq 1/n$ and approach zero. However, an upper comparison with a divergent series is logically inconclusive: a smaller positive series may converge or diverge.

Step 2: Separate the average and oscillatory parts.

Using $\sin^2 n = (1 - \cos 2n)/2$ gives

$$s_N = \frac{1}{2} \sum_{n=1}^N \frac{1}{n} - \frac{1}{2} \sum_{n=1}^N \frac{\cos(2n)}{n}.$$

Step 3: Control the oscillatory series.

To apply Dirichlet's Test to $\sum \cos(2n)/n$, verify both hypotheses. First, the partial sums of $\cos(2n)$ are bounded. Indeed,

$$\sum_{n=1}^N e^{2in} = e^{2i} \frac{1 - e^{2iN}}{1 - e^{2i}},$$

whose magnitude is at most $2/|1 - e^{2i}|$; taking real parts preserves boundedness. Second, $1/n$ is positive, decreasing, and tends to zero. Dirichlet's Test therefore shows that $\sum \cos(2n)/n$ converges.

Step 4: Combine the two parts.

The first part of s_N is one-half of the harmonic partial sum and tends to $+\infty$, while the oscillatory part approaches a finite limit. Subtracting a bounded quantity cannot stop the harmonic growth. Hence

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} \text{ diverges.}}$$

The needed extra structure is that $\sin^2 n$ has positive average $1/2$, not merely that it is bounded.

Question 4

Determine whether $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}$ converges or diverges.

- (a) Identify the reciprocal-power benchmark suggested by the leading terms.
- (b) Compute the limit-comparison ratio with that benchmark.
- (c) State the conclusion and give a direct upper comparison valid for $n \geq 1$.

Question 4 – Solution**Step 1: Identify a benchmark.**

The terms are positive. Comparing the highest powers gives $(2n)/(n^3) = 2/n^2$, so the natural benchmark is the convergent p -series $b_n = 1/n^2$.

Step 2: Compute the limit-comparison ratio.

$$\lim_{n \rightarrow \infty} \frac{(2n+1)/(n^3+1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{2n^3+n^2}{n^3+1} = 2.$$

The ratio limit is 2, which is finite and strictly positive. These are exactly the hypotheses of the Limit Comparison Test. Since $\sum 1/n^2$ converges ($p = 2 > 1$), the given series converges.

Step 3: Verify with a direct inequality.

For $n \geq 1$, $2n+1 \leq 3n$ and $n^3+1 \geq n^3$, so

$$0 < \frac{2n+1}{n^3+1} \leq \frac{3}{n^2}.$$

Because $\sum 3/n^2$ converges, the Direct Comparison Test independently confirms the conclusion. The lower-order terms affect numerical values but not tail behavior.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$ converges or diverges.

- (a) Simplify the factorial quotient completely before selecting a test.
- (b) Express the N th partial sum using harmonic numbers.
- (c) Classify the series and explain why its factorial notation is deceptive.

Question 5 – Solution

Step 1: Simplify before selecting a test.

Use $(n+1)! = (n+1)n!$ to cancel the common factorial:

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Step 2: Rewrite the partial sums.

Thus

$$s_N = \sum_{n=1}^N \frac{1}{n+1} = \sum_{k=2}^{N+1} \frac{1}{k} = H_{N+1} - 1,$$

where H_m is the m th harmonic number.

Step 3: Apply the known harmonic behavior.

Because $H_{N+1} \rightarrow \infty$, subtracting the fixed number 1 does not change the divergence. Therefore the partial sums are unbounded and

$$\boxed{\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} \text{ diverges.}}$$

Step 4: Interpret and verify.

The terms $1/(n+1)$ do approach zero, so the n th-term test alone is inconclusive. The factorial notation suggests rapid growth, but all of $n!$ cancels, leaving exactly a shifted harmonic series. Removing or shifting finitely many harmonic terms never changes convergence or divergence.

Question 6

Determine whether $\sum_{n=1}^{\infty} e^{-\sqrt{n}}$ converges or diverges.

- (a) Verify the necessary nth-term condition.
- (b) Group indices into blocks $k^2 \leq n < (k+1)^2$ and bound the sum of each block from above.
- (c) Compare the block bounds with a convergent series.

Question 6 – Solution

Step 1: Check the necessary condition.

Let $a_n = e^{-\sqrt{n}}$. Because $\sqrt{n} \rightarrow \infty$, the exponent tends to $-\infty$ and $a_n \rightarrow 0$. The nth-term condition holds but does not prove convergence.

Step 2: Organize the tail into square blocks.

In the block $k^2 \leq n < (k+1)^2$, there are exactly $2k+1$ indices. Also $\sqrt{n} \geq k$, so the decreasing exponential satisfies $e^{-\sqrt{n}} \leq e^{-k}$. Therefore

$$\sum_{n=k^2}^{(k+1)^2-1} e^{-\sqrt{n}} \leq (2k+1)e^{-k}.$$

Step 3: Classify the series of block bounds.

It remains to check $\sum_{k=1}^{\infty} (2k+1)e^{-k}$. Let $q = e^{-1} < 1$. Then

$$2 \sum_{k=1}^{\infty} kq^k + \sum_{k=1}^{\infty} q^k,$$

Both series converge: $\sum q^k$ is geometric, and $\sum kq^k = q/(1-q)^2$ follows by differentiating the geometric series.

Step 4: Conclude by comparison.

The original nonnegative series is bounded above block by block by a convergent series, so its increasing partial sums are bounded and

$$\sum_{n=1}^{\infty} e^{-\sqrt{n}} \text{ converges.}$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \arctan n$ converges or diverges.

- (a) Evaluate the limit of the summand.
- (b) Apply the nth-term test and describe the growth of the partial sums.
- (c) Explain why no comparison or ratio test is needed.

Question 7 – Solution

Step 1: Apply the nth-term test first.

The summand has limit

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0.$$

Because the limit is not zero, the series diverges immediately by the nth-term test.

Step 2: Describe how the partial sums diverge.

Since $\arctan n \rightarrow \pi/2 > 1$, the definition of a limit guarantees an N_0 such that $\arctan n > 1$ for all $n \geq N_0$. Then for $N \geq N_0$,

$$s_N \geq s_{N_0-1} + (N - N_0 + 1),$$

so $s_N \rightarrow +\infty$ at least linearly.

Step 3: Justify why the test is decisive.

Every convergent series must have terms tending to zero: if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. Since this necessary condition fails, comparison, ratio, or integral tests are unnecessary.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ converges or diverges.

- (a) Verify that the terms approach zero.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain a direct comparison.
- (c) Confirm the quadratic scale by computing a limit-comparison ratio.

Question 8 – Solution**Step 1: Check the terms.**

Since $1/n \rightarrow 0$ and cosine is continuous, $1 - \cos(1/n) \rightarrow 1 - \cos 0 = 0$. The necessary condition holds, so a convergence test is still needed.

Step 2: Build a direct comparison.

The standard small-angle inequality

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

holds for every real x . Substituting $x = 1/n$ yields

$$0 \leq 1 - \cos(1/n) \leq \frac{1}{2n^2}.$$

Both sides are nonnegative, and $\sum 1/(2n^2)$ is a convergent p -series with $p = 2 > 1$. The Direct Comparison Test therefore proves convergence.

Step 3: Confirm the asymptotic scale.

The standard Taylor limit confirms more precisely that

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Thus the summand is asymptotic to $1/(2n^2)$. The finite positive ratio also satisfies the hypotheses of the Limit Comparison Test, independently confirming the conclusion.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ converges or diverges.

- (a) Verify the nth-term condition.
- (b) Perform a limit comparison with the harmonic series.
- (c) Give a direct lower comparison that independently proves divergence.

Question 9 – Solution

Step 1: Check the terms and select a benchmark.

Dividing by n^2 shows $n/(n^2 + 1) = (1/n)/(1 + 1/n^2) \rightarrow 0$, so the n th-term test is inconclusive. The dominant-power behavior suggests the harmonic benchmark $b_n = 1/n$.

Step 2: Perform limit comparison.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

The limit equals 1, which is finite and strictly positive. Since both series have positive terms and $\sum 1/n$ diverges, the Limit Comparison Test proves that the given series diverges.

Step 3: Verify with direct comparison.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$, so taking positive reciprocals and multiplying by n gives

$$\frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The smaller series $\sum 1/(2n)$ diverges; therefore the larger given positive-term series also diverges. This direction of comparison is essential when proving divergence.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$ converges or diverges.

- (a) Show that the denominator is positive for every $n \geq 1$.
- (b) Use $-1 \leq \sin n \leq 1$ to produce a simple upper comparison for the tail.
- (c) Confirm the classification by limit comparison with $1/n^2$.

Question 10 – Solution**Step 1: Check positivity and the domain.**

For $n = 1$, $1 + \sin 1 > 0$. For $n \geq 2$, use $\sin n \geq -1$ to obtain

$$n^2 + \sin n \geq n^2 - 1 > 0,$$

so every term is defined and positive.

Step 2: Create an upper bound for the tail.

For $n \geq 2$,

$$n^2 - 1 \geq \frac{3}{4}n^2,$$

because $n^2 \geq 4$. Hence

$$0 < \frac{1}{n^2 + \sin n} \leq \frac{1}{n^2 - 1} \leq \frac{4}{3n^2}.$$

The comparison series is a constant multiple of the convergent p -series $\sum 1/n^2$. The Direct Comparison Test therefore proves convergence of the tail, and adding the finite first term cannot change that conclusion.

Step 3: Confirm the dominant behavior.

Limit comparison gives the sharper asymptotic statement

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + \sin n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \sin n} = 1,$$

because dividing numerator and denominator by n^2 produces $1/(1 + \sin n/n^2)$, and $|\sin n|/n^2 \leq 1/n^2 \rightarrow 0$. The ratio limit 1 confirms that the oscillation in the denominator is asymptotically negligible.

Question 1

Consider the Basel series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- (a) Prove that the series converges using the p -series test.
- (b) State Euler's classical evaluation of its exact sum and give a decimal approximation.
- (c) Use an integral estimate to bound the remainder after N terms, and explain why convergence alone does not produce the exact constant.

Question 1 – Solution**Step 1: Classify the series.**

This is a p -series $\sum 1/n^p$ with $p = 2 > 1$. Therefore it converges. This test establishes existence of a finite sum but does not determine that sum.

Step 2: State the classical special value.

Euler's solution of the Basel problem gives the non-elementary identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934.$$

The equality with $\pi^2/6$ requires additional machinery, such as Euler's product for $\sin x$ or Fourier series; it does not follow from the p -series test alone.

Step 3: Quantify convergence.

Because $f(x) = 1/x^2$ is positive and decreasing,

$$\int_{N+1}^{\infty} \frac{dx}{x^2} \leq R_N \leq \int_N^{\infty} \frac{dx}{x^2},$$

so

$$\frac{1}{N+1} \leq R_N \leq \frac{1}{N}.$$

Thus the partial sums approach $\pi^2/6$ from below, with an error of order $1/N$.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- (a) Prove convergence and compare its speed with the Basel series.
- (b) State Euler's classical value $\zeta(4)$.
- (c) Using that value, compute the even-indexed and odd-indexed fourth-power subseries separately.

Question 2 – Solution**Step 1: Establish convergence.**

The series is a p -series with $p = 4 > 1$, so it converges. Moreover, $1/n^4 \leq 1/n^2$ for $n \geq 1$, so its tail decays substantially faster than the Basel-series tail.

Step 2: State the special value accurately.

Euler's classical evaluation is

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082323.$$

As with $\zeta(2)$, the p -series test proves convergence but not this exact constant.

Step 3: Extract the even contribution.

For even indices $n = 2k$,

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{1440}.$$

Step 4: Extract the odd contribution.

Subtracting the even terms from the full series gives

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{90} - \frac{\pi^4}{1440} = \frac{\pi^4}{96}.$$

The even and odd pieces recombine to $\pi^4/90$, providing an algebraic check.

Question 3

Consider $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

- (a) Identify the geometric data and derive the N th partial sum.
- (b) Find the sum and exact remainder.
- (c) Interpret the result as repeatedly adding half of the remaining distance to 2.

Question 3 – Solution**Step 1: Identify the series.**

The expansion is $1 + 1/2 + 1/4 + \cdots$, so the first term is $a = 1$ and the common ratio is $r = 1/2$. Since $|r| < 1$, the series converges.

Step 2: Derive the finite partial sum.

With indices 0 through N , there are $N + 1$ terms. The finite geometric formula gives

$$s_N = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2 - 2^{-N}.$$

Because $2^{-N} \rightarrow 0$,

$$S = \lim_{N \rightarrow \infty} s_N = 2.$$

Step 3: Compute and interpret the remainder.

The exact gap is $R_N = S - s_N = 2^{-N}$. Starting at $s_0 = 1$, each new term equals one-half of the current distance to 2; the gap is halved at every step. The process never exceeds 2, but it can get arbitrarily close.

Question 4

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the cancellation in the N th partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 4 – Solution**Step 1: Decompose one summand.**

Write $1/[n(n+1)] = A/n + B/(n+1)$. Multiplication by $n(n+1)$ gives $1 = A(n+1) + Bn$, so $A = 1$ and $B = -1$. Hence

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Step 2: Telescope a finite sum.

$$\begin{aligned} s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

All interior reciprocal terms cancel; only the two boundary terms remain.

Step 3: Pass to the limit and measure the error.

Since $1/(N+1) \rightarrow 0$, the series sums to $S = 1$. The exact remainder is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As a check, $s_N - s_{N-1} = 1/[N(N+1)]$, the original N th term.

Question 5

Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify the Alternating Series Test hypotheses.
- (b) Decide whether the convergence is absolute or conditional.
- (c) Derive the value $\ln 2$ from the power series for $\ln(1+x)$ and state an error bound.

Question 5 – Solution**Step 1: Prove convergence.**

Let $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Therefore the Alternating Series Test proves that $\sum (-1)^{n-1} b_n$ converges.

Step 2: Test absolute convergence.

Taking absolute values gives

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

the divergent harmonic series. Hence the original series is conditionally convergent, not absolutely convergent.

Step 3: Identify the sum.

For $-1 < x \leq 1$, the endpoint-valid power-series identity is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

Setting $x = 1$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2.$$

Step 4: Quantify the approximation.

The Alternating Series Estimation Theorem gives $|S - s_N| \leq b_{N+1} = 1/(N+1)$. Odd partial sums lie above $\ln 2$ and even partial sums lie below it.

Question 6

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

- (a) Explain why this selects exactly the odd-indexed terms of the Basel series.
- (b) Compute the even-indexed contribution by scaling $\zeta(2)$.
- (c) Subtract to obtain the odd-square sum and verify that the two pieces recombine correctly.

Question 6 – Solution**Step 1: Partition the Basel series.**

Every positive integer is either even or odd, and the terms are positive, so

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Step 2: Evaluate the even contribution.

Using Euler's value $\zeta(2) = \pi^2/6$,

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

Step 3: Subtract to isolate the odd terms.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$

Step 4: Verify the decomposition.

The even portion is one-fourth of the full sum, while the odd portion is three-fourths. Their sum is $(1/4 + 3/4)\zeta(2) = \zeta(2)$, as required.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$.

- (a) Factor the constant scale from the summand.
- (b) Use the Basel sum to evaluate the series.
- (c) Relate the answer to the odd-square contribution and the full series.

Question 7 – Solution

Step 1: Simplify the scaled denominator.

Since $(2n)^2 = 4n^2$,

$$\frac{1}{(2n)^2} = \frac{1}{4} \frac{1}{n^2}.$$

Step 2: Use linearity of convergent series.

The Basel series converges, so its constant factor may be pulled outside:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \boxed{\frac{\pi^2}{24}}.$$

Step 3: Interpret the result.

The substitution $m = 2n$ selects precisely the even-indexed Basel terms. Their total is one-fourth of the complete sum because squaring the index contributes the scale factor $2^2 = 4$.

Step 4: Check against the odd contribution.

Adding the odd-square value $\pi^2/8$ gives

$$\frac{\pi^2}{24} + \frac{\pi^2}{8} = \frac{\pi^2}{6},$$

recovering the full Basel sum.

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric power series inside its interval of convergence.
- (b) Align the powers and evaluate at $x = 1/2$.
- (c) Find the finite partial sum and exact remainder.

Question 8 – Solution

Step 1: Begin with a generating function.

For $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Power series may be differentiated term by term at interior points, giving

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Step 2: Match and evaluate the desired series.

Multiplying by x yields

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Since $|1/2| < 1$, substitution is valid:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \boxed{2}.$$

Step 3: Quantify the finite approximation.

Differentiating the finite geometric identity gives

$$s_N = \sum_{n=1}^N \frac{n}{2^n} = 2 - \frac{N+2}{2^N}, \quad R_N = \frac{N+2}{2^N}.$$

The remainder is positive and tends to zero, so the partial sums approach 2 from below.

Question 9

Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

- (a) Differentiate the geometric generating function twice.
- (b) Explain why $n^2 = n(n-1) + n$ requires restoring a first-derivative contribution.
- (c) Derive a closed generating function and evaluate it at $x = 1/2$.

Question 9 – Solution**Step 1: Differentiate twice.**

For $|x| < 1$, $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Two derivatives give

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiplying by x^2 produces

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Step 2: Restore the missing linear part.

Because $n^2 = n(n-1) + n$, add the first-derivative identity $\sum nx^n = x/(1-x)^2$:

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}.$$

Step 3: Evaluate at the requested point.

Since $x = 1/2$ lies inside the interval of convergence,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{(1/2)^3} = \boxed{6}.$$

Omitting the $\sum nx^n$ term would compute $n(n-1)$ rather than n^2 .

Question 10

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the two-step telescoping cancellation in a finite partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 10 – Solution

Step 1: Decompose the gap-two denominator.

Solving $1/[n(n+2)] = A/n + B/(n+2)$ gives $A = 1/2$ and $B = -1/2$. Thus

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite sum.

$$\begin{aligned} s_N &= \frac{1}{2} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right) - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N+2} \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right). \end{aligned}$$

The cancellation forms two interlaced chains because the indices differ by two.

Step 3: Take the limit.

Both ending reciprocals tend to zero, so

$$S = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \boxed{\frac{3}{4}}.$$

Step 4: Find the exact remainder.

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

As a check, subtracting consecutive partial-sum formulas reproduces $1/[N(N+2)]$.

Question 1

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

- (a) Define a continuous function f with $f(n) = 1/n^{3/2}$ and verify that it is positive, continuous, and decreasing on the required interval.
- (b) Use the Integral Test to determine whether the series converges or diverges.
- (c) If s_N is the N th partial sum and $R_N = S - s_N$, derive both an upper and a lower bound for R_N .

Show the improper-integral limit and all antiderivative work.

Question 1 – Solution

Step 1: Choose the continuous function.

Let $f(x) = x^{-3/2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{3}{2}x^{-5/2} < 0,$$

so it is decreasing. Also $f(n) = 1/n^{3/2}$.

Step 2: Evaluate the improper integral.

$$\int_1^{\infty} x^{-3/2} dx = \lim_{b \rightarrow \infty} \left[-2x^{-1/2} \right]_1^b = 2.$$

The integral converges, so the series converges by the Integral Test.

Step 3: Bound the remainder.

For a positive decreasing f ,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Therefore

$$\boxed{\frac{2}{\sqrt{N+1}} \leq R_N \leq \frac{2}{\sqrt{N}}}.$$

The test proves convergence; these inequalities additionally measure the truncation error.

Question 2

Consider the logarithmic series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

- (a) Explain why the series begins at $n = 2$, and verify all three Integral Test hypotheses for $f(x) = 1/(x \ln x)$.
- (b) Evaluate the associated improper integral using $u = \ln x$.
- (c) State the convergence or divergence conclusion and describe the growth rate of the truncated integral.

The n th-term test alone is not a complete solution.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \ln x)$ for $x \geq 2$. It is continuous and positive because $x > 1$. Moreover,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0,$$

so f is decreasing, and $f(n)$ equals the summand.

Step 2: Evaluate the improper integral.

With $u = \ln x$ and $du = dx/x$,

$$\int_2^b \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln b} \frac{du}{u} = \ln(\ln b) - \ln(\ln 2).$$

As $b \rightarrow \infty$, $\ln(\ln b) \rightarrow \infty$. Thus the improper integral diverges.

Step 3: Apply the test.

All hypotheses hold, so the Integral Test gives

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

The growth scale $\ln \ln b$ explains why divergence is extremely slow.

Question 3

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- (a) Verify that $f(x) = 1/[x(\ln x)^2]$ is positive, continuous, and decreasing for $x \geq 2$.
- (b) Apply the Integral Test, showing the substitution and evaluation of the improper integral.
- (c) Derive two-sided Integral Test bounds for the remainder R_N .
- (d) Briefly compare the result with $\sum 1/(n \ln n)$.

Question 3 – Solution**Step 1: Verify the hypotheses.**

Define $f(x) = 1/[x(\ln x)^2]$ for $x \geq 2$. It is continuous and positive, and

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

Let $u = \ln x$, $du = dx/x$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Hence the series converges. The extra logarithm power changes $\int du/u$ into the convergent p -integral $\int du/u^2$.

Step 3: Estimate the remainder.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

These bounds follow by integrating from $N+1$ and N , respectively.

Question 4

Consider

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}.$$

- (a) For $f(x) = x/(x^2 + 1)^2$, verify positivity, continuity, and decreasing behavior on $[1, \infty)$.
- (b) Evaluate the associated improper integral using an appropriate substitution.
- (c) Determine convergence and obtain upper and lower bounds for R_N .

Clearly distinguish the value of the integral from the value of the series.

Question 4 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = x/(x^2 + 1)^2$ for $x \geq 1$. It is continuous and positive. Differentiation gives

$$f'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3} < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the integral.

With $u = x^2 + 1$ and $du = 2x \, dx$,

$$\int_1^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int_2^\infty u^{-2} \, du = \frac{1}{4}.$$

The integral converges, so the series converges.

Step 3: Bound the tail.

Since

$$\int_A^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2(A^2 + 1)},$$

the remainder satisfies

$$\boxed{\frac{1}{2((N+1)^2 + 1)} \leq R_N \leq \frac{1}{2(N^2 + 1)}}.$$

Question 5

Consider the nested-logarithm series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}.$$

- (a) Explain why starting at $n = 3$ makes the terms positive and well defined.
- (b) Verify the Integral Test hypotheses for the corresponding function.
- (c) Evaluate the improper integral using two successive logarithmic substitutions.
- (d) State the verdict and identify the slow growth expression responsible for it.

Question 5 – Solution

Step 1: Check the domain and hypotheses.

Start at $n = 3$ so that $\ln n > 1$ and $\ln(\ln n) > 0$. Let

$$f(x) = \frac{1}{x \ln x \ln(\ln x)} \quad (x \geq 3).$$

The denominator is positive, continuous, and strictly increasing, so f is positive, continuous, and decreasing.

Step 2: Perform two substitutions.

First let $u = \ln x$, $du = dx/x$; then let $v = \ln u$, $dv = du/u$:

$$\int_3^b \frac{dx}{x \ln x \ln(\ln x)} = \int_{\ln 3}^{\ln b} \frac{du}{u \ln u} = [\ln(\ln u)]_{\ln 3}^{\ln b}.$$

Thus the antiderivative grows as $\ln(\ln(\ln b))$, which tends to infinity.

Step 3: Conclude.

By the Integral Test,

$$\boxed{\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)} \text{ diverges.}}$$

This is another borderline series: three nested logarithms still grow without bound.

Question 6

Consider the rapidly decaying series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

- (a) Verify that $f(x) = xe^{-x^2}$ satisfies the Integral Test hypotheses on $[1, \infty)$.
- (b) Evaluate the improper integral with a substitution and classify the series.
- (c) Derive upper and lower bounds for the remainder R_N .
- (d) Explain what the bound says about the speed of convergence.

Question 6 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = xe^{-x^2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = e^{-x^2}(1 - 2x^2) < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the improper integral.

With $u = x^2$ and $du = 2x \, dx$,

$$\int_1^\infty xe^{-x^2} \, dx = \frac{1}{2} \int_1^\infty e^{-u} \, du = \frac{1}{2e}.$$

Thus the series converges by the Integral Test.

Step 3: Bound the remainder.

Since $\int_A^\infty xe^{-x^2} \, dx = \frac{1}{2}e^{-A^2}$,

$$\boxed{\frac{1}{2}e^{-(N+1)^2} \leq R_N \leq \frac{1}{2}e^{-N^2}}.$$

The Gaussian factor makes the tail decrease exceptionally rapidly.

Question 7

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}.$$

- (a) Verify positivity, continuity, and decreasing behavior for $f(x) = 1/(x^2 + 4)$.
- (b) Use the Integral Test to classify the series, showing the arctangent antiderivative and improper limit.
- (c) Write explicit upper and lower bounds for the remainder after N terms.

Your remainder bounds should contain N and inverse tangent functions.

Question 7 – Solution

Step 1: Verify the hypotheses.

Define $f(x) = 1/(x^2 + 4)$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{2x}{(x^2 + 4)^2} < 0,$$

so it decreases.

Step 2: Test the improper integral.

$$\int_1^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\arctan\left(\frac{x}{2}\right) \right]_1^\infty = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{1}{2} \right) < \infty.$$

Therefore the series converges.

Step 3: Estimate the remainder.

For $A > 0$,

$$\int_A^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{A}{2} \right).$$

Consequently,

$$\frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N+1}{2} \right) \leq R_N \leq \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N}{2} \right).$$

Question 8

Consider

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = \ln x/x^2$ on $[2, \infty)$.
- (b) Evaluate the improper integral using integration by parts.
- (c) Determine convergence and derive two-sided bounds for R_N .
- (d) Explain how the result illustrates polynomial growth dominating logarithmic growth.

Question 8 – Solution

Step 1: Verify the hypotheses on the tail.

Let $f(x) = \ln x/x^2$ for $x \geq 2$. It is continuous and positive. Also

$$f'(x) = \frac{1 - 2 \ln x}{x^3} < 0 \quad (x \geq 2),$$

so it decreases.

Step 2: Integrate by parts.

Take $u = \ln x$ and $dv = x^{-2}dx$. Then $du = dx/x$ and $v = -1/x$:

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Hence

$$\int_2^\infty \frac{\ln x}{x^2} dx = \frac{\ln 2 + 1}{2} < \infty,$$

and the series converges.

Step 3: Estimate the remainder.

Since the tail integral from A equals $(\ln A + 1)/A$,

$$\boxed{\frac{\ln(N+1) + 1}{N+1} \leq R_N \leq \frac{\ln N + 1}{N}}.$$

Quadratic denominator growth dominates the logarithmic numerator.

Question 9

Consider the shifted p -series

$$\sum_{n=1}^{\infty} \frac{1}{(n+3)^{5/4}}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = (x+3)^{-5/4}$.
- (b) Evaluate the associated improper integral and classify the series.
- (c) Obtain two-sided bounds for R_N .
- (d) Explain why a fixed horizontal shift does not alter the underlying p -series verdict.

Question 9 – Solution**Step 1: Verify the hypotheses.**

Set $f(x) = (x + 3)^{-5/4}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{5}{4}(x + 3)^{-9/4} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

$$\int_1^{\infty} (x + 3)^{-5/4} dx = \left[-4(x + 3)^{-1/4} \right]_1^{\infty} = \frac{4}{\sqrt{2}} < \infty.$$

Thus the shifted p -series converges. The horizontal shift changes initial values but not the tail exponent $p = 5/4 > 1$.

Step 3: Bound the remainder.

$$\boxed{\frac{4}{(N + 4)^{1/4}} \leq R_N \leq \frac{4}{(N + 3)^{1/4}}}.$$

Although convergent, the exponent is close to 1, so the error decreases slowly.

Question 10

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}.$$

- (a) Verify the domain and all Integral Test hypotheses for $f(x) = 1/[x\sqrt{\ln x}]$.
- (b) Evaluate the improper integral using $u = \ln x$.
- (c) Determine convergence or divergence.
- (d) Place this series within the family $\sum 1/[n(\ln n)^p]$ and state the threshold value of p .

Question 10 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/[x\sqrt{\ln x}]$ for $x \geq 2$. It is continuous and positive. Its denominator $x\sqrt{\ln x}$ is increasing, so f is decreasing. Equivalently,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}} < 0.$$

Step 2: Evaluate the improper integral.

Put $u = \ln x$, so $du = dx/x$:

$$\int_2^b \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\ln b} u^{-1/2} du = 2\sqrt{\ln b} - 2\sqrt{\ln 2}.$$

This tends to infinity, so the series diverges by the Integral Test.

Step 3: Identify the threshold.

More generally,

$$\sum \frac{1}{n(\ln n)^p}$$

converges for $p > 1$ and diverges for $p \leq 1$. Here $p = 1/2$, which lies on the divergent side.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n}$.

- (a) Identify the dominant power and select a standard benchmark.
- (b) Prove a direct-comparison inequality with the correct direction.
- (c) Classify the series and confirm the result by limit comparison.

State why the benchmark converges or diverges.

Question 1 – Solution

Step 1: Select the benchmark.

The denominator behaves like n^2 , so use $b_n = 1/n^2$, a convergent p -series with $p = 2$.

Step 2: Prove direct comparison.

Since $n^2 + 3n \geq n^2 > 0$,

$$0 < \frac{1}{n^2 + 3n} \leq \frac{1}{n^2}.$$

The Direct Comparison Test proves convergence.

Step 3: Verify by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + 3n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 + 3/n} = 1.$$

The finite positive limit confirms that both series share the same verdict.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$.

- (a) Use dominant powers to select a harmonic benchmark.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and give an independent direct lower comparison.

Question 2 – Solution

Step 1: Select a benchmark.

The leading-term quotient is $2n/n^2 = 2/n$, so choose $b_n = 1/n$. All terms are positive.

Step 2: Compute the ratio limit.

$$L = \lim_{n \rightarrow \infty} \frac{(2n+1)/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{n^2+1} = 2.$$

Because $0 < L < \infty$ and the harmonic series diverges, the given series diverges.

Step 3: Check the direction directly.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$ and $2n + 1 \geq 2n$, so

$$\frac{2n+1}{n^2+1} \geq \frac{2n}{2n^2} = \frac{1}{n}.$$

A series larger than the divergent harmonic series must diverge.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ (radians).

- (a) Establish a pointwise trigonometric bound.
- (b) Compare with a standard p -series.
- (c) Classify the series and explain why no averaging argument is needed.

Question 3 – Solution

Step 1: Bound the numerator.

For every real number t , $0 \leq \sin^2 t \leq 1$. Hence for each positive integer n ,

$$0 \leq \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}.$$

Step 2: Classify the benchmark.

The series $\sum 1/n^2$ converges because it is a p -series with $p = 2 > 1$.

Step 3: Apply Direct Comparison.

The given nonnegative series is bounded term by term above by a convergent series, so it converges. No information about the average value or periodic distribution of $\sin^2 n$ is required.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$.

- (a) Select a benchmark using dominant powers.
- (b) Perform limit comparison.
- (c) Prove divergence again using a direct lower bound.

Question 4 – Solution

Step 1: Select the harmonic benchmark.

Since $\sqrt{n^2 + 1}$ behaves like n , choose $b_n = 1/n$.

Step 2: Compute the ratio.

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n^2}} = 1.$$

The finite positive ratio and divergence of $\sum 1/n$ imply divergence.

Step 3: Give a direct lower bound.

Since $n^2 + 1 \leq 2n^2$ for $n \geq 1$,

$$\frac{1}{\sqrt{n^2 + 1}} \geq \frac{1}{\sqrt{2}n}.$$

The smaller comparison series diverges, independently confirming the verdict.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4}$.

- (a) Choose a benchmark from the dominant powers.
- (b) Compute the limiting ratio and verify all Limit Comparison Test conditions.
- (c) Classify the series and give a direct upper bound.

Question 5 – Solution

Step 1: Select the benchmark.

The leading quotient $n^3/n^5 = 1/n^2$ suggests $b_n = 1/n^2$, a convergent p -series.

Step 2: Compute the ratio limit.

$$\lim_{n \rightarrow \infty} \frac{n^3/(n^5 + 4)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^5}{n^5 + 4} = 1.$$

All terms are positive and the limit lies in $(0, \infty)$, so Limit Comparison proves convergence.

Step 3: Verify directly.

Since $n^5 + 4 \geq n^5$,

$$0 < \frac{n^3}{n^5 + 4} \leq \frac{1}{n^2}.$$

This upper comparison with a convergent series gives the same conclusion.

Question 6

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$.

- (a) Prove useful upper and lower quadratic bounds for the denominator.
- (b) Convert them into bounds for the summand, noting reciprocal inequality directions.
- (c) Classify the series and confirm by limit comparison.

Question 6 – Solution

Step 1: Bound the denominator.

Clearly $n^2 - n + 1 \leq n^2$. Also

$$n^2 - n + 1 \geq \frac{n^2}{2} \iff n^2 - 2n + 2 = (n - 1)^2 + 1 \geq 0.$$

Thus $n^2/2 \leq n^2 - n + 1 \leq n^2$.

Step 2: Take reciprocals carefully.

All expressions are positive, so reciprocals reverse the inequalities:

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n + 1} \leq \frac{2}{n^2}.$$

The upper bound is the decisive one for convergence. Since $\sum 2/n^2$ converges, Direct Comparison proves convergence.

Step 3: Confirm asymptotically.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 - n + 1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n + 1/n^2} = 1.$$

Question 7

Consider $\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2}$.

- (a) Select a geometric benchmark from the dominant exponential terms.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and identify the benchmark's common ratio.

Question 7 – Solution

Step 1: Select a geometric benchmark.

The dominant quotient is $3^n/5^n = (3/5)^n$. Let $b_n = (3/5)^n$, whose common ratio $3/5$ has absolute value less than 1; hence $\sum b_n$ converges.

Step 2: Compute the limiting ratio.

$$\frac{(3^n + 1)/(5^n - 2)}{(3/5)^n} = \frac{1 + 3^{-n}}{1 - 2 \cdot 5^{-n}} \longrightarrow 1.$$

The original terms and benchmark are positive for $n \geq 1$, and the ratio limit is finite and positive.

Step 3: Apply Limit Comparison.

The two series have the same convergence behavior. Since the benchmark is geometric and convergent,

$$\boxed{\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2} \text{ converges.}}$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$.

- (a) Prove a direct lower bound using $\sqrt{n} \leq n$.
- (b) Use the correct comparison direction to classify the series.
- (c) Confirm the conclusion by limit comparison with $1/n$.

Question 8 – Solution

Step 1: Bound the denominator.

For $n \geq 1$, $\sqrt{n} \leq n$, so

$$n + \sqrt{n} \leq 2n.$$

Because both denominators are positive, taking reciprocals reverses the inequality:

$$\frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}.$$

Step 2: Apply Direct Comparison.

The smaller series $\sum 1/(2n)$ diverges, so the larger given positive-term series diverges.

Step 3: Confirm by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n + \sqrt{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/\sqrt{n}} = 1.$$

This finite positive limit confirms harmonic tail behavior.

Question 9

Consider $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

- (a) Prove a usable power bound for $\ln n$.
- (b) Compare the summand with a convergent p -series.
- (c) Explain why limit comparison with $1/n^3$ is inconclusive.

Question 9 – Solution

Step 1: Bound the logarithm.

We claim $\ln x \leq \sqrt{x}$ for $x \geq 1$. Let $h(x) = \sqrt{x} - \ln x$. Then

$$h'(x) = \frac{\sqrt{x} - 2}{2x},$$

so the minimum occurs at $x = 4$, where $h(4) = 2 - \ln 4 > 0$. The claim follows.

Step 2: Compare directly.

For $n \geq 1$,

$$0 \leq \frac{\ln n}{n^3} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

The benchmark is a convergent p -series with $p = 5/2 > 1$, so Direct Comparison proves convergence.

Step 3: Explain benchmark choice.

Comparison with $1/n^3$ by ratios gives $(\ln n/n^3)/(1/n^3) = \ln n \rightarrow \infty$, outside the standard finite-positive Limit Comparison case. A slightly larger convergent power benchmark is more useful.

Question 10

Consider $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

- (a) Show why limit comparison with $1/n$ and $1/n^p$ does not immediately decide the series.
- (b) Select a natural logarithmic benchmark or use the Integral Test.
- (c) Classify the series and state an explicit tail bound.

Question 10 – Solution

Step 1: Audit ordinary power benchmarks.

Relative to $1/n$,

$$\frac{1/[n(\ln n)^2]}{1/n} = \frac{1}{(\ln n)^2} \rightarrow 0,$$

which does not transfer the harmonic series' divergence. For any $p > 1$, the ratio to $1/n^p$ is $n^{p-1}/(\ln n)^2 \rightarrow \infty$, which does not transfer that p -series' convergence.

Step 2: Use the natural logarithmic test.

Let $f(x) = 1/[x(\ln x)^2]$. It is positive, continuous, and decreasing for $x \geq 2$. With $u = \ln x$,

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Thus the series converges by the Integral Test.

Step 3: State the tail estimate.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

The correct benchmark retains the logarithm rather than discarding it.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify every Alternating Series Test hypothesis.
- (b) Test absolute convergence and classify the series.
- (c) State its familiar value and give the error bound after N terms.

Question 1 – Solution

Step 1: Verify the AST hypotheses.

Write $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Hence the series converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n},$$

which diverges. Therefore convergence is conditional.

Step 3: State the value and error.

The power series for $\ln(1+x)$ at $x=1$ gives $S = \ln 2$. The Alternating Series Estimation Theorem gives

$$|S - s_N| \leq b_{N+1} = \frac{1}{N+1}.$$

Odd and even partial sums approach $\ln 2$ from opposite sides.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

- (a) Verify that the Alternating Series Test applies.
- (b) Determine whether the series converges absolutely or conditionally.
- (c) Give an error bound for s_N and explain which conclusion is stronger.

Question 2 – Solution**Step 1: Apply the AST.**

For $b_n = 1/n^2$, we have $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2},$$

a convergent p -series with $p = 2$. Hence the original series converges absolutely, a stronger conclusion than the AST alone provides.

Step 3: Bound the alternating error.

$$|S - s_N| \leq \frac{1}{(N+1)^2}.$$

In fact $S = -\pi^2/12$, though this exact value is not needed for classification.

Question 3

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$.

- (a) Evaluate the limit of the term magnitude.
- (b) Apply the nth-term test before considering the AST.
- (c) Explain exactly which AST condition fails.

Question 3 – Solution

Step 1: Simplify the magnitude.

$$b_n = \frac{n+1}{n} = 1 + \frac{1}{n} \rightarrow 1.$$

Thus the series terms $a_n = (-1)^n b_n$ alternate near ± 1 and do not approach zero.

Step 2: Apply the nth-term test.

Every convergent series must satisfy $a_n \rightarrow 0$. Here the even subsequence tends to 1 and the odd subsequence tends to -1 , so the term sequence has no limit and the series diverges.

Step 3: Audit the AST.

Although b_n is positive and decreasing, the required condition $b_n \rightarrow 0$ fails. Alternating signs cannot compensate for terms of persistent size.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify the AST hypotheses and classify absolute versus conditional convergence.
- (b) Find the smallest N for which the next-term bound b_{N+1} is strictly below 0.01, thereby guaranteeing $|S - s_N| < 0.01$.
- (c) Explain the role of the strict inequality.

Question 4 – Solution**Step 1: Verify convergence.**

With $b_n = 1/\sqrt{n}$, positivity is clear, b_n decreases, and $b_n \rightarrow 0$. The AST proves convergence. The absolute series $\sum 1/\sqrt{n}$ diverges ($p = 1/2$), so convergence is conditional.

Step 2: Apply the error estimate.

$$|S - s_N| \leq b_{N+1} = \frac{1}{\sqrt{N+1}}.$$

To guarantee an error strictly below 0.01 it is sufficient that

$$\frac{1}{\sqrt{N+1}} < \frac{1}{100} \iff N+1 > 10000.$$

Thus the smallest integer furnished by this bound is $\boxed{N = 10000}$.

Step 3: Verify strictness.

At $N = 9999$, the bound equals 0.01, not less than it. At $N = 10000$, it is $1/\sqrt{10001} < 0.01$.

Question 5

Consider $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$.

- (a) Use a continuous derivative to prove eventual decrease of the magnitude.
- (b) Verify the zero-limit condition and apply the AST.
- (c) Test absolute convergence and classify the series.

Question 5 – Solution**Step 1: Prove eventual decrease.**

Let $f(x) = \ln x/x$. Then

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \quad (x > e).$$

Hence $b_n = \ln n/n$ decreases for integers $n \geq 3$. Eventual decrease is sufficient; finitely many initial terms do not affect convergence.

Step 2: Check the limit.

L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Therefore the AST proves convergence.

Step 3: Test absolute convergence.

For $n \geq 3$, $\ln n \geq 1$, so $\ln n/n \geq 1/n$. The absolute-value series diverges by comparison with the harmonic series. Thus convergence is conditional.

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n + (-1)^n}$.

- (a) Explain the corrected starting index and why the usual AST magnitude is not decreasing.
- (b) Pair consecutive even and odd terms to test convergence.
- (c) Test absolute convergence and classify the series.

Question 6 – Solution

Step 1: Check the domain and AST template.

At $n = 1$ the denominator is $1 - 1 = 0$, so the series must start at 2. Magnitudes satisfy $b_{2k} = 1/(2k+1)$ and $b_{2k+1} = 1/(2k)$; hence $b_{2k+1} > b_{2k}$ infinitely often, so the standard monotonic AST does not apply.

Step 2: Pair terms.

$$a_{2k} + a_{2k+1} = \frac{1}{2k+1} - \frac{1}{2k} = -\frac{1}{2k(2k+1)}.$$

The paired series converges absolutely by comparison with $1/(4k^2)$. Since the unpaired final term tends to zero, both even and odd partial-sum subsequences approach the same limit; the original series converges.

Step 3: Test absolute convergence.

$$|a_{2k}| + |a_{2k+1}| = \frac{1}{2k+1} + \frac{1}{2k} \geq \frac{1}{2k}.$$

Thus the absolute series diverges, so convergence is conditional.

Question 7

Consider $\sum_{n=1}^{\infty} (-1)^n e^{-n}$.

- (a) Verify the AST hypotheses.
- (b) Test absolute convergence using a geometric series.
- (c) Find the exact sum and classify convergence.

Question 7 – Solution**Step 1: Verify the AST.**

With $b_n = e^{-n} > 0$, we have $b_{n+1} = e^{-1}b_n < b_n$ and $b_n \rightarrow 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum |(-1)^n e^{-n}| = \sum (e^{-1})^n,$$

a geometric series with ratio $e^{-1} < 1$. Hence convergence is absolute.

Step 3: Find the sum.

The original series is geometric with first term $-e^{-1}$ and ratio $-e^{-1}$:

$$S = \frac{-e^{-1}}{1 + e^{-1}} = -\frac{1}{e + 1}.$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$.

- (a) Verify decrease without differentiating.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 8 – Solution**Step 1: Verify monotonicity.**

Since $n + 2 > n + 1$ and logarithm is strictly increasing,

$$\ln(n + 2) > \ln(n + 1) > 0.$$

Taking positive reciprocals reverses the inequality, so $b_{n+1} < b_n$ for $b_n = 1/\ln(n + 1)$.

Step 2: Check the limit and apply AST.

Because $\ln(n + 1) \rightarrow \infty$, $b_n \rightarrow 0$. Thus the alternating series converges.

Step 3: Test absolute convergence.

Since $\ln(n + 1) \leq n$ for $n \geq 1$,

$$\frac{1}{\ln(n + 1)} \geq \frac{1}{n}.$$

The absolute series diverges by comparison with the harmonic series. Therefore convergence is conditional.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$.

- (a) Verify the AST hypotheses using monotonicity of sine.
- (b) Test absolute convergence by limit comparison with $1/n$.
- (c) Classify the series and state the alternating error bound.

Question 9 – Solution**Step 1: Verify the AST.**

For $n \geq 1$, $1/n \in (0, 1]$ and $b_n = \sin(1/n) > 0$. Since $1/(n+1) < 1/n$ and sine is increasing on $[0, 1]$, $b_{n+1} < b_n$. Continuity gives $b_n \rightarrow \sin 0 = 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

The harmonic series diverges, so Limit Comparison shows $\sum \sin(1/n)$ diverges. Hence convergence is conditional.

Step 3: State the error bound.

$$|S - s_N| \leq \sin\left(\frac{1}{N+1}\right) < \frac{1}{N+1}.$$

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$.

- (a) Use a derivative to verify decreasing magnitudes.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 10 – Solution**Step 1: Verify monotonicity.**

Let $f(x) = x/(x^2 + 1)$. Then

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0 \quad (x \geq 1).$$

Thus $b_n = n/(n^2 + 1)$ is nonincreasing and positive.

Step 2: Check the limit and apply AST.

$$b_n = \frac{1/n}{1 + 1/n^2} \longrightarrow 0.$$

Therefore the alternating series converges.

Step 3: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

By comparison with the divergent harmonic series, the absolute series diverges. Hence convergence is conditional.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$.

- (a) Test the absolute-value series first.
- (b) Classify the original series and identify the strongest efficient theorem.
- (c) Give an upper bound for the absolute tail after N terms.

Question 1 – Solution

Step 1: Remove the signs.

$$\sum \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum \frac{1}{n^{3/2}}.$$

This is a convergent p -series because $p = 3/2 > 1$.

Step 2: Classify convergence.

Absolute convergence implies ordinary convergence, so the original series converges absolutely. Although the Alternating Series Test also proves ordinary convergence, it gives a weaker conclusion and is unnecessary here.

Step 3: Bound the absolute tail.

Since $x^{-3/2}$ is positive and decreasing,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{3/2}} \leq \int_N^{\infty} x^{-3/2} dx = \frac{2}{\sqrt{N}}.$$

This also bounds the signed remainder in magnitude.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

- (a) Prove ordinary convergence.
- (b) Test absolute convergence separately.
- (c) Classify the series, state its sum, and give an alternating error bound.

Question 2 – Solution

Step 1: Prove ordinary convergence.

For $b_n = 1/n$, we have $b_n > 0$, b_n decreases, and $b_n \rightarrow 0$. The Alternating Series Test therefore proves convergence.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n},$$

which diverges. Thus the original series is conditionally convergent.

Step 3: State the value and error.

This sign convention is the negative alternating harmonic series, so

$$S = -\ln 2, \quad |S - s_N| \leq \frac{1}{N+1}.$$

Absolute convergence fails because removing the cancellation exposes the harmonic series.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

- (a) Bound the absolute value of each term.
- (b) Apply Direct Comparison to classify absolute convergence.
- (c) Explain what this implies about the original oscillatory series.

Question 3 – Solution

Step 1: Remove oscillation by an inequality.

Since $|\cos n| \leq 1$,

$$0 \leq \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2}.$$

Step 2: Compare absolute values.

The benchmark $\sum 1/n^2$ converges. Direct Comparison gives $\sum |\cos n|/n^2 < \infty$.

Step 3: State the strongest conclusion.

The series converges absolutely, and therefore converges ordinarily. No cancellation or special information about the values of $\cos n$ is needed.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{\sin n}{n}$.

- (a) Prove ordinary convergence using Dirichlet's Test.
- (b) Prove the absolute-value series diverges.
- (c) Classify the series precisely.

Question 4 – Solution

Step 1: Bound the numerator partial sums.

From the finite geometric sum of e^{in} ,

$$\left| \sum_{n=1}^N e^{in} \right| \leq \frac{2}{|1 - e^i|}.$$

Taking imaginary parts shows that $\sum_{n=1}^N \sin n$ is uniformly bounded.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to zero. Hence $\sum (\sin n)/n$ converges.

Step 3: Rule out absolute convergence.

Since $0 \leq |\sin n| \leq 1$, we have $|\sin n| \geq \sin^2 n$. Section 4-4 established that $\sum \sin^2 n/n$ diverges; therefore

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \infty.$$

Thus the series is conditionally convergent.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify ordinary convergence.
- (b) Test the absolute-value series with the p -series rule.
- (c) Classify convergence and give an error bound.

Question 5 – Solution

Step 1: Apply the AST.

With $b_n = 1/\sqrt{n}$, positivity, decrease, and $b_n \rightarrow 0$ all hold. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{n^{1/2}},$$

which diverges because $p = 1/2 \leq 1$.

Step 3: Classify and estimate.

The convergence is conditional, and

$$|S - s_N| \leq \frac{1}{\sqrt{N+1}}.$$

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$.

- (a) Verify ordinary convergence with the AST.
- (b) Test absolute convergence using the logarithmic benchmark.
- (c) Classify the series and state an alternating error estimate.

Question 6 – Solution**Step 1: Verify the AST.**

Let $b_n = 1/(n \ln n)$. For $n \geq 2$ it is positive. Its denominator is increasing, so b_n decreases, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

The absolute series is $\sum 1/(n \ln n)$. For $f(x) = 1/(x \ln x)$,

$$\int_2^b f(x) dx = \ln(\ln b) - \ln(\ln 2) \rightarrow \infty.$$

The Integral Test proves divergence.

Step 3: Classify and estimate.

The original series is conditionally convergent, with

$$|S - s_N| \leq \frac{1}{(N+1) \ln(N+1)}.$$

Question 7

Consider $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$.

- (a) Apply the Ratio Test to the absolute-value series.
- (b) Classify convergence.
- (c) Identify the exact sum using the exponential series.

Question 7 – Solution**Step 1: Test absolute values.**Let $b_n = 2^n/n!$. Then

$$\frac{b_{n+1}}{b_n} = \frac{2}{n+1} \rightarrow 0 < 1.$$

The Ratio Test proves that $\sum b_n$ converges.**Step 2: Classify.**Therefore the signed series converges absolutely. More generally, replacing 2 by any fixed real base c gives ratio $|c|/(n+1) \rightarrow 0$.**Step 3: Find the sum.**From $e^x = \sum_{n=0}^{\infty} x^n/n!$, substitute $x = -2$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2}.$$

Question 8

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$.

- (a) Simplify the ratio of consecutive absolute terms.
- (b) Apply the Ratio Test and classify convergence.
- (c) Explain the effect of the doubled factorial.

Question 8 – Solution

Step 1: Form the absolute ratio.

For $b_n = n!/(2n)!$,

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} = \frac{1}{2(2n+1)}.$$

Step 2: Apply the Ratio Test.

The ratio tends to $0 < 1$, so $\sum b_n$ converges. Hence the original series converges absolutely.

Step 3: Interpret.

Each step in $(2n)!$ introduces two new denominator factors but only one new numerator factor. This drives the ratio rapidly to zero.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^{-n}$.

- (a) Find the limit of the magnitude.
- (b) Apply the nth-term test.
- (c) Explain why alternating signs cannot rescue the series.

Question 9 – Solution

Step 1: Find the magnitude limit.

The standard exponential limit gives

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e, \quad \left(1 + \frac{1}{n}\right)^{-n} \rightarrow e^{-1} \neq 0.$$

Step 2: Examine the signed terms.

Even-indexed terms approach e^{-1} and odd-indexed terms approach $-e^{-1}$. Thus the terms do not approach zero.

Step 3: Apply the nth-term test.

A necessary condition for series convergence fails, so the series diverges. The AST is unavailable because its magnitude condition $b_n \rightarrow 0$ fails.

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$.

- (a) Prove a useful power bound for $\ln n$.
- (b) Use it to test absolute convergence.
- (c) Classify the original series and explain why the AST is not needed.

Question 10 – Solution**Step 1: Dominate the logarithm.**

For $x \geq 1$, $\ln x \leq \sqrt{x}$. One proof minimizes $h(x) = \sqrt{x} - \ln x$; its minimum is $h(4) = 2 - \ln 4 > 0$.

Step 2: Test absolute convergence.

$$0 \leq \frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$

The upper benchmark is a convergent p -series, so Direct Comparison proves convergence of the absolute-value series.

Step 3: Classify.

The original series converges absolutely. The AST could prove ordinary convergence after checking eventual decrease, but absolute comparison is faster and stronger.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n!}{3^n}$.

- (a) Compute a_{n+1}/a_n .
- (b) Decide whether the series converges.
- (c) Explain what the calculation says about factorial growth compared with 3^n .

Question 1 – Solution**Step 1: Identify the term.**

Let $a_n = n!/3^n$. Its terms are positive, so absolute values do not change the ratio.

Step 2: Simplify before taking the limit.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} = \frac{n+1}{3}.$$

Step 3: Interpret the ratio.

Thus $L = \lim_{n \rightarrow \infty} (n+1)/3 = \infty > 1$. The terms eventually increase; indeed, $a_{n+1} > a_n$ for $n > 2$. Therefore a_n cannot approach 0.

Conclusion.

The series **diverges** by the Ratio Test (and by the nth-term test). The unbounded ratio shows that $n!$ eventually grows faster than 3^n .

Question 2

Consider $\sum_{n=0}^{\infty} \frac{3^n}{n!}$.

- (a) Apply the Ratio Test and classify the series.
- (b) Recognize the resulting series and find its exact sum.

Question 2 – Solution

Step 1: Form the ratio.

With $a_n = 3^n/n!$,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \frac{3}{n+1}.$$

Step 2: Take the limit.

$L = \lim_{n \rightarrow \infty} 3/(n+1) = 0 < 1$. The Ratio Test proves absolute convergence.

Step 3: Identify the sum.

Since $e^x = \sum_{n=0}^{\infty} x^n/n!$, substituting $x = 3$ gives

$$\boxed{\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3}.$$

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.

- (a) Compute and simplify a_{n+1}/a_n .
- (b) Apply the Ratio Test.
- (c) Explain how the limit compares polynomial and exponential growth.

Question 3 – Solution

Step 1: Compute the exact ratio.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2.$$

Step 2: Evaluate the limit.

$$L = \frac{1}{2}(1)^2 = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely**. Its consecutive terms eventually behave as though multiplied by $1/2$. The polynomial factor changes the ratio by a factor tending to 1, while 2^n supplies the decisive factor $1/2$.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 5^n}$. Compute the consecutive-term ratio carefully, apply the Ratio Test, and state the convergence classification.

Question 4 – Solution

Step 1: Name the term.

Let $a_n = (2n)! / [(n!)^2 5^n]$.

Step 2: Expand only the new factorial factors.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)!)^2 5^{n+1}} \frac{(n!)^2 5^n}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{5(n+1)^2} = \frac{2(2n+1)}{5(n+1)}.\end{aligned}$$

Step 3: Take the limit.

$$L = \lim_{n \rightarrow \infty} \frac{2(2n+1)}{5(n+1)} = \frac{4}{5} < 1.$$

Conclusion.

The positive-term series **converges**. The factorial expression is $\binom{2n}{n}$, but no approximation is needed.

Question 5

Use the Ratio Test, without Stirling's formula, to determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges. Show all algebra used to find the ratio limit.

Question 5 – Solution

Step 1: Form and simplify the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}.\end{aligned}$$

Step 2: Use the standard exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

Conclusion.

The series **converges**. This gives the decisive geometric-rate factor $1/e$ directly, without an asymptotic formula for $n!$.

Question 6

For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $x \in \mathbb{R}$:

- (a) Apply the Ratio Test to the absolute values.
- (b) Find every real x for which the series converges.
- (c) State the exact sum.

Question 6 – Solution**Step 1: Treat x as fixed.**For $a_n = x^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n+1}.$$

For $x = 0$ the series plainly converges; the same final limit covers every fixed nonzero x .**Step 2: Take the limit.**

$$L = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1 \quad \text{for every fixed } x \in \mathbb{R}.$$

Conclusion.

The series converges absolutely for all real x , so its interval of convergence is $(-\infty, \infty)$. By the exponential-series identity,

$$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}.$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^{10}}$ converges or diverges.

- (a) Simplify a_{n+1}/a_n .
- (b) Use its limiting behavior to justify the conclusion.
- (c) Explain why the nth-term test also applies.

Question 7 – Solution

Step 1: Compute the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{10}} \frac{n^{10}}{n!} = (n+1) \left(\frac{n}{n+1} \right)^{10} \\ &= n \left(1 + \frac{1}{n} \right)^{-9}.\end{aligned}$$

Step 2: Evaluate its behavior.

The factor $(1 + 1/n)^{-9} \rightarrow 1$, while $n \rightarrow \infty$. Hence $L = \infty > 1$.

Conclusion.

The series **diverges**. Its terms eventually grow rather than approach zero, so the nth-term divergence test confirms the result. A factorial eventually dominates every fixed power n^{10} .

Question 8

Analyze $\sum_{n=1}^{\infty} \frac{2^{n^2}}{3^{n^2}}$. Rewrite its term using a single base, compute the consecutive-term ratio, and classify the series.

Question 8 – Solution

Step 1: Combine the powers.

$$a_n = (2/3)^{n^2}.$$

Step 2: Compute the ratio.

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{3}\right)^{(n+1)^2 - n^2} = \left(\frac{2}{3}\right)^{2n+1}.$$

Because $0 < 2/3 < 1$, this expression tends to 0.

Conclusion.

The ratio limit is $L = 0 < 1$, so the series **converges**. Its quadratic exponent makes the terms decay faster than those of $\sum (2/3)^n$.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ converges or diverges. Reduce the consecutive-term ratio completely before taking its limit.

Question 9 – Solution

Step 1: Form the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} = \frac{(n+1)^{n-1}}{n^n} \\ &= \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n-1}.\end{aligned}$$

Step 2: Separate the limiting factors.

We have $(1 + 1/n)^{n-1} \rightarrow e$ and $1/n \rightarrow 0$. Therefore $L = 0$.

Conclusion.

Since $L = 0 < 1$, the positive-term series **converges**. The simplified ratio resolves the competition between n^n and $(n!)^2$ without guesswork.

Question 10

Use the Ratio Test to analyze $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$. Use logarithms and a standard logarithmic expansion to evaluate the ratio limit.

Question 10 – Solution

Step 1: Describe the term logarithmically.

Let $a_n = (n/(n+1))^{n^2}$. Then

$$\log a_n = -n^2 \log \left(1 + \frac{1}{n} \right).$$

Using $\log(1+t) = t - t^2/2 + O(t^3)$,

$$\log a_n = -n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n + \frac{1}{2} + O(n^{-1}).$$

Step 2: Find the ratio limit.

$$\log \left(\frac{a_{n+1}}{a_n} \right) = \log a_{n+1} - \log a_n = -1 + O(n^{-1}) \rightarrow -1.$$

Exponentiating gives $L = e^{-1} < 1$.

Conclusion.

The series **converges**. More specifically, $a_n \sim e^{1/2}e^{-n}$, so its terms have geometric-scale decay.

Question 1

Consider

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{5^n n^n}.$$

- (a) Rewrite the summand as a single quantity raised to the n th power.
- (b) Apply the Root Test and classify the series.

Question 1 – Solution**Step 1: Rewrite the term.**

Let

$$a_n = \frac{(2n+1)^n}{5^n n^n} = \left(\frac{2n+1}{5n} \right)^n = \left(\frac{2}{5} + \frac{1}{5n} \right)^n.$$

Step 2: Take the n th root.Since $a_n > 0$,

$$\sqrt[n]{|a_n|} = \frac{2n+1}{5n} = \frac{2}{5} + \frac{1}{5n}.$$

Therefore $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 2/5 < 1$.**Conclusion.**

The series **converges absolutely** by the Root Test. The slowly varying $1/(5n)$ correction disappears in the limit, leaving geometric-scale decay with factor $2/5$.

Question 2

Analyze

$$\sum_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{2} \right)^n.$$

- (a) Examine the even and odd terms separately.
- (b) Apply the nth-term divergence test.
- (c) Confirm the conclusion using the limsup form of the Root Test.

Question 2 – Solution**Step 1: Split by parity.**If $n = 2k$, then

$$a_{2k} = \left(1 + \frac{1}{2}\right)^{2k} = \left(\frac{3}{2}\right)^{2k} \rightarrow \infty.$$

If $n = 2k + 1$, then $a_{2k+1} = (1/2)^{2k+1} \rightarrow 0$.**Step 2: Apply the necessary condition.**

A convergent series must have $a_n \rightarrow 0$. The even subsequence is unbounded, so the full term sequence does not approach zero. Hence the series **diverges**.

Step 3: Check with roots.

$$\sqrt[n]{|a_n|} = 1 + \frac{(-1)^n}{2}, \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{3}{2} > 1.$$

The limsup Root Test gives the same result; an ordinary root limit does not exist.

Question 3

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} (3^n + 2^n)^{1/n} x^n.$$

Use the Root Test for the interior and test both endpoints separately.

Question 3 – Solution

Step 1: Apply the root to the entire term.

For $a_n = (3^n + 2^n)^{1/n} x^n$,

$$\sqrt[n]{|a_n|} = (3^n + 2^n)^{1/n^2} |x|.$$

Write $3^n + 2^n = 3^n[1 + (2/3)^n]$. Then

$$(3^n + 2^n)^{1/n^2} = 3^{1/n} \left[1 + \left(\frac{2}{3} \right)^n \right]^{1/n^2} \longrightarrow 1.$$

Thus the Root Test limit is $L = |x|$: convergence holds for $|x| < 1$ and divergence for $|x| > 1$.

Step 2: Test the endpoints.

Since $(3^n + 2^n)^{1/n} \rightarrow 3$, the terms at $x = 1$ approach 3, and at $x = -1$ their magnitudes approach 3. Neither endpoint passes the nth-term test.

Conclusion.

The radius is $\boxed{R = 1}$ and the interval is $\boxed{(-1, 1)}$.

Question 4

Use the Root Test to determine whether $\sum_{n=1}^{\infty} \frac{n^5}{4^n}$ converges. Explain explicitly why the polynomial factor does not affect the final root limit.

Question 4 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\frac{n^5}{4^n}} = \frac{n^{5/n}}{4}.$$

Step 2: Evaluate the polynomial-root factor.

Take logarithms:

$$\log(n^{5/n}) = \frac{5 \log n}{n} \rightarrow 0.$$

Exponentiating gives $n^{5/n} \rightarrow e^0 = 1$.

Step 3: Apply the test.

$$L = \lim_{n \rightarrow \infty} \frac{n^{5/n}}{4} = \frac{1}{4} < 1.$$

Therefore the series **converges absolutely**. Under an n th root, any fixed power n^p becomes $n^{p/n} \rightarrow 1$, so the exponential denominator determines the result.

Question 5

Determine whether

$$\sum_{n=2}^{\infty} \left(\frac{\log n}{n} \right)^n$$

converges or diverges. Compute the Root Test limit and interpret what a limit of zero implies.

Question 5 – Solution

Step 1: Take the n th root.

Since $n \geq 2$,

$$\sqrt[n]{\left| \left(\frac{\log n}{n} \right)^n \right|} = \frac{\log n}{n}.$$

Step 2: Evaluate the limit.

By l'Hopital's Rule for the corresponding continuous quotient,

$$\lim_{x \rightarrow \infty} \frac{\log x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Hence $L = 0 < 1$.

Conclusion.

The series **converges absolutely**. A root limit of zero means that for every fixed $0 < r < 1$, the terms eventually satisfy $a_n \leq r^n$; thus the tail is eventually dominated by a geometric series with any chosen ratio r .

Question 6

Analyze $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n} \right)^n$ using the Root Test. State the root limit and the convergence classification.

Question 6 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\left|\left(\frac{n+1}{2n}\right)^n\right|} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

Step 2: Evaluate the limit.

$$L = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely** by the Root Test. Although the base varies with n , it approaches $1/2$, so the terms eventually have geometric-scale decay. For example, once $n \geq 5$, the root is at most $3/5$, so $a_n \leq (3/5)^n$.

Question 7

Use the Root Test to determine whether

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{\sqrt{n}}\right)^{n^2}$$

converges. Use logarithms to evaluate the limit of the n th root.

Question 7 – Solution

Step 1: Take the n th root.

$$r_n = \sqrt[n]{a_n} = \left(1 - \frac{1}{\sqrt{n}}\right)^n.$$

Step 2: Take logarithms.

Since $\log(1 - u) \leq -u$ for $0 < u < 1$,

$$\log r_n = n \log \left(1 - \frac{1}{\sqrt{n}}\right) \leq -\sqrt{n} \longrightarrow -\infty.$$

Thus $0 \leq r_n \leq e^{-\sqrt{n}} \rightarrow 0$. (The expansion $\log(1 - u) = -u - u^2/2 + O(u^3)$ gives the sharper relation $\log r_n = -\sqrt{n} - 1/2 + O(n^{-1/2})$.)

Conclusion.

The root limit is $L = 0 < 1$, so the series **converges absolutely** and does so faster than any fixed geometric rate.

Question 8

Analyze

$$\sum_{n=1}^{\infty} \left(\frac{2 + \sin n}{4} \right)^n.$$

Use the limsup form of the Root Test to handle the oscillating base, and explain why an ordinary root limit is unavailable.

Question 8 – Solution**Step 1: Take the root.**Since $1 \leq 2 + \sin n \leq 3$,

$$\sqrt[n]{|a_n|} = \frac{2 + \sin n}{4} \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

This sequence oscillates rather than approaching one value.

Step 2: Compute the limsup.Integer angles modulo 2π are dense because $1/(2\pi)$ is irrational. Consequently, $\sin n$ comes arbitrarily close to 1, so

$$L = \limsup_{n \rightarrow \infty} \frac{2 + \sin n}{4} = \frac{3}{4} < 1.$$

(For convergence alone, the uniform bound $\sqrt[n]{|a_n|} \leq 3/4$ already suffices.)**Conclusion.**The series **converges absolutely** by the limsup Root Test.

Question 9

Determine whether $\sum_{n=1}^{\infty} n^{-n}$ converges or diverges. Use the Root Test and explain why the exponent n makes this test especially efficient.

Question 9 – Solution

Step 1: Rewrite the term.

We have $a_n = n^{-n} = (1/n)^n$.

Step 2: Take the n th root.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{1}{n}\right)^n} = \frac{1}{n}.$$

Therefore

$$L = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1.$$

Conclusion.

The series **converges absolutely**. Rather than complicating the calculation, the exponent n is exactly canceled by the n th root. Moreover, for any $0 < r < 1$, eventually $1/n < r$, so $n^{-n} < r^n$ and the tail is dominated by a geometric series.

Question 10

Let $c \in \mathbb{R}$. Determine exactly which values of c make

$$\sum_{n=1}^{\infty} c^{n^2}$$

converge. Apply the Root Test where decisive and inspect the boundary cases separately.

Question 10 – Solution

Step 1: Compute the root expression.

$$\sqrt[n]{|c^{n^2}|} = |c|^n.$$

Step 2: Separate the parameter cases.

- If $|c| < 1$, then $|c|^n \rightarrow 0 < 1$, so the series converges absolutely.
- If $|c| > 1$, then $|c|^n \rightarrow \infty$, and the terms c^{n^2} do not approach zero.
- If $|c| = 1$, the Root Test gives the boundary value 1. Directly, $|c^{n^2}| = 1$, so the terms again fail to approach zero.

Conclusion.

The series converges exactly when $|c| < 1$, equivalently $-1 < c < 1$. Both endpoints diverge.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

- (a) Choose an efficient convergence test and justify the choice.
- (b) Determine whether the series converges.
- (c) If possible, use a known power-series identity to find its exact sum.

Question 1 – Solution**Step 1: Prove convergence efficiently.**For $a_n = n/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3n} \longrightarrow \frac{1}{3} < 1.$$

The Ratio Test is quick and proves convergence.

Step 2: Choose a more informative method for the sum.

Begin with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiate and multiply by x :

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

At $x = 1/3$,

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \boxed{\frac{3}{4}}.$$

Strategy note.

The Ratio Test classifies the series; generating-function recognition gives both convergence and the exact value.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges or diverges. Explain why the Integral Test is especially well suited to this expression.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \log x)$ for $x \geq 2$. It is positive and continuous. Also,

$$f'(x) = -\frac{\log x + 1}{x^2(\log x)^2} < 0,$$

so f is decreasing.

Step 2: Evaluate the improper integral.

The integrand contains the differential $dx/x = d(\log x)$, suggesting $u = \log x$:

$$\int_2^b \frac{dx}{x \log x} = \int_{\log 2}^{\log b} \frac{du}{u} = \log(\log b) - \log(\log 2).$$

As $b \rightarrow \infty$, this tends to ∞ .

Conclusion.

The series **diverges** by the Integral Test. The substitution exposes the antiderivative $\log(\log x)$ immediately, which is why this test is more natural than a ratio or root test here.

Question 3

Classify $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ as absolutely convergent, conditionally convergent, or divergent. Use separate tests for ordinary and absolute convergence.

Question 3 – Solution

Step 1: Test the alternating series.

Put $b_n = 1/\sqrt{n}$. Then $b_n > 0$, $b_{n+1} < b_n$, and

$$\lim_{n \rightarrow \infty} b_n = 0.$$

Therefore $\sum (-1)^n b_n$ converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$$

This is a divergent p -series because $p = 1/2 \leq 1$.

Conclusion.

The original series is **conditionally convergent**. One test establishes convergence; the second shows that convergence depends essentially on cancellation between alternating signs.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n}$.

- (a) Show why the Ratio Test is inconclusive.
- (b) Use a standard central-binomial-coefficient asymptotic to reach a conclusion.

Question 4 – Solution

Step 1: Try the natural factorial test.

If $a_n = (2n)!/[(n!)^2 4^n]$, then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = \frac{2n+1}{2n+2} \rightarrow 1.$$

The Ratio Test is therefore inconclusive.

Step 2: Replace it with asymptotic comparison.

Stirling's formula yields the standard estimate

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Consequently,

$$a_n = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

Equivalently, $a_n/(1/\sqrt{n}) \rightarrow 1/\sqrt{\pi} > 0$.

Conclusion.

Since $\sum n^{-1/2}$ diverges, the given series **diverges** by Limit Comparison. A ratio limit of 1 signaled that a finer growth estimate was needed.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \cos n}$ converges. For the tail $n \geq 2$, use only the bound $-1 \leq \cos n \leq 1$; no exact trigonometric values are needed.

Question 5 – Solution

Step 1: Ensure the terms are defined and positive.

For $n = 1$, $\cos 1 > 0$ because $0 < 1 < \pi/2$. For $n \geq 2$, $n^2 + \cos n \geq n^2 - 1 > 0$.

Step 2: Build a simple upper comparison.

For $n \geq 2$,

$$n^2 + \cos n \geq n^2 - 1 \geq \frac{1}{2}n^2.$$

Taking reciprocals reverses the denominator comparison:

$$0 < \frac{1}{n^2 + \cos n} \leq \frac{2}{n^2}.$$

Step 3: Apply the Comparison Test.

The series $\sum 2/n^2$ converges because it is a constant multiple of a p -series with $p = 2 > 1$.

Conclusion.

The given series **converges**. A direct comparison is more efficient than ratio or root tests because the bounded trigonometric perturbation does not change the dominant n^2 denominator.

Question 6

Analyze $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$.

- (a) Simplify the factorial quotient.
- (b) Decide whether telescoping or comparison is more informative.
- (c) Determine convergence and, if possible, the exact sum.

Question 6 – Solution

Step 1: Cancel factorials.

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}.$$

This already resembles a telescoping difference.

Step 2: Decompose and form partial sums.

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\begin{aligned} S_N &= \sum_{n=1}^N \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} - \frac{1}{N+2}. \end{aligned}$$

Step 3: Pass to the limit.

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \lim_{N \rightarrow \infty} S_N = \boxed{\frac{1}{2}}.$$

Strategy note.

Comparison with $1/n^2$ proves convergence, but telescoping is better because it also gives the exact sum.

Question 7

Classify $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ as absolutely convergent, conditionally convergent, or divergent. Use bounded partial sums for the sine factor.

Question 7 – Solution**Step 1: Verify bounded partial sums.**

The finite identity

$$\sum_{n=1}^N \sin n = \frac{\sin(N/2) \sin((N+1)/2)}{\sin(1/2)}$$

shows that $\left| \sum_{n=1}^N \sin n \right| \leq 1/|\sin(1/2)|$.**Step 2: Apply Dirichlet's Test.**The sequence $1/n$ decreases to 0, and the partial sums of $\sin n$ are bounded. Hence $\sum \sin n/n$ converges.**Step 3: Check absolute convergence.**Since $|\sin n| \geq \sin^2 n$,

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \frac{1}{2} \sum \frac{1}{n} - \frac{1}{2} \sum \frac{\cos(2n)}{n}.$$

The cosine series converges by Dirichlet's Test while the harmonic series diverges, so the absolute-value series diverges.

Conclusion.The series is **conditionally convergent**.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$ converges. Explain why the Root Test is the natural first choice.

Question 8 – Solution

Step 1: Recognize the growing exponent.

Let $a_n = (1 + 1/n)^{-n^2}$. Taking an n th root cancels one power of n :

$$\sqrt[n]{a_n} = \left(1 + \frac{1}{n}\right)^{-n}.$$

Step 2: Use the exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = e^{-1} < 1.$$

Conclusion.

The series **converges** by the Root Test. This method is especially efficient because the summand is already a varying base raised to a multiple of n ; the root exposes its limiting geometric rate directly.

Question 9

Determine whether $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges. Compare the Integral Test with Cauchy Condensation and identify the shorter calculation.

Question 9 – Solution**Step 1: Use the Integral Test.**

The function $f(x) = \log x/x^2$ is positive and continuous for $x \geq 2$. Also,

$$f'(x) = \frac{1 - 2 \log x}{x^3} < 0 \quad (x \geq 2).$$

Integration by parts gives

$$\int_2^\infty \frac{\log x}{x^2} dx = \left[-\frac{\log x + 1}{x} \right]_2^\infty = \frac{\log 2 + 1}{2} < \infty.$$

Thus the series converges.

Step 2: Compare with condensation.

Because the terms decrease,

$$2^k a_{2^k} = 2^k \frac{\log(2^k)}{(2^k)^2} = \frac{k \log 2}{2^k},$$

whose series converges by the Ratio Test. Condensation also works, but the integral is slightly shorter and matches the logarithm-over-power form directly.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^3 - 2}$ converges or diverges. Identify the dominant powers and use Limit Comparison with an appropriate benchmark series.

Question 10 – Solution**Step 1: Identify the benchmark.**

The dominant-power quotient is

$$\frac{3n^2}{n^3} = \frac{3}{n},$$

so compare with the harmonic term $b_n = 1/n$. A finite number of initial terms, including the negative $n = 1$ term, does not affect convergence.

Step 2: Compute the limit.

For $n \geq 2$ both sequences are positive, and

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{3n^2 + 1}{n^3 - 2} \cdot n \\ &= \lim_{n \rightarrow \infty} \frac{3 + 1/n^2}{1 - 2/n^3} = 3.\end{aligned}$$

The limit is finite and strictly positive.

Conclusion.

Since $\sum 1/n$ diverges, the given series **diverges** by the Limit Comparison Test. Dominant powers correctly predicted harmonic-size terms.

Question 1

Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ and $S_N = \sum_{n=1}^N (-1)^{n-1}/n$.

- (a) State the Alternating Series Estimation Theorem for $R_N = S - S_N$.
- (b) Find the least final index N for which the next-term bound $1/(N+1)$ is strictly below 10^{-3} .

Question 1 – Solution

Step 1: Verify the theorem applies.

The magnitudes $b_n = 1/n$ decrease to 0, so the alternating series converges and

$$|R_N| \leq b_{N+1} = \frac{1}{N+1}.$$

Step 2: Solve the strict inequality.

$$\frac{1}{N+1} < 10^{-3} \iff N+1 > 1000 \iff N > 999.$$

The least integer is $\boxed{N = 1000}$.

Step 3: Check minimality.

For $N = 999$, the displayed non-strict bound gives $|R_{999}| \leq 1/1000 = 10^{-3}$, so this bound is not strictly below the tolerance. (The actual remainder is strictly smaller than the next term.) For $N = 1000$, $|R_{1000}| \leq 1/1001 < 10^{-3}$.

Conclusion.

Use the first 1000 terms, ending at index 1000.

Question 2

For $S = \sum_{n=1}^{\infty} 1/n^2$, let $S_N = \sum_{n=1}^N 1/n^2$ and $R_N = S - S_N$. Use both sides of the Integral Test remainder estimate to bracket R_N , then bracket S in terms of S_N .

Question 2 – Solution

Step 1: State the two-sided estimate.

For a positive decreasing function f with $a_n = f(n)$,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Here $f(x) = x^{-2}$ is positive, continuous, and decreasing.

Step 2: Evaluate both integrals.

$$\int_{N+1}^{\infty} \frac{dx}{x^2} = \frac{1}{N+1}, \quad \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

Thus

$$\boxed{\frac{1}{N+1} \leq R_N \leq \frac{1}{N}}.$$

Step 3: Convert the tail bracket to a sum bracket.

Since $S = S_N + R_N$,

$$\boxed{S_N + \frac{1}{N+1} \leq S \leq S_N + \frac{1}{N}}.$$

The bracket width is $1/[N(N+1)]$.

Question 3

Let $S = \sum_{n=1}^{\infty} 1/n^3$. Using the Integral Test upper remainder bound, find the least final index N certified to give absolute error $|R_N| < \frac{1}{2}10^{-4}$. Explain whether this tolerance alone guarantees identical rounding to four decimal places.

Question 3 – Solution

Step 1: Obtain an upper tail bound.

With $f(x) = x^{-3}$,

$$0 < R_N \leq \int_N^{\infty} x^{-3} dx = \left[-\frac{1}{2x^2} \right]_N^{\infty} = \frac{1}{2N^2}.$$

Step 2: Impose the absolute-error tolerance.

The requested error is below $0.00005 = 5 \times 10^{-5}$:

$$\frac{1}{2N^2} < 5 \times 10^{-5} \iff N^2 > 10,000 \iff N > 100.$$

Thus the least integer certified by this bound is $\boxed{N = 101}$.

Step 3: Check the strict endpoint.

At $N = 100$, the bound equals 5×10^{-5} , so it does not guarantee a strict error smaller than half a unit in the fourth decimal place. At $N = 101$, it does.

This absolute-error tolerance alone does not guarantee identical rounding: two nearby numbers can lie on opposite sides of a rounding threshold.

Question 4

Approximate $e^{-1} = \sum_{n=0}^{\infty} (-1)^n/n!$ using $S_N = \sum_{n=0}^N (-1)^n/n!$. Find the least final index N that guarantees correct rounding through six decimal places.

Question 4 – Solution**Step 1: Use the alternating remainder bound.**

The magnitudes $1/n!$ decrease to 0, so

$$|R_N| \leq \frac{1}{(N+1)!}.$$

The tolerance $|R_N| < 5 \times 10^{-7}$ suggests a candidate; correct rounding also requires a bracket within a single rounding interval.

Step 2: Find the first qualifying factorial.

$$9! = 362,880, \quad 10! = 3,628,800.$$

Hence

$$\frac{1}{9!} \approx 2.76 \times 10^{-6} > 5 \times 10^{-7}, \quad \frac{1}{10!} \approx 2.76 \times 10^{-7} < 5 \times 10^{-7}.$$

Therefore $N+1=10$, so $\boxed{N=9}$.

Step 3: Certify the rounding.

The alternating sign gives

$$0.3678791887 < S < S_9 + 1/10! < 0.3678794643.$$

Both endpoints lie in $[0.3678785, 0.3678795)$ and round to 0.367879. Also $S_8 \approx 0.3678819444$ and $S_7 \approx 0.3678571429$; the earlier even and odd partial sums lie still farther away. Hence no smaller final index gives this rounded value.

Conclusion.

Ten terms (indices 0 through 9) suffice. They give $S_9 \approx 0.3678791887$, which rounds to 0.367879, the same six-decimal value as e^{-1} .

Question 5

For $S = \sum_{n=0}^{\infty} (1/3)^n$, define $S_N = \sum_{n=0}^N (1/3)^n$.

- (a) Compute the tail $R_N = S - S_N$ exactly.
- (b) Compare it with the standard geometric-tail estimate based on the first omitted term.

Question 5 – Solution

Step 1: Identify the first omitted term.

Because S_N ends at index N , the tail begins at $n = N + 1$:

$$R_N = \sum_{n=N+1}^{\infty} \left(\frac{1}{3}\right)^n.$$

Step 2: Sum the geometric tail.

$$R_N = \frac{(1/3)^{N+1}}{1 - 1/3} = \frac{1}{2 \cdot 3^N}.$$

Thus the exact error is $\boxed{R_N = 3^{-N}/2}$.

Step 3: Compare with the general bound.

For a geometric tail with ratio magnitude $r < 1$,

$$|R_N| \leq \frac{|a_{N+1}|}{1 - r}.$$

Here $a_{N+1} = 3^{-(N+1)}$ and $r = 1/3$, producing $1/(2 \cdot 3^N)$. Because every term is positive and the ratio is exactly constant, the usual “bound” equals the exact tail.

Question 6

The Leibniz series is $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. If exactly M terms are used (indices 0 through $M-1$), find the least M certified by a next-term bound strictly below 10^{-4} .

Question 6 – Solution

Step 1: Identify the first omitted term.

With M terms, the final included index is $M - 1$, so the first omitted term has index M and magnitude $1/(2M + 1)$.

Step 2: Apply the alternating estimate.

$$|R| \leq \frac{1}{2M + 1}.$$

We need

$$\frac{1}{2M + 1} < 10^{-4} \iff 2M + 1 > 10,000 \iff M > 4999.5.$$

Thus the least integer is $\boxed{M = 5000}$.

Step 3: Verify minimality.

With 4999 terms the bound is $1/9999 > 10^{-4}$; with 5000 terms it is $1/10001 < 10^{-4}$. This large count illustrates the slow convergence of the Leibniz series.

Question 7

For $S = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, derive a two-sided Integral Test estimate for $R_N = S - S_N$. Then determine what the upper bound requires to guarantee $R_N < 0.01$ and interpret the result.

Question 7 – Solution**Step 1: Verify the setup.**

The function $f(x) = 1/[x(\log x)^2]$ is positive and decreasing for $x \geq 2$.

Step 2: Integrate the tail.

With $u = \log x$,

$$\int_A^\infty \frac{dx}{x(\log x)^2} = \int_{\log A}^\infty u^{-2} du = \frac{1}{\log A}.$$

Therefore

$$\boxed{\frac{1}{\log(N+1)} \leq R_N \leq \frac{1}{\log N}}.$$

Step 3: Solve the requested tolerance.

$$\frac{1}{\log N} < 0.01 \iff \log N > 100 \iff N > e^{100}.$$

Thus the least certified integer is $\lfloor e^{100} \rfloor + 1$, an astronomically large index (about 2.69×10^{43}).

Interpretation.

The lower and upper bounds are asymptotically equal, so this is not merely a loose estimate: the series itself converges extraordinarily slowly.

Question 8

Let $S = \sum_{n=1}^{\infty} ne^{-n}$ and $R_N = \sum_{n=N+1}^{\infty} ne^{-n}$. Use the Integral Test remainder theorem to give a two-sided bound for R_N .

Question 8 – Solution**Step 1: Check monotonicity.**For $f(x) = xe^{-x}$,

$$f'(x) = e^{-x}(1 - x) \leq 0 \quad (x \geq 1).$$

Thus f is positive, continuous, and decreasing on the needed interval.**Step 2: Find the antiderivative.**

Integration by parts gives

$$\int_A^\infty xe^{-x} dx = (A + 1)e^{-A}.$$

Step 3: Apply both remainder bounds.

$$\int_{N+1}^\infty xe^{-x} dx \leq R_N \leq \int_N^\infty xe^{-x} dx,$$

so

$$\boxed{(N + 2)e^{-(N+1)} \leq R_N \leq (N + 1)e^{-N}}.$$

The upper bound is a decreasing integral majorant for the full omitted tail; no extra first-term adjustment is needed under this standard formulation.

Question 9

Let $S = \sum_{n=1}^{\infty} (-1)^n/n^2$ and $S_N = \sum_{n=1}^N (-1)^n/n^2$.

- (a) Give the alternating-series error bound.
- (b) Compare it with an Integral Test bound for the tail of $\sum 1/n^2$.
- (c) Evaluate both bounds at $N = 100$.

Question 9 – Solution**Step 1: Use cancellation.**

Since $1/n^2$ decreases to 0, the alternating estimate gives

$$|S - S_N| \leq \frac{1}{(N+1)^2}.$$

Step 2: Ignore signs and bound the absolute tail.

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

This also bounds the signed error, but it discards cancellation and is therefore much weaker.

Step 3: Compare numerically.

At $N = 100$,

$$\frac{1}{101^2} = \frac{1}{10201} \approx 9.80 \times 10^{-5}, \quad \frac{1}{100} = 10^{-2}.$$

The alternating bound is over 100 times smaller. In general, it is order N^{-2} rather than N^{-1} .

Question 10

For $S = \sum_{n=1}^{\infty} 2^{-n^2}$ and $S_N = \sum_{n=1}^N 2^{-n^2}$, bound $R_N = S - S_N$ by a geometric series. Clearly identify its first term and common ratio.

Question 10 – Solution**Step 1: Reindex the tail.**

Write $n = N + 1 + k$, where $k \geq 0$. Then

$$(N + 1 + k)^2 = (N + 1)^2 + k(2N + 2) + k^2 \geq (N + 1)^2 + k(2N + 3),$$

because $k^2 \geq k$ for integers $k \geq 0$.

Step 2: Compare term by term.

$$2^{-(N+1+k)^2} \leq 2^{-(N+1)^2} \left(2^{-(2N+3)} \right)^k.$$

The comparison series has first term $2^{-(N+1)^2}$ and ratio $q = 2^{-(2N+3)}$.

Step 3: Sum the geometric majorant.

$$0 < R_N \leq \frac{2^{-(N+1)^2}}{1 - 2^{-(2N+3)}}.$$

The bound is very close to the first omitted term because q becomes extremely small.

Question 1

Find the center, radius of convergence, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}.$$

Test $x = -3$ and $x = 3$ separately and identify each resulting endpoint series.

Question 1 – Solution

Step 1: Recognize the geometric series.

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n.$$

It converges exactly when $|x/3| < 1$, or $|x| < 3$. Thus the center is 0 and $R = 3$.

Step 2: Test the endpoints.

- At $x = 3$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$. Its terms do not approach 0, so it diverges.

Conclusion.

The interval is $\boxed{(-3, 3)}$. Neither endpoint is included.

Question 2

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{nx^n}{4^n}$. Explain why the polynomial factor n does not change the radius but does determine the endpoint behavior.

Question 2 – Solution**Step 1: Use the Ratio Test.**For $a_n = nx^n/4^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} \frac{|x|}{4} \longrightarrow \frac{|x|}{4}.$$

Convergence requires $|x|/4 < 1$, so $|x| < 4$ and $R = 4$.**Step 2: Test the endpoints.**

- At $x = 4$, the terms are n , which do not approach 0.
- At $x = -4$, the terms are $n(-1)^n$, whose magnitudes grow.

Both endpoint series diverge by the nth-term test.

Conclusion.

The interval is $\boxed{(-4, 4)}$. The factor $(n+1)/n \rightarrow 1$, so n does not alter the radius; at the endpoints it prevents the terms from tending to zero.

Question 3

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n$. Identify the harmonic or alternating harmonic series arising at each endpoint.

Question 3 – Solution

Step 1: Find the interior interval.

$$\left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = |x| \frac{n}{n+1} \longrightarrow |x|.$$

The Ratio Test gives convergence for $|x| < 1$ and divergence for $|x| > 1$, so $R = 1$.

Step 2: Test $x = 1$.

The series becomes $\sum 1/n$, the divergent harmonic series.

Step 3: Test $x = -1$.

The series becomes $\sum (-1)^n/n$, which converges by the Alternating Series Test (but not absolutely).

Conclusion.

The interval is $\boxed{[-1, 1)}$. Convergence at the left endpoint is conditional.

Question 4

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n^2$. Explain why its radius matches Question 3 while both endpoints now converge.

Question 4 – Solution

Step 1: Find the radius.

$$\left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \left(\frac{n}{n+1} \right)^2 \rightarrow |x|.$$

Thus the interior condition is $|x| < 1$ and $R = 1$.

Step 2: Test the endpoints.

- At $x = 1$, the series is $\sum 1/n^2$, which converges.
- At $x = -1$, the absolute-value series is again $\sum 1/n^2$, so the series converges absolutely.

Conclusion.

The interval is $\boxed{[-1, 1]}$. Both this series and $\sum x^n/n$ have coefficient roots tending to 1, hence the same radius, but the stronger n^{-2} decay makes both boundary series absolutely convergent.

Question 5

Determine the radius and set of convergence of $\sum_{n=0}^{\infty} n!x^n$. Use the Ratio Test to show what happens for every nonzero x .

Question 5 – Solution

Step 1: Fix $x \neq 0$ and form the ratio.

With $a_n = n!x^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = (n+1)|x| \longrightarrow \infty.$$

The Ratio Test therefore gives divergence for every fixed nonzero x . Indeed, the terms eventually increase in magnitude and cannot tend to zero.

Step 2: Check $x = 0$.

At $x = 0$, the series has only its $n = 0$ term nonzero (using $0^0 = 1$ in the power-series convention), so it converges to 1.

Conclusion.

The radius is $\boxed{R = 0}$ and the set of convergence is the single point $\boxed{\{0\}}$. This is not a nondegenerate interval.

Question 6

Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} x^n/n!$. Explain why the factorial denominator allows every real x .

Question 6 – Solution

Step 1: Apply the Ratio Test for a fixed x .

$$\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \rightarrow 0.$$

For every fixed real x , the limit is $0 < 1$.

Step 2: State the convergence set.

The series converges absolutely for all $x \in \mathbb{R}$. Thus

$$\boxed{R = \infty}, \quad \boxed{(-\infty, \infty)}.$$

There are no finite endpoints to test.

Step 3: Interpret the factorial.

Each ratio contains the growing divisor $n+1$, which eventually dominates any fixed $|x|$. The series is the familiar expansion $e^x = \sum x^n/n!$.

Question 7

Find the center, radius, and interval of convergence of $\sum_{n=0}^{\infty} (x-2)^n/5^n$. Verify the behavior at both physical endpoints.

Question 7 – Solution

Step 1: Read the geometric structure.

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{5} \right)^n$$

has center $c = 2$ and converges when

$$\left| \frac{x-2}{5} \right| < 1 \iff |x-2| < 5 \iff -3 < x < 7.$$

Thus $R = 5$.

Step 2: Test the endpoints.

- At $x = 7$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$, which also diverges because its terms do not tend to zero.

Conclusion.

The interval is $\boxed{(-3, 7)}$, symmetric about the center 2.

Question 8

Find the center, radius, and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}.$$

Track the center carefully and determine which physical endpoint produces the alternating harmonic series.

Question 8 – Solution**Step 1: Find the interior condition.**

The series is centered at $c = -1$. For $a_n = (-1)^n(x+1)^n/n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = |x+1| \frac{n}{n+1} \longrightarrow |x+1|.$$

Thus $|x+1| < 1$, so $-2 < x < 0$ and $R = 1$.

Step 2: Test $x = -2$.

Here $x+1 = -1$, so

$$\frac{(-1)^n(-1)^n}{n} = \frac{1}{n}.$$

The harmonic series diverges; $x = -2$ is excluded.

Step 3: Test $x = 0$.

Here $x+1 = 1$, producing $\sum (-1)^n/n$, which converges conditionally.

Conclusion.

The interval is $\boxed{(-2, 0]}$.

Question 9

Find the center, radius, and interval of convergence of $\sum_{n=1}^{\infty} \frac{n^2(x-3)^n}{2^n}$. Explain the different roles of the exponential and polynomial factors.

Question 9 – Solution**Step 1: Apply the Ratio Test.**

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n} \right)^2 \frac{|x-3|}{2} \rightarrow \frac{|x-3|}{2}.$$

Thus $|x-3| < 2$, giving center $c = 3$, radius $R = 2$, and preliminary interval $1 < x < 5$.

Step 2: Test the endpoints.

- At $x = 5$, the terms reduce to n^2 , so the series diverges.
- At $x = 1$, the terms reduce to $n^2(-1)^n$, whose magnitudes do not approach zero.

Conclusion.

The interval is $\boxed{(1, 5)}$. The exponential coefficient 2^{-n} fixes the radius; the polynomial n^2 has root tending to 1 but makes both boundary terms fail the nth-term test.

Question 10

Find the interval of convergence and sum function of $\sum_{n=0}^{\infty} (x^2)^n$. Treat it as a geometric series in x^2 and test $x = \pm 1$.

Question 10 – Solution**Step 1: Apply the geometric criterion.**

The common ratio is $r = x^2$. Therefore

$$|r| < 1 \iff |x^2| < 1 \iff x^2 < 1 \iff |x| < 1.$$

Step 2: Test the endpoints.

At both $x = 1$ and $x = -1$, we have $x^2 = 1$, so the series becomes $\sum 1$ and diverges.

Step 3: Find the sum.

For $|x| < 1$, the geometric formula gives

$$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1 - x^2}.$$

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$. As a power series in x , only even powers occur, but its center is still 0 and its radius is 1.

Question 1

Starting from the geometric-series formula, find a power-series representation centered at 0 for $f(x) = 1/(1 - x)$. State its radius and interval of convergence, including endpoint tests.

Question 1 – Solution

Step 1: Use the geometric prototype.

For a geometric series,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (|r| < 1).$$

Set $r = x$ to obtain

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Carry over the convergence condition.

The requirement $|r| < 1$ becomes $|x| < 1$, so the radius is $R = 1$.

Step 3: Test the endpoints.

At $x = 1$, the series is $\sum 1$; at $x = -1$, it is $\sum (-1)^n$. In both cases the terms fail to approach zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$. The identity holds only there, even though the rational function is defined at some points outside that interval.

Question 2

Obtain a power series centered at 0 for $f(x) = 1/(1+x)$ by substituting into the geometric series. State its radius and interval of convergence and test both endpoints directly.

Question 2 – Solution

Step 1: Make the sign substitution.

Replace x by $-x$ in $1/(1-x) = \sum x^n$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Step 2: Translate the restriction.

The geometric condition is $|-x| < 1$, equivalent to $|x| < 1$, so $R = 1$.

Step 3: Test the endpoints.

- At $x = 1$, the series is $\sum (-1)^n$, which diverges by the nth-term test.
- At $x = -1$, it is $\sum 1$, which diverges.

Conclusion.

The interval is $\boxed{(-1, 1)}$. Although $1/(1+x)$ is finite at $x = 1$, its Taylor series about 0 does not converge there.

Question 3

Differentiate the geometric series term by term to find a power series for $1/(1-x)^2$. Reindex it in powers x^n and state its full interval of convergence.

Question 3 – Solution**Step 1: Begin inside the radius.**For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

A power series may be differentiated term by term at every point strictly inside its radius.

Step 2: Differentiate.

$$\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Let $k = n - 1$:

$$\boxed{\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k}.$$

Step 3: Check the boundary.Differentiation preserves the radius $R = 1$, but endpoints must be retested. At $x = 1$ the terms are $k + 1$; at $x = -1$ their magnitudes are $k + 1$. Neither tends to zero.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 4

Derive a power series for $\log(1 + x)$ by integrating a geometric series. Determine the constant of integration and the exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .Term-by-term integration is valid when $|x| < 1$:

$$\begin{aligned} \log(1+x) - \log 1 &= \int_0^x \frac{dt}{1+t} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

Thus

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

The lower limit makes the constant 0.

Step 3: Test endpoints.At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series is $-\sum 1/n$ and diverges.**Conclusion.** $R = 1$ and the interval is $\boxed{(-1, 1]}$.

Question 5

Derive the Maclaurin series for $\arctan x$ by integrating the series for $1/(1+x^2)$. Explain the odd powers and determine the endpoint behavior.

Question 5 – Solution**Step 1: Expand the derivative.**With geometric ratio $-t^2$,

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad |t| < 1.$$

Step 2: Integrate from 0 to x .

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}, \quad |x| < 1.$$

Integrating the even powers t^{2n} produces only odd powers x^{2n+1} ; the value at 0 fixes the constant.**Step 3: Test endpoints.**At $x = 1$, the alternating series converges to $\pi/4$. At $x = -1$, it converges to $-\pi/4$. Both are conditional.**Conclusion.**The radius is $R = 1$ and the interval is $\boxed{[-1, 1]}$.

Question 6

Use the differentiated geometric identity to find a power series for $x/(1-x)^2$. State the index, radius, and exact interval of convergence.

Question 6 – Solution

Step 1: Differentiate the geometric series.

For $|x| < 1$,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Step 2: Multiply by x .

$$\boxed{\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n}.$$

Multiplication by x shifts every exponent upward by one but does not change the radius.

Step 3: Test the boundary.

The radius remains $R = 1$. At $x = 1$, the terms are n ; at $x = -1$, they are $n(-1)^n$. Neither term sequence approaches zero.

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$.

Question 7

Find a Maclaurin series for $1/(1-x^2)$ by recognizing a geometric series. Explain why only even powers occur and determine the interval of convergence.

Question 7 – Solution**Step 1: Identify the ratio.**Apply $1/(1-r) = \sum r^n$ with $r = x^2$:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Only even powers occur because each power of the ratio x^2 adds two to the exponent.**Step 2: Translate the geometric restriction.**

$$|x^2| < 1 \iff |x| < 1.$$

Hence $R = 1$.**Step 3: Test the endpoints.**At both $x = 1$ and $x = -1$, every term x^{2n} equals 1. Thus the series diverges at both endpoints.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 8

Find a power series for $F(x) = \int_0^x \frac{dt}{1-t^4}$. Justify term-by-term integration, then determine the radius and exact interval of convergence of the resulting series.

Question 8 – Solution**Step 1: Expand the integrand.**For $|t| < 1$,

$$\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n}.$$

On every closed subinterval inside $(-1, 1)$, this power series converges uniformly, so it may be integrated term by term.

Step 2: Integrate.

$$F(x) = \sum_{n=0}^{\infty} \int_0^x t^{4n} dt = \boxed{\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}}.$$

Step 3: Test the endpoints.

Integration preserves radius $R = 1$. At $x = 1$, the series is $\sum 1/(4n+1)$, which diverges by Limit Comparison with $1/n$. At $x = -1$, it is the negative of the same divergent positive series.

Conclusion.The interval is $\boxed{(-1, 1)}$.

Question 9

Find a Maclaurin series for $1/(2 - x)$ by first factoring the denominator. State the radius and interval of convergence, including endpoint tests.

Question 9 – Solution

Step 1: Put the function in geometric form.

$$\frac{1}{2-x} = \frac{1}{2} \frac{1}{1-x/2}.$$

Using the geometric identity with $r = x/2$ gives

$$\frac{1}{2-x} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}.$$

Step 2: Translate the restriction.

The condition $|x/2| < 1$ becomes $|x| < 2$, so $R = 2$.

Step 3: Test endpoints.

At $x = 2$, each term equals $1/2$. At $x = -2$, the terms alternate between $\pm 1/2$. Neither tends to zero, so both series diverge.

Conclusion.

The interval is $\boxed{(-2, 2)}$.

Question 10

Find a Maclaurin series for $x^2/(1 + x^3)$. Identify the geometric ratio, explain the gaps between exponents, and determine the interval of convergence.

Question 10 – Solution**Step 1: Expand the reciprocal.**Use the geometric ratio $r = -x^3$:

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \quad |x^3| < 1.$$

Step 2: Multiply by x^2 .

$$\boxed{\frac{x^2}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}}.$$

The exponents are 2, 5, 8, ... because multiplying by each additional ratio contributes three powers of x .

Step 3: Determine convergence.

The interior condition is $|x| < 1$, so $R = 1$. At $x = 1$, terms are $(-1)^n$; at $x = -1$, every term is 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 1

Construct the Maclaurin series for e^x from its derivatives. Prove that it converges to e^x for every real x , and give a uniform Lagrange-remainder bound for the degree- N polynomial on $|x| \leq 1$.

Question 1 – Solution

Step 1: Compute the coefficients.

Every derivative of $f(x) = e^x$ is e^x , so $f^{(n)}(0) = 1$. Hence

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n!}, \quad \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

Step 2: Bound the remainder.

Taylor's theorem gives

$$R_N(x) = \frac{e^c x^{N+1}}{(N+1)!}$$

for some c between 0 and x . If $|x| \leq 1$, then $c \leq 1$, so

$$\boxed{|R_N(x)| \leq \frac{e|x|^{N+1}}{(N+1)!} \leq \frac{e}{(N+1)!}}.$$

Step 3: Justify equality with the function.

For any fixed real x , $e^c \leq e^{|x|}$ and $e^{|x|}|x|^{N+1}/(N+1)! \rightarrow 0$ by the Ratio Test. Thus $R_N(x) \rightarrow 0$ for every x , proving the series represents e^x on all of $\mathbb{R}$.

Question 2

Derive the Maclaurin series for $\sin x$ from its derivative cycle. Explain why only odd powers occur, give the degree-7 polynomial, and state the convergence domain.

Question 2 – Solution

Step 1: List the derivative cycle.

$$\sin x, \quad \cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

At $x = 0$, the values are $0, 1, 0, -1, 0, \dots$. All even-order derivatives vanish, while the odd derivatives alternate between 1 and -1 .

Step 2: Form the series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The requested polynomial is

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Step 3: Establish convergence.

The absolute-value ratio of consecutive displayed terms is

$$\frac{x^2}{(2n+2)(2n+3)} \longrightarrow 0$$

for every fixed x . Taylor's remainder is bounded by $|x|^{m+1}/(m+1)!$ because all derivatives of sine have magnitude at most 1, so it tends to 0. Thus the series equals $\sin x$ for every real x .

Question 3

Derive the Maclaurin series for $\cos x$ from its derivative cycle. Explain why every odd coefficient vanishes, give the degree-6 polynomial, and state the convergence domain.

Question 3 – Solution

Step 1: Evaluate the derivative cycle at zero.

$$\cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

gives values $1, 0, -1, 0, 1, \dots$ at 0. Therefore every odd-order coefficient is zero, and the even coefficients alternate in sign.

Step 2: Write the series and polynomial.

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}, \quad P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Step 3: Justify convergence to cosine.

Consecutive absolute terms have ratio

$$\frac{x^2}{(2n+1)(2n+2)} \rightarrow 0,$$

so the series converges for all x . Since every derivative of cosine has magnitude at most 1, the Lagrange remainder is at most $|x|^{m+1}/(m+1)! \rightarrow 0$. Hence the series equals $\cos x$ on $(-\infty, \infty)$.

Question 4

Derive the Maclaurin series for $\log(1 + x)$ by integrating a geometric series. Give its first five nonzero terms and determine its exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}}.$$

The first five nonzero terms are

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

Step 3: Determine endpoints.

The interior radius is 1. At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series becomes $-\sum 1/n$ and diverges.

Conclusion.The interval is $\boxed{(-1, 1]}$.

Question 5

Obtain the Maclaurin series and degree-4 Taylor polynomial for e^{2x} by substitution into the series for e^u . State the convergence domain and justify it.

Question 5 – Solution

Step 1: Start with the parent series.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (u \in \mathbb{R}).$$

Set $u = 2x$:

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

Step 2: Extract the requested polynomial.

$$P_4(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

Step 3: Check convergence.

For $a_n = (2x)^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2|x|}{n+1} \rightarrow 0$$

for every real x . Hence the radius is infinite and the series represents e^{2x} on all of $\mathbb{R}$.

Question 6

Use $\cosh x = (e^x + e^{-x})/2$ to derive its Maclaurin series. Explain why all odd powers cancel, give the degree-6 polynomial, and state the convergence domain.

Question 6 – Solution

Step 1: Expand both exponentials.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

When the series are added, odd-power coefficients cancel and even-power coefficients double.

Step 2: Divide by two.

$$\boxed{\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}, \quad P_6(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}.$$

Step 3: State convergence.

Both exponential series converge absolutely for every real x , so their sum and scalar multiple do also. Thus the Maclaurin series for $\cosh x$ has infinite radius and represents the function on $(-\infty, \infty)$.

Question 7

Derive the Maclaurin series for xe^x by multiplying a known series by x . Give two equivalent index forms, the first five nonzero terms, and the convergence domain.

Question 7 – Solution

Step 1: Multiply the exponential series.

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}.$$

Multiplication by x raises every exponent by one but leaves the coefficient denominator tied to the original index.

Step 2: Reindex if powers x^k are preferred.

Set $k = n + 1$:

$$xe^x = \boxed{\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}}.$$

The first five nonzero terms are

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}.$$

Step 3: State convergence.

Multiplying an everywhere-convergent series by x does not change its infinite radius. Therefore the identity holds for every real x .

Question 8

Derive the Maclaurin series for $f(x) = 1/(1 - x)$ from Taylor coefficients, then connect it with the finite geometric-sum identity. Determine the exact interval of convergence.

Question 8 – Solution**Step 1: Compute Taylor coefficients.**

Repeated differentiation gives

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad f^{(n)}(0) = n!.$$

Thus the Maclaurin coefficient is $f^{(n)}(0)/n! = 1$, and

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Verify using finite sums.

$$S_N = 1 + x + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then $x^{N+1} \rightarrow 0$, so $S_N \rightarrow 1/(1-x)$. This proves the Taylor series and geometric identity are the same construction.

Step 3: Check endpoints.

At $x = 1$ the terms are 1; at $x = -1$ they alternate between ± 1 . Both diverge by the nth-term test.

Conclusion.

$R = 1$ and the interval is $\boxed{(-1, 1)}$.

Question 9

Use the generalized binomial coefficients to find the first four nonzero terms of the Maclaurin series for $\sqrt{1+x}$. State the radius and determine convergence at both endpoints.

Question 9 – Solution

Step 1: Use the binomial pattern.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

With $\alpha = 1/2$,

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 2: State radius and endpoints.

The binomial series has $R = 1$. Its coefficients have magnitude asymptotic to a constant times $n^{-3/2}$, so it converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$, with sums $\sqrt{2}$ and 0 at the endpoints.

Question 10

Use complex exponentials to find the first four nonzero Maclaurin terms of $e^x \cos x$. Give a general coefficient formula and state the convergence domain.

Question 10 – Solution

Step 1: Encode the product as a real part.

$$e^x \cos x = \operatorname{Re} (e^x e^{ix}) = \operatorname{Re} (e^{(1+i)x}).$$

Step 2: Expand the complex exponential.

$$e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(1+i)^n x^n}{n!},$$

so the real coefficient of x^n is $\operatorname{Re}((1+i)^n)/n!$. Since $1+i = \sqrt{2}e^{i\pi/4}$,

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{2^{n/2} \cos(n\pi/4)}{n!} x^n.$$

Step 3: Extract nonzero terms.

The x^2 coefficient vanishes. The first four nonzero terms are

$$\boxed{1 + x - \frac{x^3}{3} - \frac{x^4}{6}} \quad \left(\text{next term } -\frac{x^5}{30} \right).$$

The factorial denominator gives infinite radius, so the series represents the function for every real x .

Question 1

Approximate $e^{0.1}$ using the cubic Maclaurin polynomial for e^x . Then use the Lagrange remainder to give a rigorous upper bound for the absolute error.

Question 1 – Solution

Step 1: Write the cubic polynomial.

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Step 2: Evaluate at $x = 0.1$.

$$\begin{aligned} P_3(0.1) &= 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} \\ &= 1 + 0.1 + 0.005 + 0.0001666667 \\ &= \boxed{1.1051666667}. \end{aligned}$$

Step 3: Bound the error.

Taylor's theorem gives

$$R_3(0.1) = \frac{e^c(0.1)^4}{4!} \quad \text{for some } 0 < c < 0.1.$$

Because $e^c \leq e^{0.1}$,

$$|R_3(0.1)| \leq \frac{e^{0.1}(0.1)^4}{24} < 4.61 \times 10^{-6}.$$

Thus $e^{0.1}$ lies within 0.00000461 of the stated approximation.

Question 2

Approximate $\sin(0.2)$ so that correct rounding through four decimal places is guaranteed. Use the lowest-degree odd Maclaurin polynomial whose error bound is below 5×10^{-5} .

Question 2 – Solution**Step 1: Check the linear approximation.**

For $x = 0.2$, the sine series is alternating with decreasing term magnitudes. Using only x gives

$$|R| \leq \frac{0.2^3}{3!} = 0.001333\ldots > 5 \times 10^{-5},$$

so the degree-1 polynomial does not provide the requested guarantee.

Step 2: Use the cubic polynomial.

$$P_3(x) = x - \frac{x^3}{3!}, \quad P_3(0.2) = 0.2 - \frac{0.008}{6} = \boxed{0.1986666667}.$$

Step 3: Bound the first omitted term.

$$|R| \leq \frac{0.2^5}{5!} = \frac{0.00032}{120} \approx 2.67 \times 10^{-6} < 5 \times 10^{-5}.$$

The alternating sign gives $0.1986666666 < \sin(0.2) < 0.1986693334$; both endpoints round to 0.1987. Therefore the smallest qualifying odd polynomial is cubic, and $\boxed{\sin(0.2) \approx 0.1987}$ to four decimal places.

Question 3

Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ by using a Taylor expansion. Identify why the constant and linear terms disappear and which coefficient determines the limit.

Question 3 – Solution

Step 1: Expand far enough to survive the cancellation.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

Step 2: Subtract the specified terms.

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

The constant 1 and linear term x cancel exactly.

Step 3: Divide by x^2 .

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + O(x^2).$$

As $x \rightarrow 0$, every remaining positive-power term tends to zero.

Conclusion.

$$\boxed{\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}}.$$

The answer is the first Taylor coefficient not removed by the numerator's subtraction.

Question 4

Solve the initial-value problem $y' = y$, $y(0) = 1$ by assuming a power-series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the coefficient recursion and identify the resulting function.

Question 4 – Solution

Step 1: Differentiate and align powers.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The original series is $y = \sum_{n=0}^{\infty} a_n x^n$.

Step 2: Match coefficients in $y' = y$.

$$(n+1)a_{n+1} = a_n, \quad a_{n+1} = \frac{a_n}{n+1}.$$

The initial condition gives $a_0 = y(0) = 1$. Repeated use of the recursion yields

$$a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \quad \text{and in general} \quad a_n = \frac{1}{n!}.$$

Conclusion.

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{e^x}.$$

The series has infinite radius by the Ratio Test, so the solution is valid for every real x .

Question 5

Estimate $I = \int_0^{0.5} e^{-x^2} dx$ by integrating the first four terms of the Maclaurin series for e^{-x^2} . Give a rigorous error bound from the first omitted integrated term.

Question 5 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

For $0 \leq x \leq 0.5$, the alternating term magnitudes decrease.

Step 2: Integrate through the x^6 term.

$$\begin{aligned} I &\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right]_0^{0.5} \\ &= 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} \\ &= \boxed{0.4612723214}. \end{aligned}$$

Step 3: Bound the integrated remainder.

The omitted integrand has magnitude at most $x^8/4!$, and the integrated alternating remainder satisfies

$$|R| \leq \int_0^{0.5} \frac{x^8}{4!} dx = \frac{0.5^9}{9 \cdot 4!} < 9.05 \times 10^{-6}.$$

Thus the estimate is accurate to within 0.00000905.

Question 6

For $0 < x \leq 1$, compare the linear approximation $L(x) = x$ with $\log(1 + x)$. Decide whether L overestimates or underestimates, and use the alternating series to bound the absolute error.

Question 6 – Solution

Step 1: Expand the logarithm.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad 0 < x \leq 1.$$

The first neglected term after the linear term is negative.

Step 2: Determine the direction.

Subtract the function from the linear approximation:

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots.$$

This is positive by the Alternating Series Test. Therefore x is an **overestimate** of $\log(1+x)$.

Step 3: Bound the error.

The alternating remainder after the first term is no larger than the first omitted magnitude:

$$0 < x - \log(1+x) \leq \frac{x^2}{2}.$$

For example, at $x = 0.2$, the error is at most 0.02; the actual error is approximately 0.01768.

Question 7

Derive one forward Euler step for $y' = f(t, y)$ from Taylor's theorem. Identify the discarded term. Then apply one step with $h = 0.1$ to $y' = y$, $y(0) = 1$, and bound its local error.

Question 7 – Solution**Step 1: Expand the exact solution in time.**Taylor's theorem about t_n gives

$$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n)$$

for some ξ_n between t_n and $t_n + h$.**Step 2: Replace the derivative using the ODE.**Since $y'(t_n) = f(t_n, y(t_n))$, dropping the quadratic remainder produces Euler's update

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}.$$

The one-step local truncation error is $h^2y''(\xi_n)/2$, hence is order h^2 .**Step 3: Apply the method.**For $y' = y$, $y_0 = 1$, and $h = 0.1$,

$$y_1 = 1 + 0.1(1) = \boxed{1.1}.$$

The exact value is $e^{0.1}$. On $[0, 0.1]$, $y''(t) = e^t \leq e^{0.1}$, so

$$|e^{0.1} - 1.1| \leq \frac{e^{0.1}(0.1)^2}{2} < 0.00553.$$

Question 8

The small-angle model replaces $\sin \theta$ by θ when θ is measured in radians. At $\theta = 0.2$, compute the relative error $|\theta - \sin \theta|/|\sin \theta|$ and explain it using the sine series.

Question 8 – Solution**Step 1: Use the local series.**

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots.$$

Thus the leading absolute error in replacing $\sin \theta$ by θ is approximately $\theta^3/6$, and the leading relative error is approximately $\theta^2/6$.

Step 2: Evaluate the exact relative error.

At $\theta = 0.2$ radians,

$$\sin(0.2) \approx 0.1986693308, \quad 0.2 - \sin(0.2) \approx 0.0013306692.$$

Therefore

$$\frac{0.2 - \sin(0.2)}{\sin(0.2)} \approx 0.0066979 = \boxed{0.6698\%}.$$

Step 3: Compare with the leading prediction.

$$\frac{\theta^2}{6} = \frac{0.04}{6} \approx 0.0066667 = 0.6667\%,$$

which is close to the exact relative error. This quantifies why the approximation is useful but not exact.

Question 9

A principal P earns annual rate r compounded n times per year, so $A_n = P(1 + r/n)^n$. Use the series for $\log(1 + u)$ to derive the continuous-compounding limit as $n \rightarrow \infty$.

Question 9 – Solution**Step 1: Take a logarithm.**

For sufficiently large n , $1 + r/n > 0$, and

$$\log\left(\frac{A_n}{P}\right) = n \log\left(1 + \frac{r}{n}\right).$$

Step 2: Insert the logarithmic expansion.

$$\begin{aligned} n \log\left(1 + \frac{r}{n}\right) &= n \left(\frac{r}{n} - \frac{r^2}{2n^2} + \frac{r^3}{3n^3} + O(n^{-4}) \right) \\ &= r - \frac{r^2}{2n} + \frac{r^3}{3n^2} + O(n^{-3}) \longrightarrow r. \end{aligned}$$

Step 3: Exponentiate.

Continuity of the exponential function gives

$$\frac{A_n}{P} \longrightarrow e^r, \quad \boxed{A_n \longrightarrow P e^r}.$$

The correction $-r^2/(2n)$ also explains the approach rate. For positive r , discrete compounding approaches the continuous value from below.

Question 10

Construct the quadratic Maclaurin model for the Gaussian e^{-x^2} . Determine where that polynomial is positive, and give an error bound for $|x| \leq 1$ using the first omitted term.

Question 10 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots.$$

Keeping terms through degree 2 gives

$$\boxed{Q_2(x) = 1 - x^2}.$$

Step 2: Analyze the model's sign.

$$Q_2(x) > 0 \iff 1 - x^2 > 0 \iff |x| < 1.$$

It vanishes at $x = \pm 1$ and becomes negative for $|x| > 1$, whereas the true Gaussian is positive for every real x . This shows the model is local.

Step 3: Bound the local error.

For $|x| \leq 1$, the series in x^2 alternates with decreasing magnitudes. Therefore

$$\boxed{|e^{-x^2} - (1 - x^2)| \leq \frac{x^4}{2}}.$$

The error is order x^4 near the origin.

Question 1

Use the generalized binomial theorem to find the first four nonzero terms of $(1 + x)^{1/2}$. State the radius and determine convergence at both endpoints.

Question 1 – Solution

Step 1: State the coefficient formula.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Step 2: Substitute $\alpha = 1/2$.

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 3: Determine the interval.

The radius is $R = 1$. The coefficient magnitudes are asymptotic to a constant times $n^{-3/2}$, so the series converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$.

Question 2

Treat $(1 + x)^{-1}$ as a generalized binomial series. Derive its coefficients, connect the result with a geometric series, and determine the exact interval of convergence.

Question 2 – Solution

Step 1: Compute the generalized coefficients.

For $n \geq 0$,

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = (-1)^n.$$

Thus

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots.$$

Step 2: Connect with geometry.

This is exactly

$$\frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n,$$

whose common ratio is $-x$. It converges when $|-x| < 1$, so $R = 1$.

Step 3: Test endpoints.

At $x = 1$, terms are $(-1)^n$; at $x = -1$, terms are 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 3

Find the first four nonzero terms of $(1 + x)^{-1/2}$ using generalized binomial coefficients. Explain the signs and determine the exact interval of convergence.

Question 3 – Solution**Step 1: Compute coefficients.**

$$\binom{-1/2}{1} = -\frac{1}{2}, \quad \binom{-1/2}{2} = \frac{3}{8}, \quad \binom{-1/2}{3} = -\frac{5}{16}.$$

Every factor in the numerator is negative, so the coefficient sign is $(-1)^n$. Hence

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots.$$

Step 2: Find the radius.

The generalized binomial series has $R = 1$.

Step 3: Test endpoints.

The coefficient magnitudes satisfy

$$\left| \binom{-1/2}{n} \right| = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

At $x = 1$ the signs alternate, so the series converges conditionally. At $x = -1$, all terms are positive and comparable with $1/\sqrt{n}$, so it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1, 1]}$.

Question 4

Expand $(1 - 2x)^{3/2}$ through the x^4 term. Track the substitution $u = -2x$ carefully, then find the radius and endpoint behavior.

Question 4 – Solution

Step 1: Expand in an intermediate variable.

$$(1 + u)^{3/2} = 1 + \frac{3}{2}u + \frac{3}{8}u^2 - \frac{1}{16}u^3 + \frac{3}{128}u^4 + \cdots.$$

Step 2: Substitute $u = -2x$.

$$\begin{aligned}(1 - 2x)^{3/2} &= 1 + \frac{3}{2}(-2x) + \frac{3}{8}(4x^2) - \frac{1}{16}(-8x^3) + \frac{3}{128}(16x^4) + \cdots \\ &= \boxed{1 - 3x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \cdots}.\end{aligned}$$

Each degree- n coefficient is multiplied by $(-2)^n$.

Step 3: Determine convergence.

The condition $|u| < 1$ gives $|x| < 1/2$, so $R = 1/2$. For exponent $3/2$, coefficient magnitudes are order $n^{-5/2}$, so both endpoint series converge absolutely.

Conclusion.

The interval is $\boxed{[-1/2, 1/2]}$.

Question 5

For $(1+x)^{1/3} = \sum_{n=0}^{\infty} c_n x^n$, derive a recurrence for c_{n+1} from c_n . Use it to compute through x^4 , then state the convergence interval.

Question 5 – Solution**Step 1: Derive the recurrence.**Since $c_n = \binom{1/3}{n}$,

$$\frac{c_{n+1}}{c_n} = \frac{(1/3) - n}{n + 1}, \quad \boxed{c_{n+1} = c_n \frac{1/3 - n}{n + 1}}, \quad c_0 = 1.$$

Step 2: Generate coefficients.

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{9}, \quad c_3 = \frac{5}{81}, \quad c_4 = -\frac{10}{243}.$$

Therefore

$$\boxed{(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \cdots}.$$

Step 3: State convergence.

The radius is 1. Since the coefficient magnitudes are order $n^{-4/3}$, both endpoint series converge absolutely.

Conclusion.The interval is $\boxed{[-1, 1]}$.

Question 6

Derive the binomial series for $(1 - x)^{-2}$ and show that it agrees with differentiating the geometric series. Determine the exact interval of convergence.

Question 6 – Solution**Step 1: Use generalized coefficients.**For $n \geq 0$,

$$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-n-1)}{n!} = (-1)^n(n+1).$$

Substitute $u = -x$ into $(1+u)^{-2}$:

$$\binom{-2}{n}(-x)^n = (-1)^n(n+1)(-1)^n x^n = (n+1)x^n.$$

Thus

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 2: Verify by differentiation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Rightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 3: Check convergence.

The radius is 1. At $x = 1$ terms are $n+1$; at $x = -1$ their magnitudes are $n+1$. Both endpoints diverge.

Conclusion.The interval is $\boxed{(-1, 1)}$.

Question 7

Approximate $\sqrt{1.04}$ using only the constant and linear terms of the binomial series for $\sqrt{1+x}$. Use the alternating-series structure to bound the omitted correction and report the approximation.

Question 7 – Solution

Step 1: Use the binomial expansion.

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots.$$

Set $x = 0.04$.

Step 2: Keep two terms.

$$\sqrt{1.04} \approx 1 + \frac{0.04}{2} = \boxed{1.02}.$$

Step 3: Bound the error.

Beginning with the quadratic term, the omitted terms alternate and decrease in magnitude at $x = 0.04$. Therefore

$$|R| \leq \frac{(0.04)^2}{8} = \boxed{0.0002}.$$

The first correction is negative, so the linear approximation is an overestimate. More precisely,

$$1.0198 \leq \sqrt{1.04} < 1.02.$$

The actual value, $1.0198039\dots$, is consistent with this bound.

Question 8

Let m be a nonnegative integer. Explain algebraically why the generalized binomial series for $(1+x)^m$ terminates, and recover the ordinary finite Binomial Theorem. Illustrate with $m = 4$.

Question 8 – Solution

Step 1: Examine the generalized coefficient.

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

If $n = m + 1$, the numerator includes the factor $m - m = 0$. For every $n > m$, it still includes that zero factor. Hence

$$\binom{m}{n} = 0 \quad (n > m).$$

Step 2: Recover the finite theorem.

The formally infinite series therefore reduces to

$$(1+x)^m = \sum_{n=0}^m \binom{m}{n} x^n.$$

Because this is a polynomial identity, it is valid for every real (and complex) x ; there is no finite radius restriction.

Step 3: Illustrate with $m = 4$.

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The next coefficient is $\binom{4}{5} = 0$, so no higher powers occur.

Question 9

Find the coefficient of x^4 in the Maclaurin series for $(1+x)^{1/2}$ directly from the generalized binomial formula. Do not compute the lower-degree coefficients.

Question 9 – Solution**Step 1: Identify the needed coefficient.**

In

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n,$$

the coefficient of x^4 is exactly $\binom{1/2}{4}$.**Step 2: Evaluate only that coefficient.**

$$\begin{aligned} \binom{1/2}{4} &= \frac{(1/2)(1/2-1)(1/2-2)(1/2-3)}{4!} \\ &= \frac{(1/2)(-1/2)(-3/2)(-5/2)}{24} \\ &= \frac{-15}{16 \cdot 24} = \boxed{-\frac{5}{128}}. \end{aligned}$$

There are three negative numerator factors, so the negative sign is expected.

Conclusion.The x^4 term is $\boxed{-5x^4/128}$. Computing preceding terms is unnecessary because generalized binomial coefficients directly target any desired power.

Question 10

Find the binomial series, radius, and exact interval of convergence for $(1 + 3x)^{-1/2}$. Test $x = -1/3$ and $x = 1/3$ separately.

Question 10 – Solution

Step 1: Substitute into the parent series.

$$(1 + u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} u^n.$$

With $u = 3x$,

$$(1 + 3x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 3^n x^n = 1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 + \cdots.$$

Step 2: Find the radius.

The condition $|u| < 1$ becomes $|3x| < 1$, so $R = 1/3$.

Step 3: Test endpoints.

At $x = 1/3$, $u = 1$, and the terms alternate with magnitude asymptotic to $1/\sqrt{\pi n}$; the series converges conditionally. At $x = -1/3$, $u = -1$ cancels the coefficient signs, leaving positive terms comparable with $1/\sqrt{n}$; it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1/3, 1/3]}$.