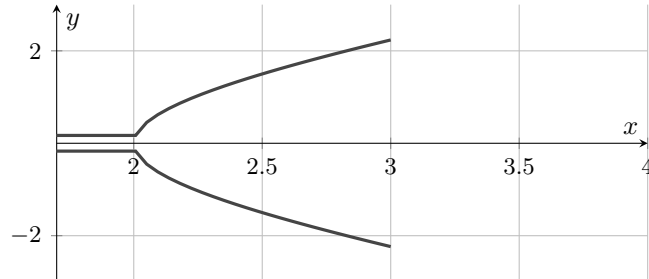


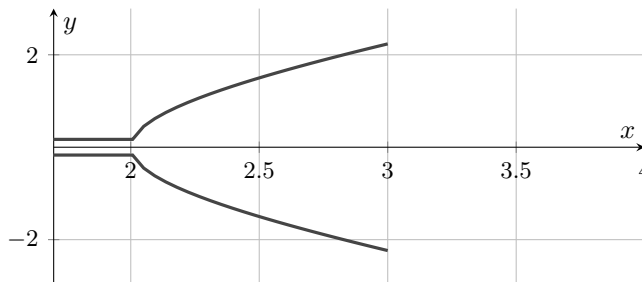
Question 1

Problem

A mechanism has $x = t + t^{-1}$ and $y = t - t^{-1}$, $t > 0$. Identify its Cartesian curve and explain which part the mechanism can reach.



Question 1 – Solution



Solution

1. Square both parametric equations:

$$x^2 = \left(t + \frac{1}{t}\right)^2 = t^2 + 2 + \frac{1}{t^2}, \quad y^2 = \left(t - \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2}.$$

2. Subtract the second equation from the first:

$$x^2 - y^2 = 4.$$

Thus the Cartesian curve is the hyperbola

$$\boxed{\frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

3. Because $t > 0$, the arithmetic–geometric mean inequality gives

$$x = t + \frac{1}{t} \geq 2.$$

Therefore, the parametrization reaches only the right-hand branch of the hyperbola.

4. To determine the direction of travel, note that

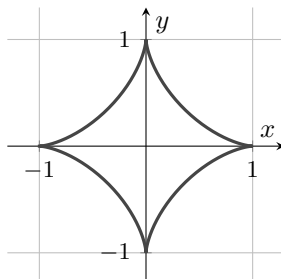
$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 1 + \frac{1}{t^2} > 0.$$

As t increases from 0 to 1, the point moves from the lower right toward $(2, 0)$. As t increases beyond 1, it moves from $(2, 0)$ toward the upper right.

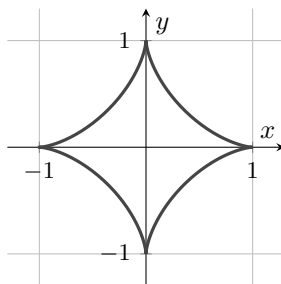
5. Hence the mechanism traces the entire right branch exactly once, moving upward as t increases.

Question 2**Problem**

The curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$, looks almost circular. Eliminate t and identify the four points where its shape is least circle-like.



Question 2 – Solution



Solution

1. Begin with

$$x = \cos^3 t, \quad y = \sin^3 t.$$

Taking each coordinate to the power $2/3$ gives

$$x^{2/3} = \cos^2 t, \quad y^{2/3} = \sin^2 t.$$

Here $x^{2/3}$ means $(\sqrt[3]{x})^2$, so the calculation is valid even when a coordinate is negative.

2. Add the two equations and use $\cos^2 t + \sin^2 t = 1$:

$$\boxed{x^{2/3} + y^{2/3} = 1}.$$

This curve is an *astroid*.

3. The curve is least circle-like where it has cusps. These occur when one trigonometric coordinate is 0 and the other is ± 1 .
4. Evaluating the parametrization at the corresponding parameter values gives

t	(x, y)
0	(1, 0)
$\pi/2$	(0, 1)
π	(-1, 0)
$3\pi/2$	(0, -1)

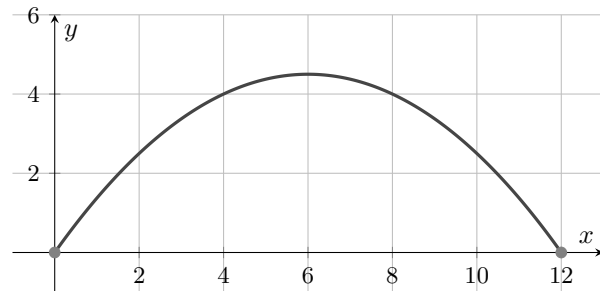
5. Therefore, the four sharp points are

$$\boxed{(1, 0), (0, 1), (-1, 0), (0, -1)}.$$

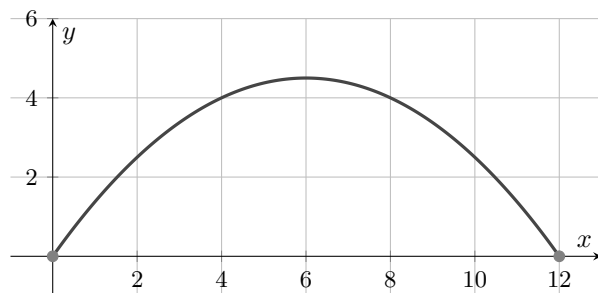
Question 3

Problem

A fountain droplet follows $x = 6t$ and $y = 9t - \frac{9}{2}t^2$. Eliminate time, find the landing point, and state why the parameter is more informative than the Cartesian equation.



Question 3 – Solution



Solution

1. Solve the horizontal equation for the parameter:

$$x = 6t \implies t = \frac{x}{6}.$$

2. Substitute $t = x/6$ into the vertical equation:

$$\begin{aligned} y &= 9\left(\frac{x}{6}\right) - \frac{9}{2}\left(\frac{x}{6}\right)^2 \\ &= \frac{3}{2}x - \frac{1}{8}x^2. \end{aligned}$$

Thus the Cartesian path is the downward-opening parabola

$$\boxed{y = \frac{3}{2}x - \frac{1}{8}x^2}.$$

3. The droplet lands when it returns to ground level, so set $y = 0$:

$$0 = \frac{3}{2}x - \frac{1}{8}x^2 = \frac{x}{8}(12 - x).$$

The solutions are $x = 0$ and $x = 12$.

4. The value $x = 0$ is the launch point. The nonzero intercept is therefore the landing point

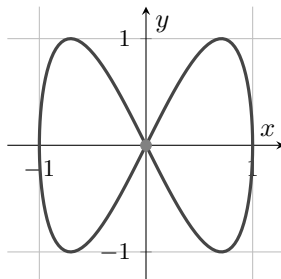
$$\boxed{(12, 0)}.$$

It occurs at $t = 12/6 = 2$.

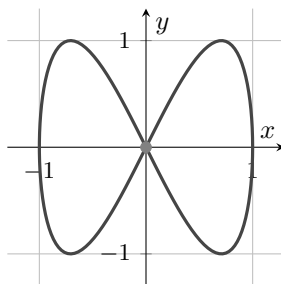
5. The Cartesian equation describes only the geometric path. The parameter t also records time, so it identifies the droplet's position and direction of motion at each instant.

Question 4**Problem**

For $x = \cos t$, $y = \sin(2t)$, $0 \leq t \leq 2\pi$, locate the self-intersection and determine how many times it is visited.



Question 4 – Solution



Solution

1. A self-intersection occurs when two different parameter values produce the same point. The graph suggests checking the origin.
2. For the point to lie at the origin, first require $x = 0$:

$$\cos t = 0.$$

On $0 \leq t \leq 2\pi$, this gives

$$t = \frac{\pi}{2} \quad \text{or} \quad t = \frac{3\pi}{2}.$$

3. Check the y -coordinate at both values:

$$y\left(\frac{\pi}{2}\right) = \sin \pi = 0, \quad y\left(\frac{3\pi}{2}\right) = \sin 3\pi = 0.$$

Hence both parameter values produce $(0, 0)$.

4. The two visits correspond to different directions. Since

$$\frac{dy}{dx} = \frac{2 \cos(2t)}{-\sin t},$$

the slopes are 2 at $t = \pi/2$ and -2 at $t = 3\pi/2$.

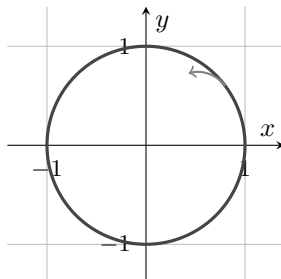
5. Therefore, the curve crosses itself at

$$\boxed{(0, 0)},$$

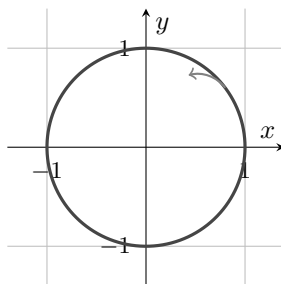
and this point is visited exactly twice on the stated interval.

Question 5**Problem**

Two students parametrize the unit circle by $(\cos t, \sin t)$ and $(\cos 2t, -\sin 2t)$ on $0 \leq t \leq 2\pi$. Compare orientation and number of traversals.



Question 5 – Solution

**Solution**

1. For the first parametrization,

$$(x, y) = (\cos t, \sin t),$$

the polar angle is t . As t increases from 0 to 2π , the angle increases through one full revolution.

2. It begins at $(1, 0)$, then moves through $(0, 1)$, $(-1, 0)$, and $(0, -1)$ before returning to $(1, 0)$. Therefore, it traces the circle counterclockwise exactly once.

3. Rewrite the second parametrization as

$$(x, y) = (\cos(2t), -\sin(2t)) = (\cos(-2t), \sin(-2t)).$$

Its polar angle is therefore $-2t$.

4. As t increases from 0 to 2π , the angle decreases from 0 to -4π . A decreasing angle produces clockwise motion, and a total angular change of 4π represents two complete revolutions.
5. Thus

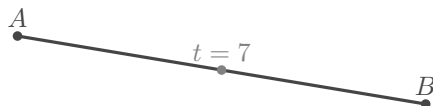
$(\cos t, \sin t)$ travels counterclockwise once,

whereas

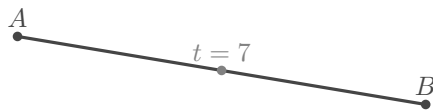
$(\cos 2t, -\sin 2t)$ travels clockwise twice.

Question 6**Problem**

Find a parametrization of the line segment from $A = (-2, 5)$ to $B = (4, -1)$ that reaches its midpoint at $t = 7$, with $6 \leq t \leq 8$.



Question 6 – Solution



Solution

1. A line segment from A to B can be parametrized by

$$\mathbf{r} = A + u(B - A), \quad 0 \leq u \leq 1.$$

2. The parameter interval must be changed from $6 \leq t \leq 8$ to $0 \leq u \leq 1$. The linear change of variables is

$$u = \frac{t - 6}{8 - 6} = \frac{t - 6}{2}.$$

3. Compute the displacement vector:

$$B - A = (4, -1) - (-2, 5) = (6, -6).$$

Therefore,

$$(x, y) = (-2, 5) + \frac{t - 6}{2}(6, -6).$$

4. Simplify each coordinate:

$$x = -2 + 3(t - 6) = 3t - 20,$$

$$y = 5 - 3(t - 6) = 23 - 3t.$$

Hence a suitable parametrization is

$$\boxed{x = 3t - 20, \quad y = 23 - 3t, \quad 6 \leq t \leq 8}.$$

5. At $t = 6$ it gives $A = (-2, 5)$, and at $t = 8$ it gives $B = (4, -1)$. At $t = 7$ it gives

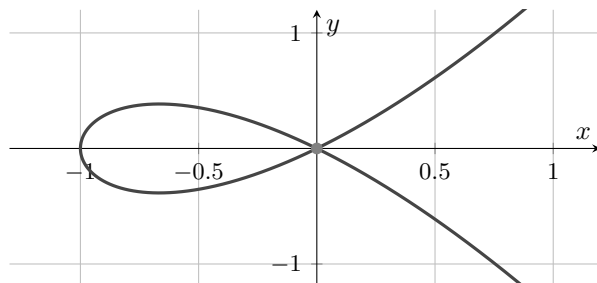
$$(x, y) = (1, 2) = \frac{A + B}{2},$$

so the midpoint condition is satisfied.

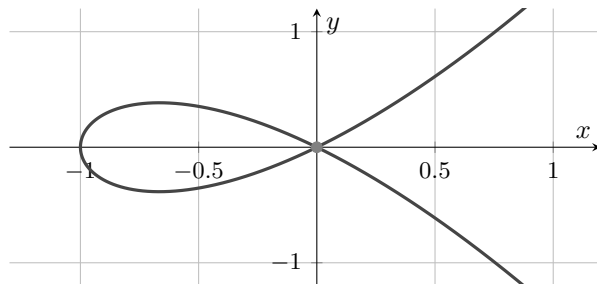
Question 7

Problem

The curve $x = t^2 - 1$, $y = t^3 - t$ crosses itself. Find the two parameter values at the crossing and describe the two tangent directions there.



Question 7 – Solution



Solution

1. Factor the coordinate equations:

$$x = t^2 - 1 = (t - 1)(t + 1), \quad y = t^3 - t = t(t - 1)(t + 1).$$

2. Both coordinates equal zero when $t = 1$ and when $t = -1$:

$$\mathbf{r}(1) = (0, 0), \quad \mathbf{r}(-1) = (0, 0).$$

Because two distinct parameter values produce the same point, the self-intersection is $(0, 0)$.

3. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1.$$

Since $dx/dt \neq 0$ at $t = \pm 1$, the tangent slope is

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}.$$

4. At $t = 1$,

$$\frac{dy}{dx} = \frac{3(1)^2 - 1}{2(1)} = 1,$$

so one tangent line is $y = x$.

5. At $t = -1$,

$$\frac{dy}{dx} = \frac{3(-1)^2 - 1}{2(-1)} = -1,$$

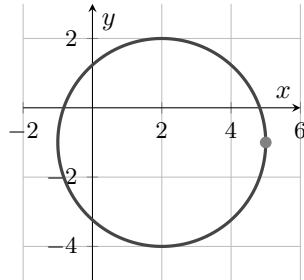
so the other tangent line is $y = -x$.

6. Therefore, the curve crosses itself at the origin with tangent directions of slopes

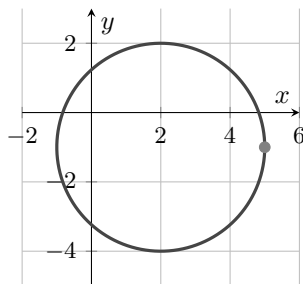
$$\boxed{1 \text{ and } -1}.$$

Question 8**Problem**

A moving point satisfies $x = 2 + 3 \cos t$, $y = -1 + 3 \sin t$. Without eliminating t , determine its center, radius, starting point, and initial direction.



Question 8 – Solution



Solution

1. Compare the equations

$$x = 2 + 3 \cos t, \quad y = -1 + 3 \sin t$$

with the standard circle parametrization

$$x = h + R \cos t, \quad y = k + R \sin t.$$

2. It follows immediately that

$$(h, k) = (2, -1) \quad \text{and} \quad R = 3.$$

Thus the curve is a circle centered at $(2, -1)$ with radius 3 .

3. Evaluate the coordinates at $t = 0$:

$$x(0) = 2 + 3 \cos 0 = 5, \quad y(0) = -1 + 3 \sin 0 = -1.$$

Therefore, the starting point is $(5, -1)$.

4. Differentiate to find the velocity:

$$\mathbf{r}'(t) = (-3 \sin t, 3 \cos t).$$

At $t = 0$,

$$\mathbf{r}'(0) = (0, 3),$$

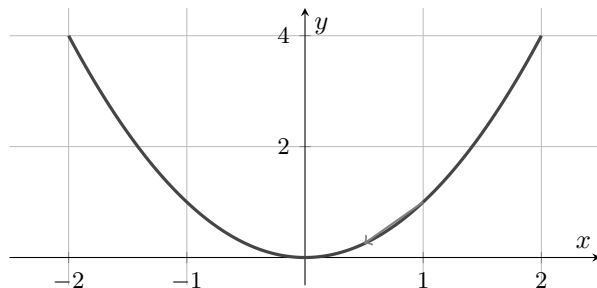
which points straight upward from the rightmost point of the circle.

5. Upward motion from the rightmost point is counterclockwise, so the initial direction is **counterclockwise**.

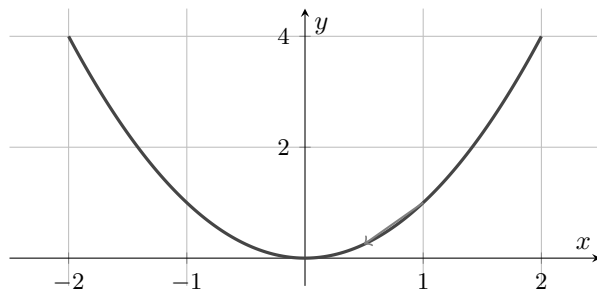
Question 9

Problem

Can the parabola $y = x^2$ be traced from right to left while the parameter increases? Give a parametrization on $-2 \leq t \leq 2$ that does so.



Question 9 – Solution



Solution

1. To trace the parabola from right to left, choose the x -coordinate so that it decreases as t increases. A simple choice is

$$x = -t.$$

2. The curve must satisfy $y = x^2$. Substituting $x = -t$ gives

$$y = (-t)^2 = t^2.$$

Hence one suitable parametrization is

$$\boxed{x = -t, \quad y = t^2, \quad -2 \leq t \leq 2}.$$

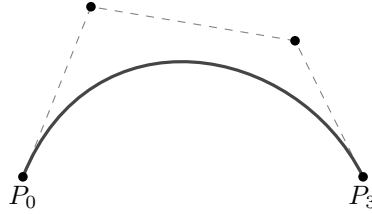
3. Verify that the curve is correct by eliminating the parameter:

$$t = -x \implies y = t^2 = (-x)^2 = x^2.$$

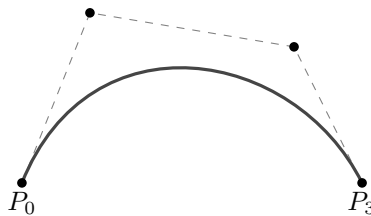
4. Verify the direction. At $t = -2$, the point is $(2, 4)$; at $t = 0$, it is $(0, 0)$; and at $t = 2$, it is $(-2, 4)$.
5. Also, $dx/dt = -1 < 0$, so x decreases continuously. Therefore, the parabola is traced from right to left as required.

Question 10**Problem**

A cubic Bézier path is $\mathbf{r}(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3$. Explain, without expanding, why it begins at P_0 and ends at P_3 .



Question 10 – Solution



Solution

1. The cubic Bézier path is

$$\mathbf{r}(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3.$$

2. Evaluate the path at $t = 0$:

$$\begin{aligned}\mathbf{r}(0) &= (1-0)^3 P_0 + 3(1-0)^2 (0) P_1 \\ &\quad + 3(1-0)(0)^2 P_2 + (0)^3 P_3 \\ &= P_0.\end{aligned}$$

Every term containing a factor of t vanishes, leaving only P_0 .

3. Evaluate the path at $t = 1$:

$$\begin{aligned}\mathbf{r}(1) &= (1-1)^3 P_0 + 3(1-1)^2 (1) P_1 \\ &\quad + 3(1-1)(1)^2 P_2 + (1)^3 P_3 \\ &= P_3.\end{aligned}$$

Every term containing a factor of $1-t$ vanishes, leaving only P_3 .

4. Therefore,

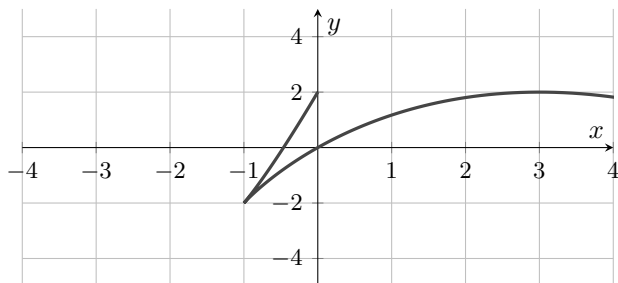
$$\boxed{\mathbf{r}(0) = P_0 \quad \text{and} \quad \mathbf{r}(1) = P_3}.$$

The path begins at the first control point and ends at the last control point.

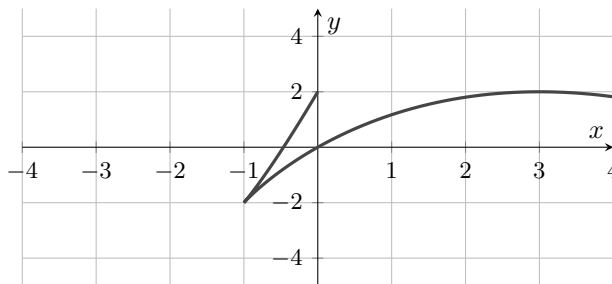
Question 1

Problem

For $x = t^2 - 2t$, $y = t^3 - 3t$, find every horizontal and vertical tangent.



Question 1 – Solution



Solution

1. Differentiate each coordinate:

$$\frac{dx}{dt} = 2t - 2 = 2(t - 1), \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1).$$

2. A horizontal tangent requires $dy/dt = 0$ while $dx/dt \neq 0$. The candidates are

$$3(t - 1)(t + 1) = 0 \implies t = -1, 1.$$

At $t = -1$, $dx/dt = -4 \neq 0$, so this value gives a horizontal tangent. The point is

$$x(-1) = 3, \quad y(-1) = 2.$$

Thus the horizontal tangent line is

$$\boxed{y = 2}.$$

3. At $t = 1$, both derivatives are zero, so the usual horizontal- or vertical-tangent tests are inconclusive. For $t \neq 1$, cancel the common factor:

$$\frac{dy}{dx} = \frac{3(t - 1)(t + 1)}{2(t - 1)} = \frac{3}{2}(t + 1).$$

Therefore,

$$\lim_{t \rightarrow 1} \frac{dy}{dx} = 3.$$

The point at $t = 1$ is $(-1, -2)$, and its tangent line is $y + 2 = 3(x + 1)$; it is neither horizontal nor vertical.

4. A vertical tangent would require $dx/dt = 0$ with $dy/dt \neq 0$. The only zero of dx/dt is $t = 1$, where dy/dt is also zero and the limiting slope is finite. Hence there are no vertical tangents.

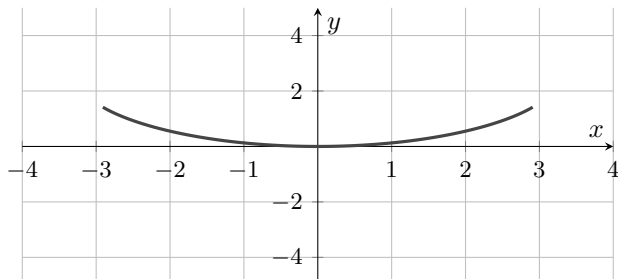
5. The complete answer is

$$\boxed{\text{horizontal tangent } y = 2 \text{ at } (3, 2); \quad \text{no vertical tangents.}}$$

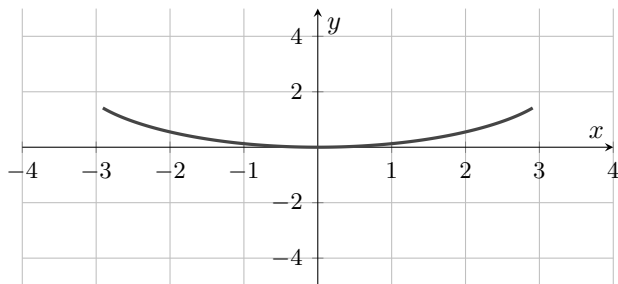
Question 2

Problem

The path $x = t + \sin t$, $y = 1 - \cos t$ models a rolling marker. Find its tangent slope at $t = \pi/2$ and interpret its sign.



Question 2 – Solution



Solution

1. Differentiate the coordinate functions:

$$\frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = \sin t.$$

2. At $t = \pi/2$,

$$\frac{dx}{dt} = 1 + \cos \frac{\pi}{2} = 1, \quad \frac{dy}{dt} = \sin \frac{\pi}{2} = 1.$$

Since $dx/dt \neq 0$, the tangent slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{1} = \boxed{1}.$$

3. The point on the path is

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2} + 1, 1\right).$$

Therefore, the tangent line is

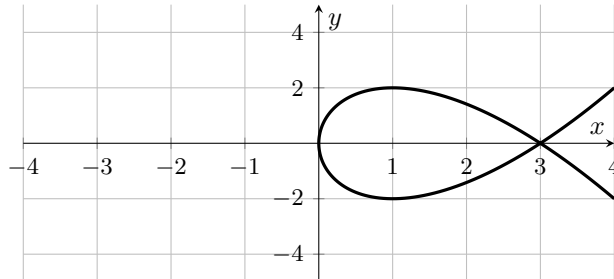
$$y - 1 = x - \left(\frac{\pi}{2} + 1\right).$$

4. Both velocity components are positive at this instant. Thus the marker is moving to the right and upward, which is consistent with the positive slope.

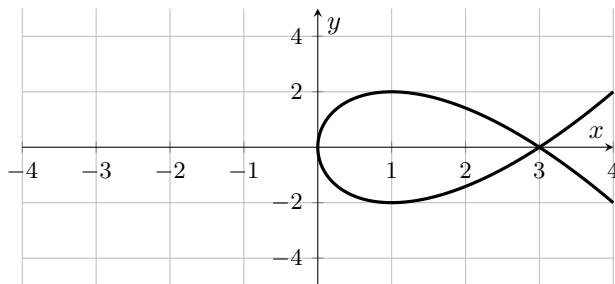
Question 3

Problem

At which points does $x = t^2$, $y = t^3 - 3t$ have tangent slope 0? Is the particle momentarily stopped there?



Question 3 – Solution



Solution

1. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1).$$

2. A tangent has slope 0 when $dy/dt = 0$ and $dx/dt \neq 0$. Solve

$$3(t-1)(t+1) = 0,$$

which gives $t = -1$ and $t = 1$.

3. Check the horizontal velocity:

$$x'(-1) = -2 \neq 0, \quad x'(1) = 2 \neq 0.$$

Thus both parameter values give genuine horizontal tangents.

4. Find the corresponding points:

$$\mathbf{r}(-1) = (1, 2), \quad \mathbf{r}(1) = (1, -2).$$

The horizontal tangent lines are therefore $y = 2$ and $y = -2$.

5. A particle is momentarily stopped only when both velocity components are zero. At $t = \pm 1$, $dy/dt = 0$ but $dx/dt = \pm 2$ is nonzero, so the particle is moving horizontally rather than stopping.

6. Hence the requested points are

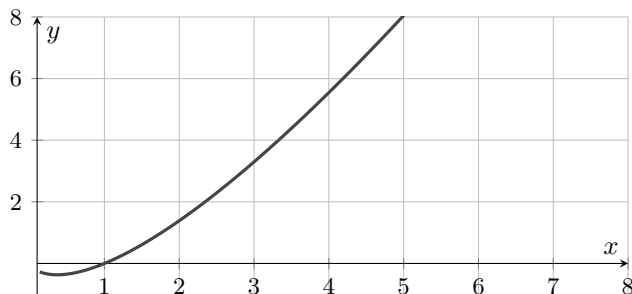
$$\boxed{(1, 2) \text{ and } (1, -2)},$$

and the particle is not momentarily stopped at either point.

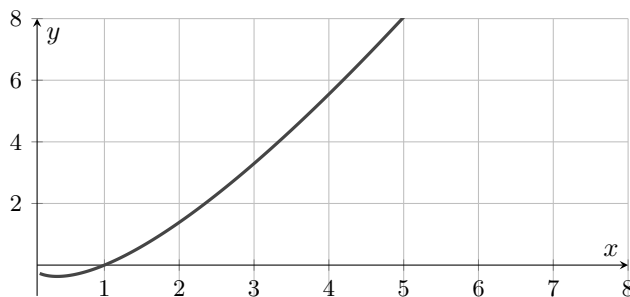
Question 4

Problem

For $x = e^t$, $y = te^t$, find the tangent line at the point whose x -coordinate is 1.



Question 4 – Solution



Solution

1. Locate the required parameter value by solving

$$x = e^t = 1.$$

Since the exponential function equals 1 only at $t = 0$, the required parameter value is $t = 0$.

2. Evaluate the y -coordinate:

$$y(0) = 0 \cdot e^0 = 0.$$

Thus the point of tangency is $(1, 0)$.

3. Differentiate both coordinates. The product rule is needed for y :

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t + te^t = e^t(1 + t).$$

4. Because $dx/dt = e^t \neq 0$, divide the derivatives:

$$\frac{dy}{dx} = \frac{e^t(1 + t)}{e^t} = 1 + t.$$

At $t = 0$, the slope is $m = 1$.

5. Use point-slope form through $(1, 0)$:

$$y - 0 = 1(x - 1).$$

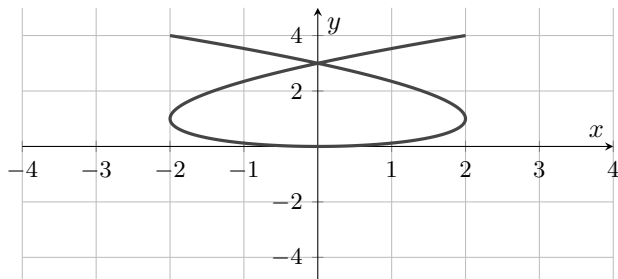
Therefore, the tangent line is

$$\boxed{y = x - 1}.$$

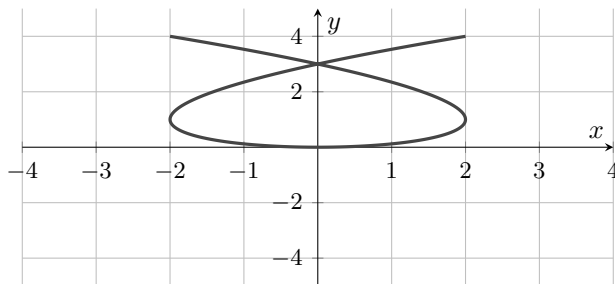
Question 5

Problem

A student claims that the two tangents to $x = t^3 - 3t$, $y = t^2$ at its self-intersection are perpendicular. Locate the crossing and test the claim.



Question 5 – Solution



Solution

- At a self-intersection, two distinct parameter values give the same y -coordinate. Since $y = t^2$, equal y -coordinates occur for parameters t and $-t$.
- Require the corresponding x -coordinates to be equal:

$$t^3 - 3t = (-t)^3 - 3(-t) = -t^3 + 3t.$$

Simplifying gives

$$2t^3 - 6t = 2t(t^2 - 3) = 0.$$

The value $t = 0$ does not give two distinct parameters. The nontrivial pair is

$$t = \pm\sqrt{3}.$$

- At either parameter value,

$$x = t(t^2 - 3) = 0, \quad y = t^2 = 3.$$

Hence the self-intersection is $\boxed{(0, 3)}$.

- Differentiate:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t.$$

At $t = \pm\sqrt{3}$, $dx/dt = 6 \neq 0$, so

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 3} = \frac{t}{3}.$$

The two slopes are

$$m_1 = \frac{1}{\sqrt{3}}, \quad m_2 = -\frac{1}{\sqrt{3}}.$$

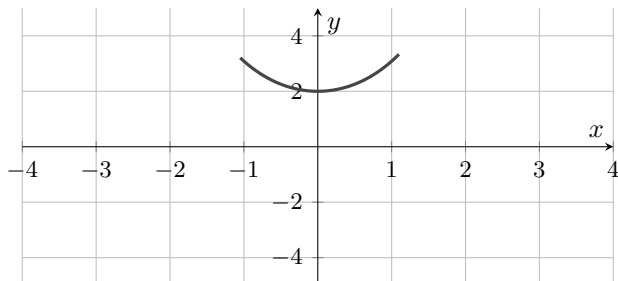
- Two nonvertical lines are perpendicular only if the product of their slopes is -1 . Here

$$m_1 m_2 = -\frac{1}{3} \neq -1.$$

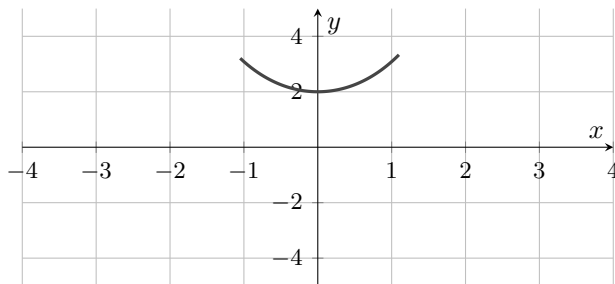
Therefore, the two tangents are $\boxed{\text{not perpendicular}}$.

Question 6**Problem**

For $x = \ln t$, $y = t + t^{-1}$, find d^2y/dx^2 at $t = 1$ and classify the local shape.



Question 6 – Solution



Solution

1. Because $x = \ln t$, the parameter domain is $t > 0$. Differentiate the coordinates:

$$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2}.$$

2. Compute the first derivative with respect to x :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - t^{-2}}{t^{-1}} = t - \frac{1}{t}.$$

3. For a parametric curve, differentiate dy/dx with respect to t and divide once more by dx/dt :

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt}.$$

Since

$$\frac{d}{dt} \left(t - \frac{1}{t} \right) = 1 + \frac{1}{t^2},$$

we obtain

$$\frac{d^2y}{dx^2} = \frac{1 + t^{-2}}{t^{-1}} = t + \frac{1}{t}.$$

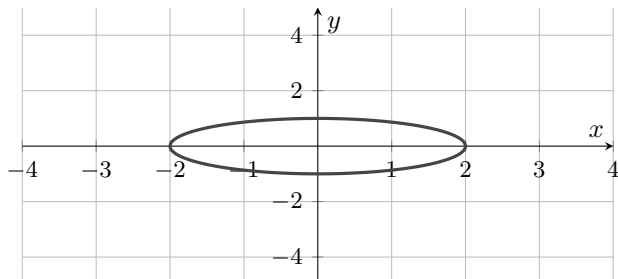
4. Evaluate at $t = 1$:

$$\left. \frac{d^2y}{dx^2} \right|_{t=1} = 1 + 1 = \boxed{2}.$$

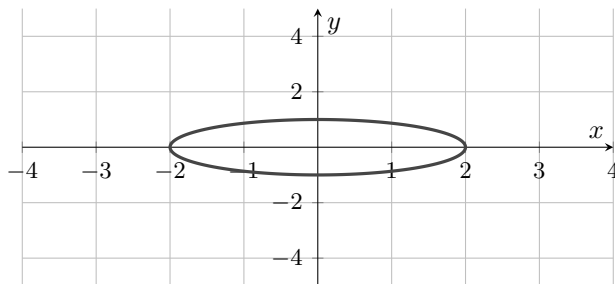
5. Because the second derivative is positive, the curve is concave up at this point. Also, $(x(1), y(1)) = (0, 2)$ and $dy/dx = 0$, so $(0, 2)$ is a local minimum of the curve when viewed as y versus x .

Question 7**Problem**

Find the angle at which the parametric curve $x = 2 \cos t$, $y = \sin t$ crosses the line $y = x/2$ in the first quadrant.



Question 7 – Solution



Solution

1. At an intersection with $y = x/2$, substitute the parametric equations:

$$\sin t = \frac{2 \cos t}{2} = \cos t.$$

In the first quadrant, this gives $t = \pi/4$.

2. The intersection point is

$$\left(2 \cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right) = \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right).$$

3. Differentiate the curve:

$$\frac{dx}{dt} = -2 \sin t, \quad \frac{dy}{dt} = \cos t.$$

Thus its tangent slope at $t = \pi/4$ is

$$m_c = \frac{\cos t}{-2 \sin t} = -\frac{1}{2}.$$

The line $y = x/2$ has slope $m_\ell = 1/2$.

4. If ϕ is the acute angle between two lines, then

$$\tan \phi = \left| \frac{m_\ell - m_c}{1 + m_\ell m_c} \right|.$$

Therefore,

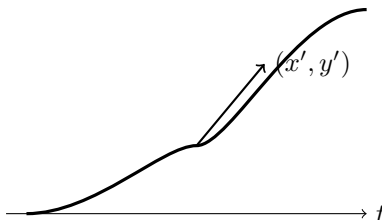
$$\tan \phi = \left| \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + (\frac{1}{2})(-\frac{1}{2})} \right| = \frac{1}{3/4} = \frac{4}{3}.$$

5. Hence the crossing angle is

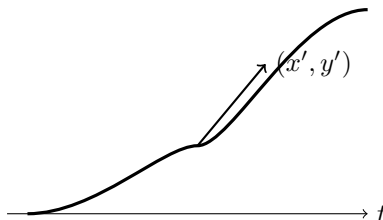
$$\phi = \arctan \left(\frac{4}{3} \right) \approx 53.13^\circ.$$

Question 8**Problem**

A particle has velocity $(x', y') = (t - 2, t^2 - 4)$. Does its path have a vertical tangent? Analyze the tangent and direction of motion at any stationary time.



Question 8 – Solution

**Solution**

1. A regular vertical tangent would require

$$x'(t) = t - 2 = 0 \quad \text{and} \quad y'(t) = t^2 - 4 \neq 0.$$

The first condition gives the only candidate, $t = 2$.

2. At $t = 2$,

$$(x'(2), y'(2)) = (0, 0).$$

Thus the particle is momentarily stopped, and the ordinary vertical-tangent test does not apply.

3. For $t \neq 2$, factor y' and simplify the slope:

$$\frac{dy}{dx} = \frac{t^2 - 4}{t - 2} = \frac{(t - 2)(t + 2)}{t - 2} = t + 2.$$

Consequently,

$$\lim_{t \rightarrow 2} \frac{dy}{dx} = 4,$$

which is finite rather than infinite. The path has a tangent of slope 4 at the stationary point, not a vertical tangent.

4. The direction also changes across $t = 2$. Just before 2, both x' and y' are negative, so the particle moves down and left. Just after 2, both are positive, so it moves up and right. At $t = 2$ itself, the velocity is zero and therefore has no instantaneous direction.

5. Hence

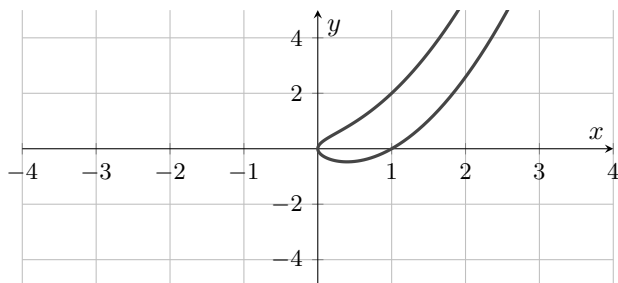
the path has no vertical tangent.

The only candidate is a stationary point where the particle reverses direction along a tangent of slope 4.

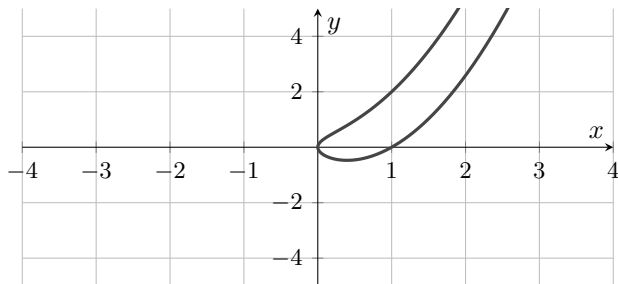
Question 9

Problem

The curve $x = t^2$, $y = t^4 - t$ passes through $(1, 0)$. Find all tangent lines there.



Question 9 – Solution



Solution

1. First find every parameter value that could produce the point $(1, 0)$. From $x = t^2 = 1$,

$$t = 1 \quad \text{or} \quad t = -1.$$

2. Check the y -coordinate for both values:

$$y(1) = 1^4 - 1 = 0, \quad y(-1) = (-1)^4 - (-1) = 2.$$

Only $t = 1$ produces $(1, 0)$, so there can be only one tangent line there.

3. Differentiate:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4t^3 - 1.$$

At $t = 1$, $dx/dt = 2 \neq 0$, and the slope is

$$\frac{dy}{dx} = \frac{4t^3 - 1}{2t} = \frac{4(1)^3 - 1}{2(1)} = \frac{3}{2}.$$

4. Apply point-slope form through $(1, 0)$:

$$y - 0 = \frac{3}{2}(x - 1).$$

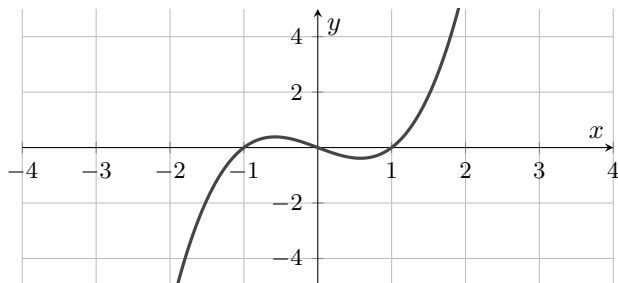
Therefore, the only tangent line at the specified point is

$$\boxed{y = \frac{3}{2}(x - 1)}.$$

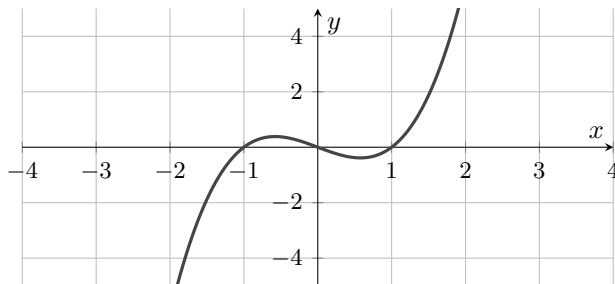
Question 10

Problem

Design a so that $x = t$, $y = t^3 - at$ meets the x -axis at $t = 1$ with slope 2. Is it possible?



Question 10 – Solution



Solution

1. To meet the x -axis at $t = 1$, the y -coordinate must be zero:

$$y(1) = 1^3 - a(1) = 1 - a = 0.$$

Therefore, the position condition requires

$$a = 1.$$

2. Now impose the slope condition. Differentiate:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3t^2 - a.$$

Since $dx/dt = 1$, the tangent slope is

$$\frac{dy}{dx} = 3t^2 - a.$$

3. At $t = 1$, requiring slope 2 gives

$$3 - a = 2 \implies a = 1.$$

Thus the position and slope conditions agree.

4. Verify the result directly. With $a = 1$,

$$y(1) = 1 - 1 = 0, \quad \left. \frac{dy}{dx} \right|_{t=1} = 3 - 1 = 2.$$

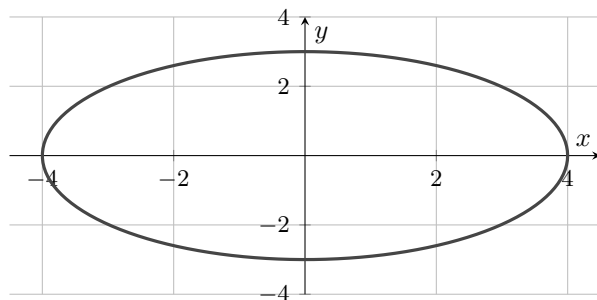
The point is $(x(1), y(1)) = (1, 0)$, and its tangent line is $y = 2(x - 1)$.

5. Hence the design is possible, and the unique value is

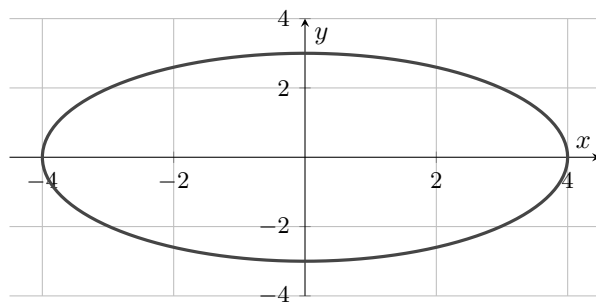
$$\boxed{a = 1}.$$

Question 1**Problem**

Use a parametric integral to find the area of the ellipse $x = 4 \cos t$, $y = 3 \sin t$.



Question 1 – Solution



Solution

1. The interval $0 \leq t \leq \pi/2$ traces the ellipse's first-quadrant arc from $(4, 0)$ to $(0, 3)$. By symmetry, the total area is four times the area in that quadrant.

2. Differentiate the horizontal coordinate:

$$\frac{dx}{dt} = -4 \sin t.$$

Because x decreases on this interval, the first-quadrant area is

$$A_1 = - \int_0^{\pi/2} y(t)x'(t) dt.$$

3. Substitute $y = 3 \sin t$ and $x' = -4 \sin t$:

$$A_1 = - \int_0^{\pi/2} (3 \sin t)(-4 \sin t) dt = 12 \int_0^{\pi/2} \sin^2 t dt.$$

4. Using $\sin^2 t = (1 - \cos 2t)/2$,

$$\int_0^{\pi/2} \sin^2 t dt = \frac{\pi}{4}.$$

Hence $A_1 = 3\pi$.

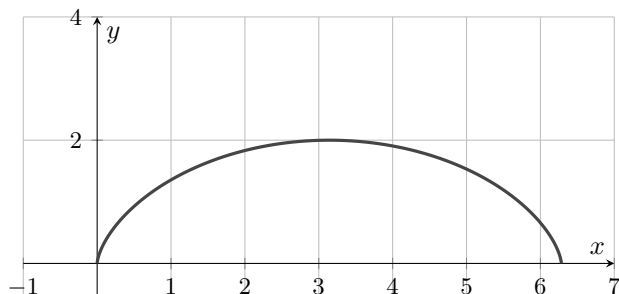
5. Multiply by four quadrants:

$$A = 4A_1 = \boxed{12\pi}.$$

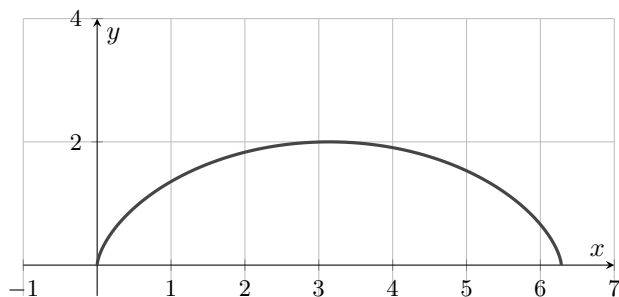
Question 2

Problem

Find the area under one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Question 2 – Solution



Solution

1. The given interval traces one arch from $(0, 0)$ to $(2\pi, 0)$. Since

$$x'(t) = 1 - \cos t \geq 0,$$

the curve moves from left to right, so the area under it is $\int y \, dx$ with the stated bounds.

2. Substitute $y(t) = 1 - \cos t$ and $dx = x'(t)dt$:

$$A = \int_0^{2\pi} y(t)x'(t) \, dt = \int_0^{2\pi} (1 - \cos t)^2 \, dt.$$

3. Expand the integrand:

$$(1 - \cos t)^2 = 1 - 2 \cos t + \cos^2 t.$$

Therefore,

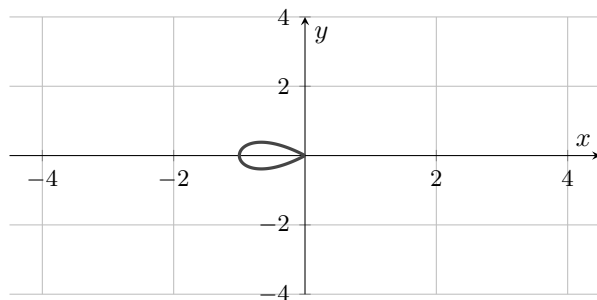
$$A = \int_0^{2\pi} 1 \, dt - 2 \int_0^{2\pi} \cos t \, dt + \int_0^{2\pi} \cos^2 t \, dt.$$

4. These three integrals equal 2π , 0, and π , respectively. Thus

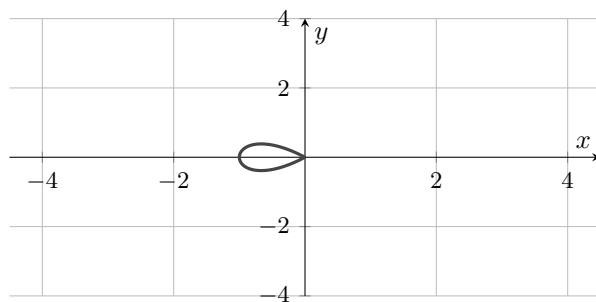
$$\boxed{A = 3\pi}.$$

Question 3**Problem**

A loop is traced by $x = t^2 - 1$, $y = t^3 - t$. Find the exact area enclosed by the loop.



Question 3 – Solution

**Solution**

1. Locate the parameter values at which the loop closes. Since

$$x = t^2 - 1, \quad y = t(t^2 - 1),$$

both $t = -1$ and $t = 1$ produce the point $(0, 0)$. The interval $-1 \leq t \leq 1$ traces the loop once.

2. Differentiate x :

$$x'(t) = 2t.$$

The signed area integral is

$$\int_{-1}^1 y(t)x'(t) dt = \int_{-1}^1 (t^3 - t)(2t) dt.$$

3. Simplify and integrate:

$$\begin{aligned} \int_{-1}^1 (2t^4 - 2t^2) dt &= \left[\frac{2}{5}t^5 - \frac{2}{3}t^3 \right]_{-1}^1 \\ &= \frac{4}{5} - \frac{4}{3} = -\frac{8}{15}. \end{aligned}$$

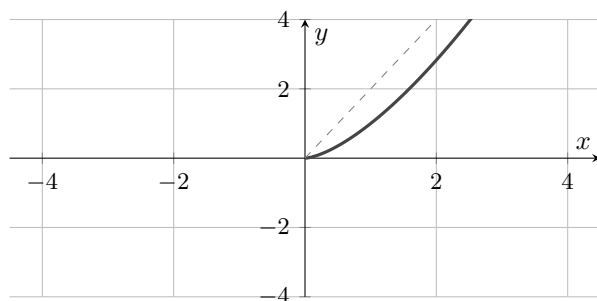
4. The negative sign records the orientation of traversal; geometric area is nonnegative. Hence

$$A = \left| -\frac{8}{15} \right| = \frac{8}{15}.$$

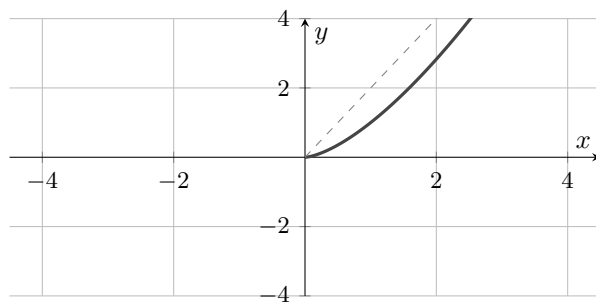
Question 4

Problem

The curve $x = t^2$, $y = t^3$ joins $(0,0)$ to $(4,8)$. Find the area between it and the chord joining those endpoints.



Question 4 – Solution



Solution

1. The endpoints correspond to $t = 0$ and $t = 2$. The chord through $(0, 0)$ and $(4, 8)$ has slope

$$m = \frac{8 - 0}{4 - 0} = 2,$$

so its equation is $y = 2x$.

2. Along the parametric curve, $x = t^2$ and $y = t^3$. The chord's height at the same x -coordinate is

$$y_{\text{chord}} = 2x = 2t^2.$$

For $0 \leq t \leq 2$, $2t^2 - t^3 = t^2(2 - t) \geq 0$, so the chord lies above the curve.

3. Since $dx/dt = 2t$, the area between them is

$$A = \int_0^2 (y_{\text{chord}} - y_{\text{curve}}) dx = \int_0^2 (2t^2 - t^3)(2t) dt.$$

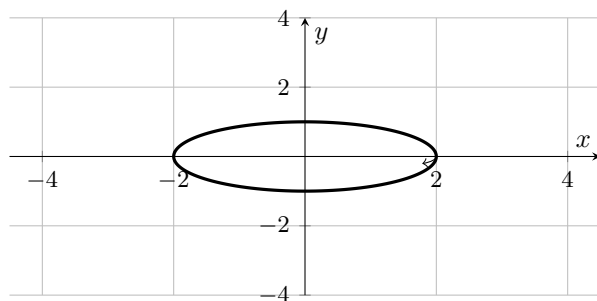
4. Evaluate:

$$\begin{aligned} A &= \int_0^2 (4t^3 - 2t^4) dt \\ &= \left[t^4 - \frac{2}{5}t^5 \right]_0^2 = 16 - \frac{64}{5} = \boxed{\frac{16}{5}}. \end{aligned}$$

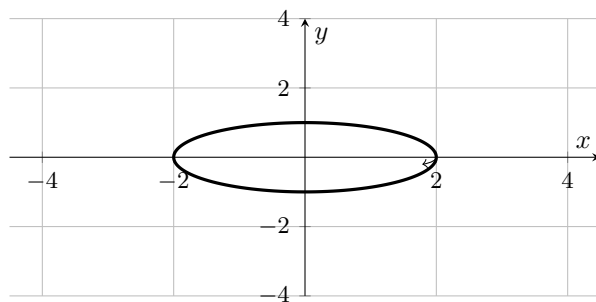
Question 5

Problem

A simple, piecewise smooth closed curve is traversed once clockwise. Explain why $\int y \, dx$ may be positive even though area cannot be negative.



Question 5 – Solution



Solution

1. A line integral around a closed curve is signed: reversing the direction of traversal reverses the sign because dx changes sign.
2. For a positively oriented, counterclockwise curve C , Green's theorem gives

$$A = - \oint_C y \, dx.$$

Thus $\oint_C y \, dx = -A$ for counterclockwise traversal.

3. If the same boundary is traversed clockwise, the line integral changes sign:

$$\oint_{-C} y \, dx = - \oint_C y \, dx = A.$$

Therefore, $\int y \, dx$ can be positive for clockwise motion; the sign describes orientation, not a negative or positive physical area by itself.

4. A direction-independent formula for geometric area is

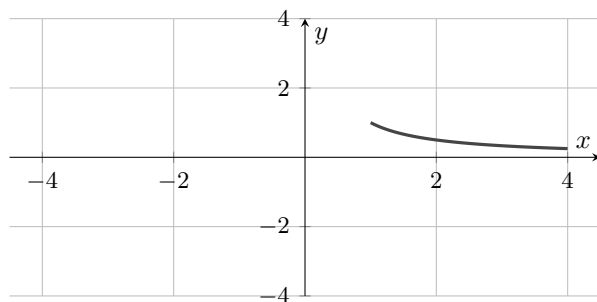
$$A = \left| \oint_C y \, dx \right|.$$

Equivalently, use $-\oint_C y \, dx$ with counterclockwise orientation or reverse the bounds when necessary.

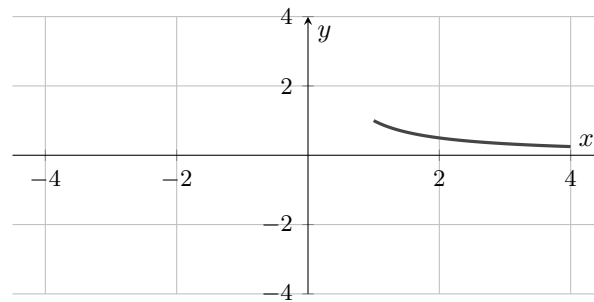
Question 6

Problem

Find the area swept under $x = e^t$, $y = e^{-t}$ from $t = 0$ to $t = \ln 4$.



Question 6 – Solution

**Solution**

1. On $0 \leq t \leq \ln 4$, $x = e^t$ increases from 1 to 4 because

$$x'(t) = e^t > 0.$$

Therefore, the requested area under the curve is directly $\int y \, dx$.

2. Substitute the parametric expressions:

$$A = \int_0^{\ln 4} y(t)x'(t) \, dt = \int_0^{\ln 4} e^{-t} e^t \, dt.$$

3. The exponential factors cancel:

$$e^{-t} e^t = 1.$$

Hence

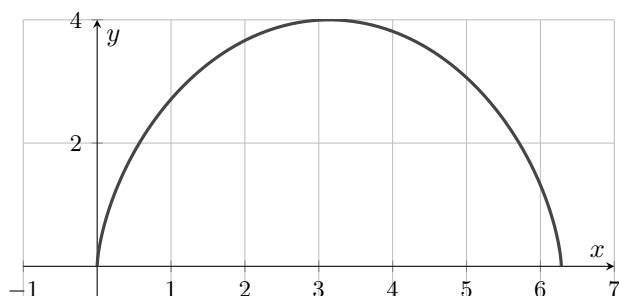
$$A = \int_0^{\ln 4} 1 \, dt = [t]_0^{\ln 4} = \boxed{\ln 4}.$$

4. As a check, eliminating t gives $y = 1/x$, and $\int_1^4 (1/x) \, dx = \ln 4$, the same result.

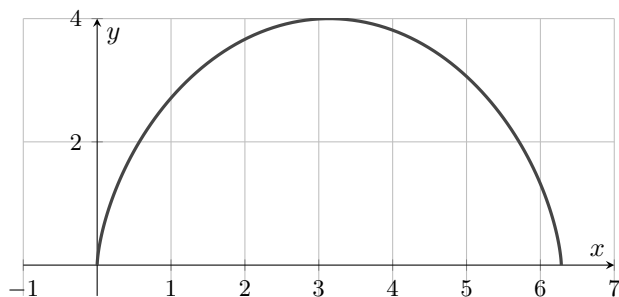
Question 7

Problem

The parametric curve $x = t - \sin t$, $y = 2 - 2 \cos t$ is a stretched cycloid. Predict, then compute, how its arch area compares with the standard cycloid.



Question 7 – Solution



Solution

1. Compared with the standard cycloid $x = t - \sin t$, $y = 1 - \cos t$, the x -coordinates are unchanged while every y -coordinate is multiplied by 2. A vertical stretch by a factor of 2 doubles area, so the predicted arch area is $2(3\pi) = 6\pi$.
2. Verify this prediction directly on $0 \leq t \leq 2\pi$. Differentiate x :

$$x'(t) = 1 - \cos t.$$

Since $x'(t) \geq 0$, the arch is traced from left to right.

3. Substitute $y = 2(1 - \cos t)$:

$$A = \int_0^{2\pi} y(t)x'(t) dt = 2 \int_0^{2\pi} (1 - \cos t)^2 dt.$$

4. From

$$\int_0^{2\pi} (1 - \cos t)^2 dt = 2\pi + \pi = 3\pi,$$

it follows that

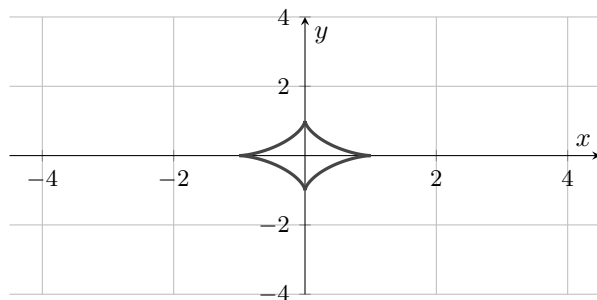
$$\boxed{A = 2(3\pi) = 6\pi}.$$

The computation agrees with the scaling prediction.

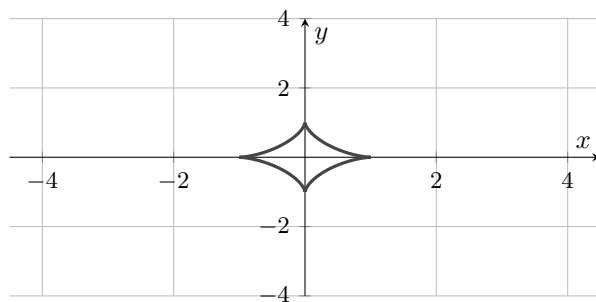
Question 8

Problem

Find the area enclosed by $x = \cos^3 t$, $y = \sin^3 t$ without first eliminating the parameter.



Question 8 – Solution

**Solution**

1. The interval $0 \leq t \leq \pi/2$ traces the first-quadrant arc from $(1, 0)$ to $(0, 1)$. By symmetry, the enclosed area is four times the first-quadrant area.
2. It is convenient to integrate with respect to y :

$$A_1 = \int x \, dy = \int_0^{\pi/2} x(t) y'(t) \, dt.$$

Since $y = \sin^3 t$,

$$y'(t) = 3 \sin^2 t \cos t.$$

3. Substitute $x = \cos^3 t$:

$$A_1 = 3 \int_0^{\pi/2} \sin^2 t \cos^4 t \, dt.$$

Using the even-power integral (or power-reduction identities),

$$\int_0^{\pi/2} \sin^2 t \cos^4 t \, dt = \frac{\pi}{32}.$$

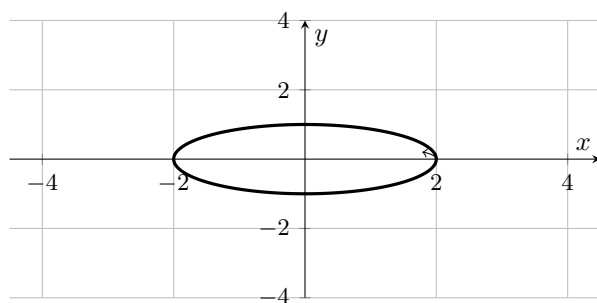
Thus $A_1 = 3\pi/32$.

4. Multiply by four quadrants:

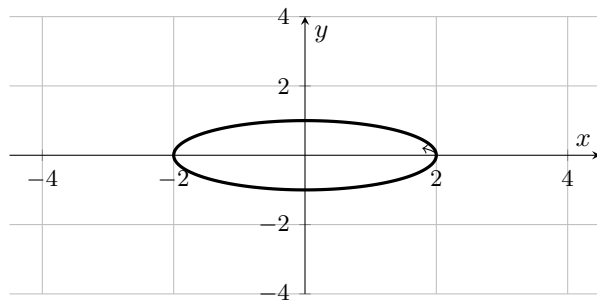
$$A = 4A_1 = 4 \left(\frac{3\pi}{32} \right) = \boxed{\frac{3\pi}{8}}.$$

Question 9**Problem**

A designer reverses a parametrization on $0 \leq t \leq 1$ by replacing t with $1 - t$. What happens to the signed area integral $\int yx' dt$?



Question 9 – Solution

**Solution**

1. Let the original parametrization be $(x(t), y(t))$ on $0 \leq t \leq 1$, with signed integral

$$I = \int_0^1 y(t)x'(t) dt.$$

2. Under the reversed parametrization, set $u = 1 - t$. The new coordinates are

$$\tilde{x}(t) = x(1 - t), \quad \tilde{y}(t) = y(1 - t).$$

By the chain rule,

$$\tilde{x}'(t) = -x'(1 - t).$$

3. The reversed signed integral is

$$\begin{aligned} \tilde{I} &= \int_0^1 \tilde{y}(t)\tilde{x}'(t) dt \\ &= - \int_0^1 y(1 - t)x'(1 - t) dt. \end{aligned}$$

Substituting $u = 1 - t$ shows that $\tilde{I} = -I$.

4. Therefore, reversing the parametrization changes the sign of the signed area integral but not its magnitude:

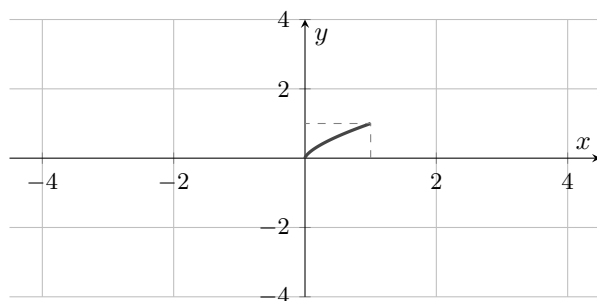
$$\boxed{\tilde{I} = -I, \quad |\tilde{I}| = |I|}.$$

The geometric area is unchanged.

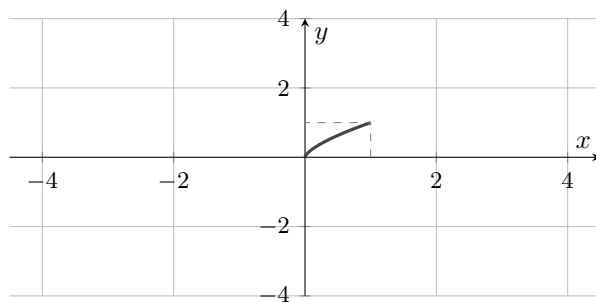
Question 10

Problem

For $x = t^3$, $y = t^2$, $0 \leq t \leq 1$, compare the area under the curve with the area to its left inside the unit square.



Question 10 – Solution

**Solution**

1. As t increases from 0 to 1, the curve runs from $(0, 0)$ to $(1, 1)$ with

$$x'(t) = 3t^2 \geq 0.$$

The area below the curve is therefore

$$A_{\text{below}} = \int_0^1 y(t)x'(t) dt.$$

2. Substitute $y = t^2$ and $x' = 3t^2$:

$$A_{\text{below}} = \int_0^1 3t^4 dt = \left[\frac{3}{5}t^5 \right]_0^1 = \boxed{\frac{3}{5}}.$$

3. The curve divides the unit square into the area below it and the area to its left (equivalently, above it). Since the square has area 1,

$$A_{\text{left}} = 1 - A_{\text{below}} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}.$$

4. This can also be checked directly with $dy = 2t dt$:

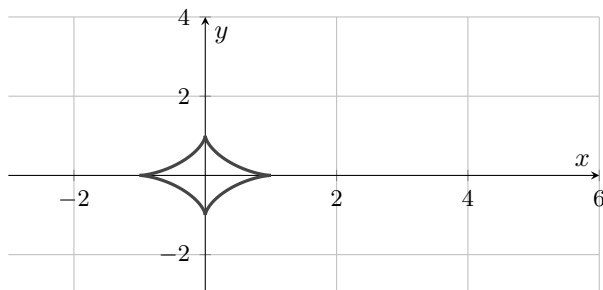
$$A_{\text{left}} = \int_0^1 x dy = \int_0^1 t^3(2t) dt = \frac{2}{5}.$$

Thus the area under the curve is larger by $1/5$.

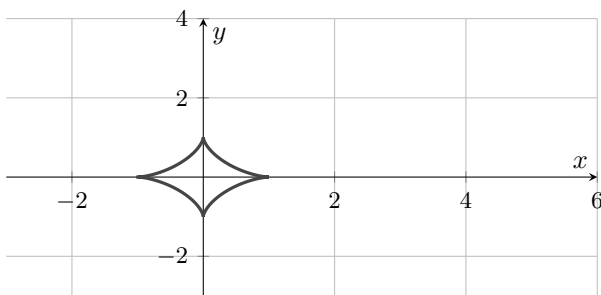
Question 1

Problem

Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$.



Question 1 – Solution



Solution

1. The astroid is symmetric about both coordinate axes, so compute the length in the first quadrant, $0 \leq t \leq \pi/2$, and multiply by four.

2. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

3. Simplify the speed:

$$\begin{aligned} \sqrt{x'^2 + y'^2} &= 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} \\ &= 3\sqrt{\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} \\ &= 3|\sin t \cos t|. \end{aligned}$$

On $0 \leq t \leq \pi/2$, both factors are nonnegative, so the speed is $3 \sin t \cos t$.

4. The first-quadrant length is

$$L_1 = 3 \int_0^{\pi/2} \sin t \cos t \, dt = \frac{3}{2} [\sin^2 t]_0^{\pi/2} = \frac{3}{2}.$$

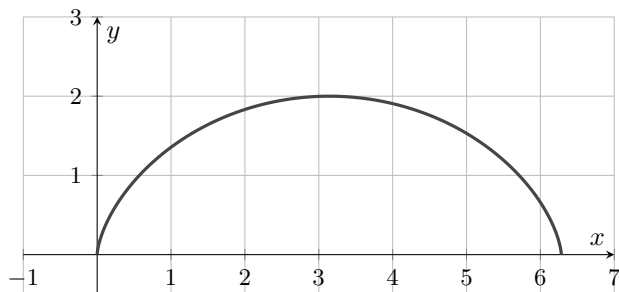
5. Therefore, the total length is

$$L = 4L_1 = 4 \left(\frac{3}{2} \right) = 6.$$

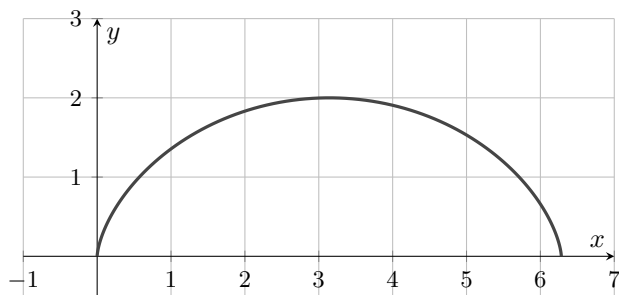
Question 2

Problem

Find the length of one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Question 2 – Solution



Solution

1. Differentiate the parametric equations:

$$x'(t) = 1 - \cos t, \quad y'(t) = \sin t.$$

2. Compute and simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= (1 - \cos t)^2 + \sin^2 t \\ &= 1 - 2 \cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2 \cos t = 4 \sin^2 \left(\frac{t}{2} \right). \end{aligned}$$

Hence the speed is

$$2 \left| \sin \left(\frac{t}{2} \right) \right|.$$

3. On $0 \leq t \leq 2\pi$, we have $0 \leq t/2 \leq \pi$, so $\sin(t/2) \geq 0$. Thus

$$L = 2 \int_0^{2\pi} \sin \left(\frac{t}{2} \right) dt.$$

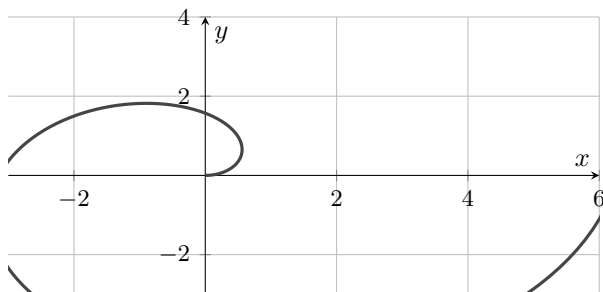
4. Evaluate the integral:

$$L = 2 \left[-2 \cos \left(\frac{t}{2} \right) \right]_0^{2\pi} = -4(\cos \pi - \cos 0) = \boxed{8}.$$

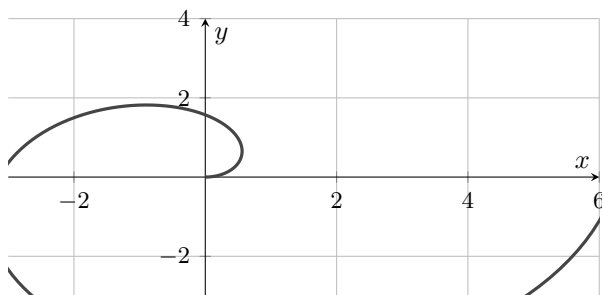
Question 3

Problem

A spiral-like path is $x = t \cos t$, $y = t \sin t$, $0 \leq t \leq 2\pi$. Set up its exact length and simplify the speed.



Question 3 – Solution



Solution

1. Apply the product rule to each coordinate:

$$x'(t) = \cos t - t \sin t, \quad y'(t) = \sin t + t \cos t.$$

2. Square and add:

$$\begin{aligned} x'^2 + y'^2 &= (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 \\ &= \cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) \\ &\quad - 2t \sin t \cos t + 2t \sin t \cos t \\ &= 1 + t^2. \end{aligned}$$

The cross terms cancel exactly.

3. Therefore, the speed is $\sqrt{1 + t^2}$, and the exact length is

$$L = \int_0^{2\pi} \sqrt{1 + t^2} dt.$$

4. If an evaluated form is desired, use

$$\int \sqrt{1 + t^2} dt = \frac{1}{2} \left(t\sqrt{1 + t^2} + \operatorname{arsinh} t \right),$$

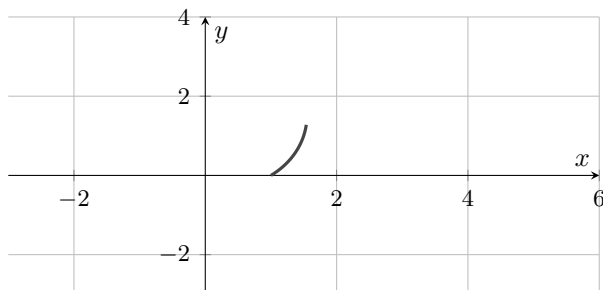
giving

$$L = \pi\sqrt{1 + 4\pi^2} + \frac{1}{2} \operatorname{arsinh}(2\pi).$$

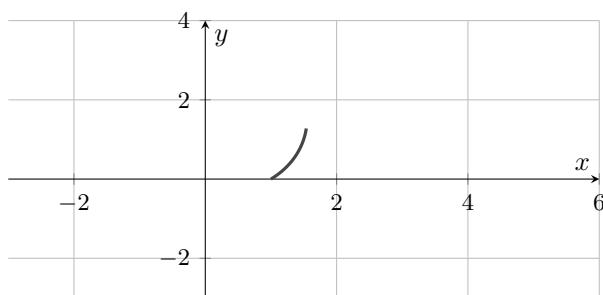
Question 4

Problem

Show that $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \ln 2$, has length $\sqrt{2}$.



Question 4 – Solution



Solution

1. Differentiate using the product rule:

$$x'(t) = e^t(\cos t - \sin t), \quad y'(t) = e^t(\sin t + \cos t).$$

2. Compute the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= e^{2t} [(\cos t - \sin t)^2 + (\sin t + \cos t)^2] \\ &= e^{2t} [2 \cos^2 t + 2 \sin^2 t] \\ &= 2e^{2t}. \end{aligned}$$

Since $e^t > 0$, the speed is $\sqrt{2}e^t$.

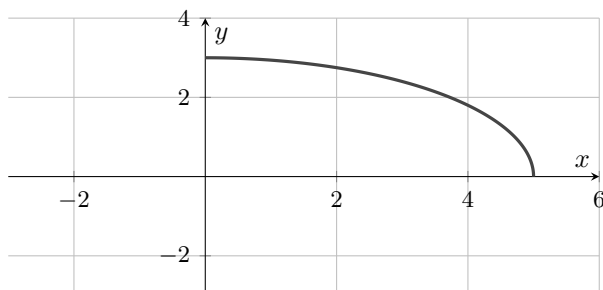
3. Set up and evaluate the length:

$$\begin{aligned} L &= \int_0^{\ln 2} \sqrt{2}e^t dt \\ &= \sqrt{2} [e^t]_0^{\ln 2} \\ &= \sqrt{2}(2 - 1) = \boxed{\sqrt{2}}. \end{aligned}$$

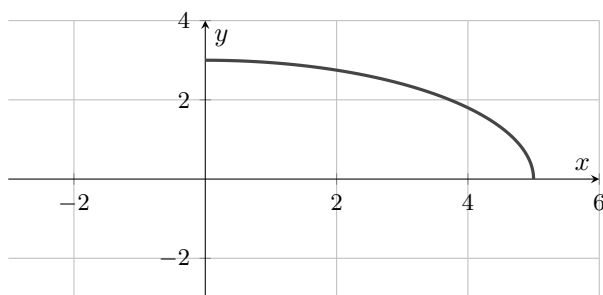
Question 5

Problem

A quarter ellipse is $x = 5 \cos t$, $y = 3 \sin t$, $0 \leq t \leq \pi/2$. Which simple quarter-circle lengths bound it?



Question 5 – Solution



Solution

1. Differentiate:

$$x'(t) = -5 \sin t, \quad y'(t) = 3 \cos t.$$

Thus the speed is

$$v(t) = \sqrt{25 \sin^2 t + 9 \cos^2 t}.$$

2. Rewrite the expression under the square root in two useful ways:

$$25 \sin^2 t + 9 \cos^2 t = 9 + 16 \sin^2 t = 25 - 16 \cos^2 t.$$

Since $0 \leq \sin^2 t, \cos^2 t \leq 1$, it follows that

$$9 \leq v(t)^2 \leq 25, \quad \text{so} \quad 3 \leq v(t) \leq 5.$$

3. Integrate these bounds over an interval of width $\pi/2$:

$$3 \int_0^{\pi/2} dt \leq L \leq 5 \int_0^{\pi/2} dt.$$

Therefore,

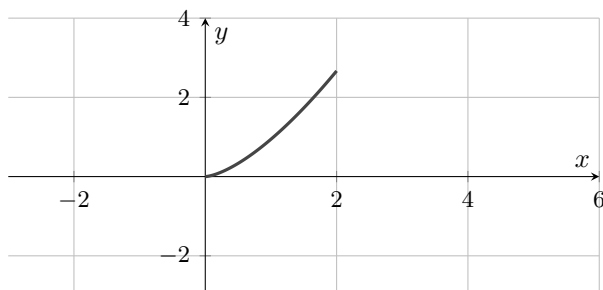
$$\boxed{\frac{3\pi}{2} \leq L \leq \frac{5\pi}{2}}.$$

These are the quarter-circumferences of circles with radii 3 and 5.

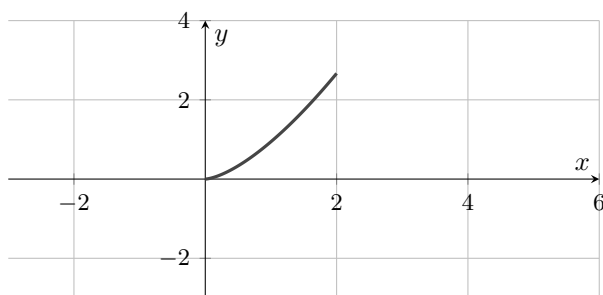
Question 6

Problem

Find the length of $x = t^2/2$, $y = t^3/3$, $0 \leq t \leq 2$.



Question 6 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = t, \quad y'(t) = t^2.$$

2. The speed is

$$\sqrt{t^2 + t^4} = \sqrt{t^2(1 + t^2)} = |t|\sqrt{1 + t^2}.$$

Since $0 \leq t \leq 2$, $|t| = t$.

3. Therefore,

$$L = \int_0^2 t\sqrt{1 + t^2} dt.$$

Let $u = 1 + t^2$, so $du = 2t dt$. Then

$$\begin{aligned} L &= \frac{1}{2} \int_1^5 u^{1/2} du \\ &= \frac{1}{3} \left[u^{3/2} \right]_1^5 \\ &= \frac{1}{3} (5^{3/2} - 1). \end{aligned}$$

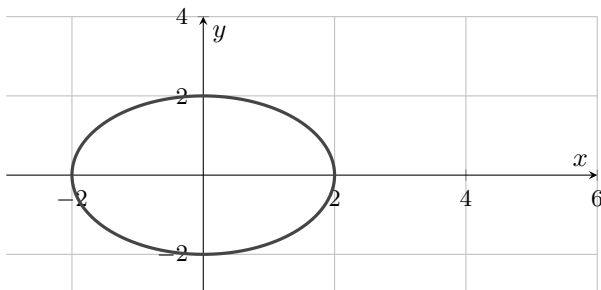
4. Hence

$$L = \frac{5\sqrt{5} - 1}{3}.$$

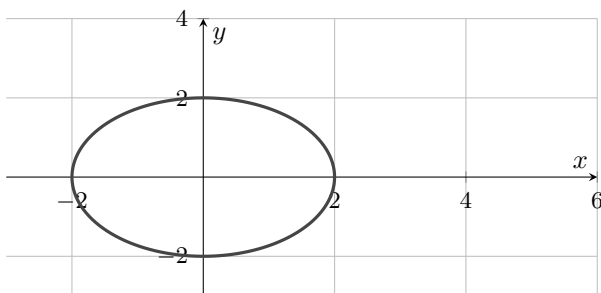
Question 7

Problem

A curve is reparametrized from t to $u = t^3$. Explain why its length is unchanged when both parameters increase.



Question 7 – Solution



Solution

1. Let the original curve be $\mathbf{r}(t)$ for $a \leq t \leq b$, and suppose $u = t^3$ increases on this interval. Then

$$t = u^{1/3}, \quad \frac{dt}{du} = \frac{1}{3u^{2/3}}$$

wherever this derivative is defined.

2. By the chain rule, the velocity in the new parameter is

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \frac{dt}{du}.$$

Taking magnitudes gives

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \left\| \frac{d\mathbf{r}}{dt} \right\| \left| \frac{dt}{du} \right|.$$

3. Because both parameters increase, $dt/du \geq 0$ and $dt = (dt/du)du$. Hence

$$\left\| \frac{d\mathbf{r}}{du} \right\| du = \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$

Integrating over corresponding endpoints yields the same value in either parameter.

4. Therefore, reparametrization changes the rate at which the curve is traced but not the geometric arc length:

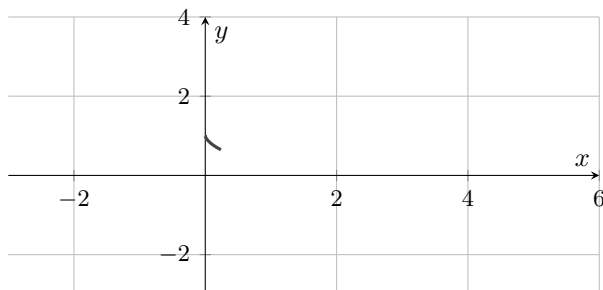
$$\boxed{L_u = L_t}.$$

A possible derivative singularity at the single point $u = 0$ does not change the value of the proper or corresponding improper integral.

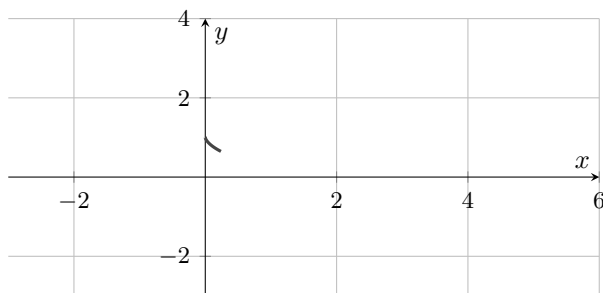
Question 8

Problem

Find the length of the tractable curve $x = t - \tanh t$, $y = \operatorname{sech} t$, $0 \leq t \leq 1$.



Question 8 – Solution



Solution

1. Differentiate and use $1 - \operatorname{sech}^2 t = \tanh^2 t$:

$$x'(t) = 1 - \operatorname{sech}^2 t = \tanh^2 t,$$

$$y'(t) = -\operatorname{sech} t \tanh t.$$

2. Simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t \\ &= \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t) \\ &= \tanh^2 t. \end{aligned}$$

Thus the speed is $|\tanh t|$.

3. On $0 \leq t \leq 1$, $\tanh t \geq 0$, so

$$L = \int_0^1 \tanh t \, dt.$$

Since $d(\ln \cosh t)/dt = \tanh t$,

$$L = [\ln(\cosh t)]_0^1 = \ln(\cosh 1) - \ln 1.$$

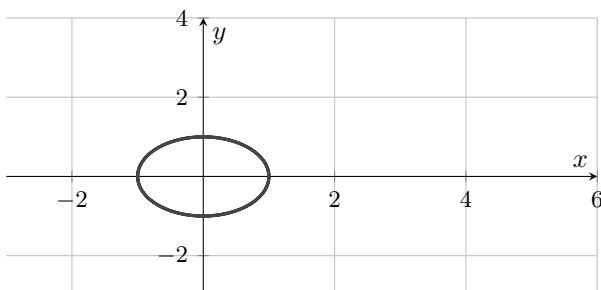
4. Therefore,

$$\boxed{L = \ln(\cosh 1)}.$$

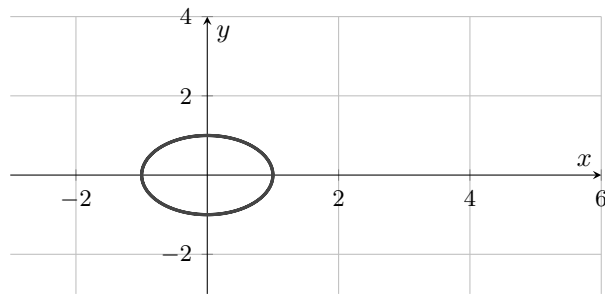
Question 9

Problem

A particle goes around the unit circle twice. Is its path length 2π or 4π ? Justify using $x = \cos 2t$, $y = \sin 2t$, $0 \leq t \leq 2\pi$.



Question 9 – Solution



Solution

1. Differentiate:

$$x'(t) = -2 \sin 2t, \quad y'(t) = 2 \cos 2t.$$

2. The speed is constant:

$$\sqrt{x'^2 + y'^2} = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2.$$

3. Integrate over the full parameter interval:

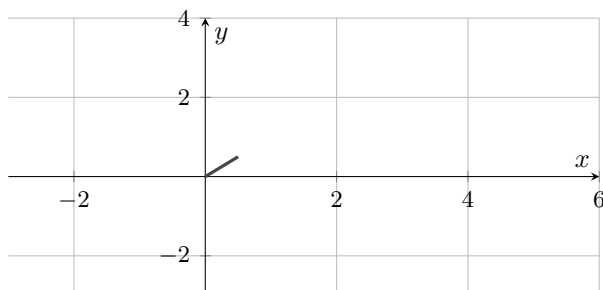
$$L = \int_0^{2\pi} 2 \, dt = \boxed{4\pi}.$$

4. The angle of the point is $2t$, which increases from 0 to 4π . Therefore, the unit circle is traversed twice. Its geometric circumference is 2π , but the particle travels that circumference twice, producing total traveled length 4π .

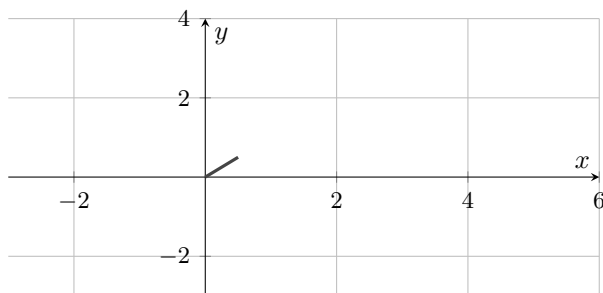
Question 10

Problem

Find $a > 0$ so that the line path $x = at$, $y = (1 - a)t$, $0 \leq t \leq 1$, has minimum length.



Question 10 – Solution



Solution

1. Differentiate the line parametrization:

$$x'(t) = a, \quad y'(t) = 1 - a.$$

The speed is constant, so

$$L(a) = \int_0^1 \sqrt{a^2 + (1-a)^2} dt = \sqrt{a^2 + (1-a)^2}.$$

2. Because the square-root function is increasing, minimizing $L(a)$ is equivalent to minimizing

$$L(a)^2 = a^2 + (1-a)^2 = 2a^2 - 2a + 1.$$

Complete the square:

$$2a^2 - 2a + 1 = 2 \left(a - \frac{1}{2} \right)^2 + \frac{1}{2}.$$

3. The squared term is smallest when $a = 1/2$, which satisfies $a > 0$. Therefore,

$$\boxed{a = \frac{1}{2}}.$$

4. At this value, the minimum length is

$$L_{\min} = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}}.$$

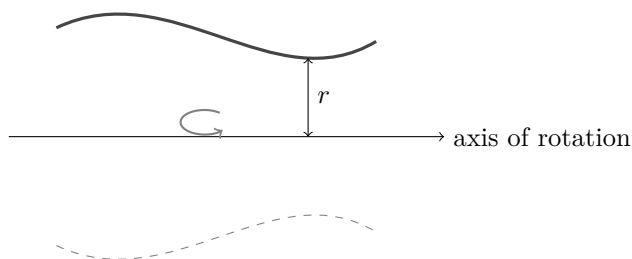
Geometrically, the endpoint $(a, 1-a)$ is then $(1/2, 1/2)$, the closest point on the line $x + y = 1$ to the origin.

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 1

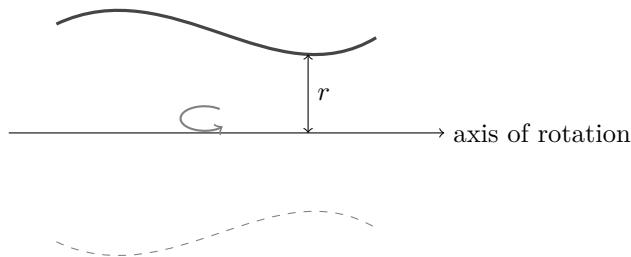
Problem

Rotate the line segment $x = t$, $y = 1 - t$, $0 \leq t \leq 1$, about the x -axis. Find the surface area and identify the surface.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 1 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 1, \quad y'(t) = -1.$$

Therefore,

$$ds = \sqrt{1^2 + (-1)^2} dt = \sqrt{2} dt.$$

2. The distance from (x, y) to the x -axis is $y = 1 - t$, which is nonnegative on $0 \leq t \leq 1$.
3. Apply the surface-area formula:

$$\begin{aligned} S &= 2\pi \int_0^1 y \, ds \\ &= 2\pi\sqrt{2} \int_0^1 (1-t) \, dt \\ &= 2\pi\sqrt{2} \left[t - \frac{t^2}{2} \right]_0^1 = \boxed{\pi\sqrt{2}}. \end{aligned}$$

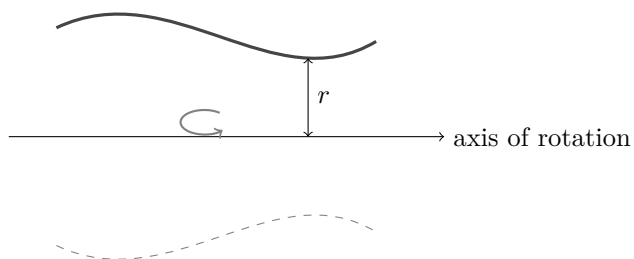
4. The segment runs from $(0, 1)$ to $(1, 0)$. Rotating it about the x -axis produces a right circular cone with radius 1, height 1, and slant height $\sqrt{2}$. The result agrees with $\pi r \ell = \pi\sqrt{2}$.

Calculus II Section 3-5: Surface Area with Parametric Equations

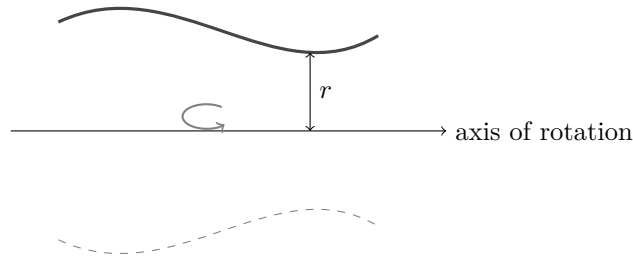
Question 2

Problem

The upper semicircle $x = R \cos t$, $y = R \sin t$, $0 \leq t \leq \pi$, rotates about the x -axis. Recover the sphere's area.



Question 2 – Solution



Solution

1. Differentiate:

$$x'(t) = -R \sin t, \quad y'(t) = R \cos t.$$

Thus

$$ds = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt = R dt.$$

2. On $0 \leq t \leq \pi$, the generating semicircle lies above the x -axis, so its radius of rotation is

$$y = R \sin t \geq 0.$$

This semicircle generates the sphere exactly once.

3. Substitute into $S = 2\pi \int y ds$:

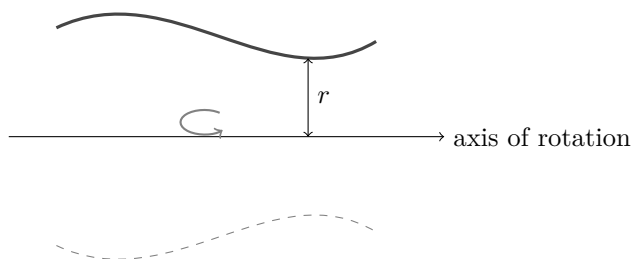
$$\begin{aligned} S &= 2\pi \int_0^\pi (R \sin t)(R dt) \\ &= 2\pi R^2 [-\cos t]_0^\pi \\ &= 2\pi R^2 (2) = \boxed{4\pi R^2}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 3

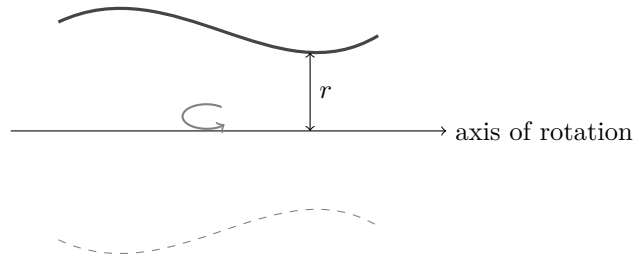
Problem

A circle $x = 3 + \cos t$, $y = \sin t$ rotates about the y -axis. Find the torus area without citing a memorized formula.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 3 – Solution



Solution

1. The interval $0 \leq t \leq 2\pi$ traces the generating circle once. Differentiate:

$$x'(t) = -\sin t, \quad y'(t) = \cos t.$$

Hence

$$ds = \sqrt{\sin^2 t + \cos^2 t} dt = dt.$$

2. The circle lies between $x = 2$ and $x = 4$, entirely to the right of the y -axis. Therefore, the radius of rotation is the nonnegative distance

$$x = 3 + \cos t.$$

3. Apply $S = 2\pi \int x ds$:

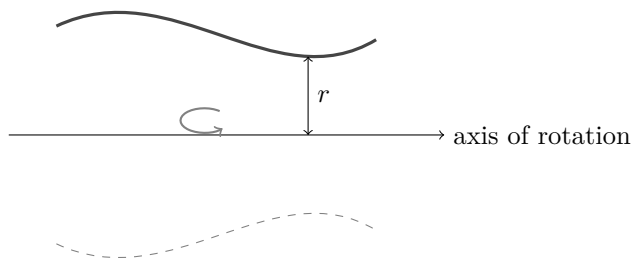
$$\begin{aligned} S &= 2\pi \int_0^{2\pi} (3 + \cos t) dt \\ &= 2\pi [3t + \sin t]_0^{2\pi} \\ &= 2\pi(6\pi) = \boxed{12\pi^2}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

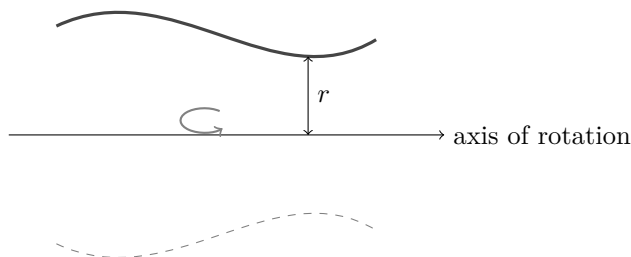
Question 4

Problem

Set up, but do not evaluate, the area formed by rotating $x = t^2$, $y = t^3$, $0 \leq t \leq 1$, about the y -axis.



Question 4 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 2t, \quad y'(t) = 3t^2.$$

2. Compute the arc-length element:

$$\begin{aligned} ds &= \sqrt{(2t)^2 + (3t^2)^2} dt \\ &= \sqrt{4t^2 + 9t^4} dt = |t|\sqrt{4 + 9t^2} dt. \end{aligned}$$

Since $0 \leq t \leq 1$, this becomes $ds = t\sqrt{4 + 9t^2} dt$.

3. Rotation is about the y -axis, so the radius is the distance $x = t^2$.
4. Therefore, without evaluating,

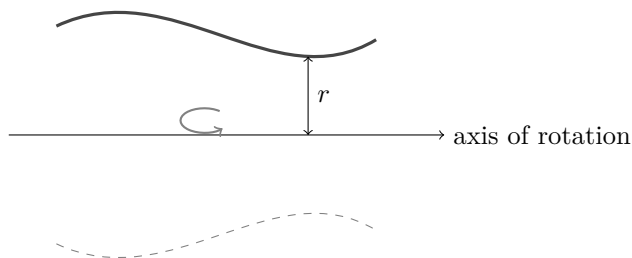
$$\begin{aligned} S &= 2\pi \int_0^1 x(t) ds \\ &= \boxed{2\pi \int_0^1 t^3 \sqrt{4 + 9t^2} dt}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

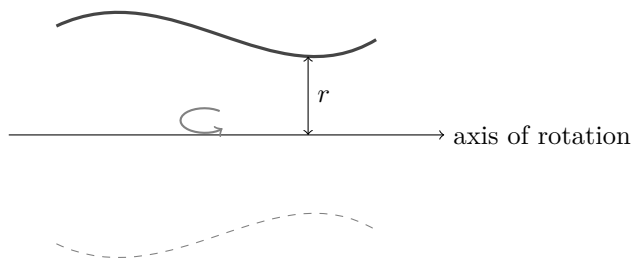
Question 5

Problem

Why is rotating $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$, about the x -axis a bad direct use of the surface formula?



Question 5 – Solution



Solution

1. The interval $0 \leq t \leq \pi$ traces the upper semicircle, while $\pi \leq t \leq 2\pi$ traces the lower semicircle.
2. When either semicircle is rotated about the x -axis, it generates the entire unit sphere. A point (x, y) and its reflection $(x, -y)$ produce the same circle of revolution because both are the same distance $|y|$ from the axis.
3. If the full interval is used with the correct radius $|y|$, then $ds = dt$ and

$$2\pi \int_0^{2\pi} |\sin t| dt = 8\pi,$$

which is twice the unit sphere's area. The formula counts the repeated sweep.

4. If one incorrectly uses signed $y = \sin t$ instead, the upper and lower contributions cancel and give zero, which is also not physical surface area.
5. Use one generating semicircle only, for example

$$S = 2\pi \int_0^\pi \sin t dt = \boxed{4\pi}.$$

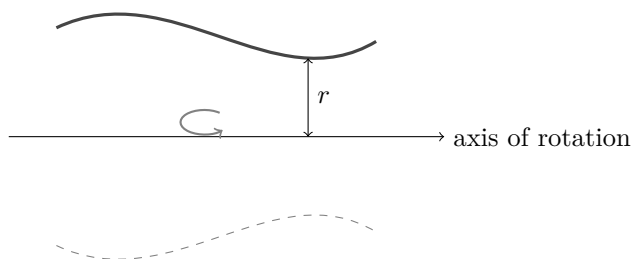
Thus the full-circle interval is unsuitable because it double-covers the surface.

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 6

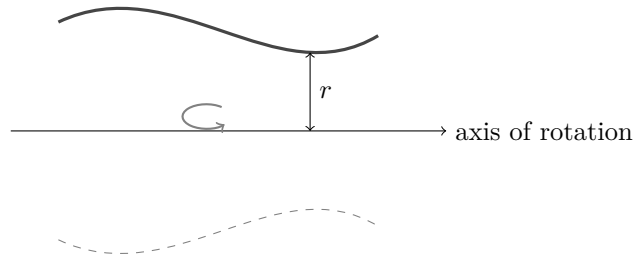
Problem

Rotate $x = e^t$, $y = e^{-t}$, $0 \leq t \leq \ln 2$, about the x -axis. Write the exact surface-area integral in simplest form.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 6 – Solution



Solution

1. Differentiate:

$$x'(t) = e^t, \quad y'(t) = -e^{-t}.$$

Therefore,

$$ds = \sqrt{e^{2t} + e^{-2t}} dt.$$

2. The curve lies above the x -axis because $y = e^{-t} > 0$, so the radius of rotation is $y = e^{-t}$.
3. Multiply the radius by ds and simplify:

$$\begin{aligned} y ds &= e^{-t} \sqrt{e^{2t} + e^{-2t}} dt \\ &= \sqrt{e^{-2t}(e^{2t} + e^{-2t})} dt \\ &= \sqrt{1 + e^{-4t}} dt. \end{aligned}$$

4. Hence the requested exact integral is

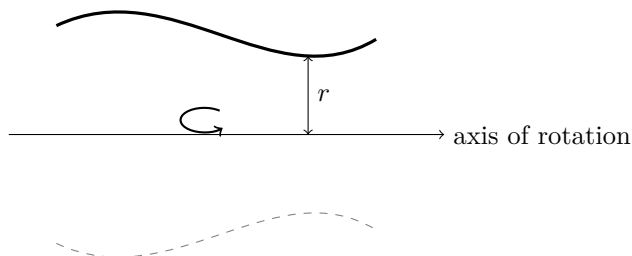
$$S = 2\pi \int_0^{\ln 2} \sqrt{1 + e^{-4t}} dt.$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 7

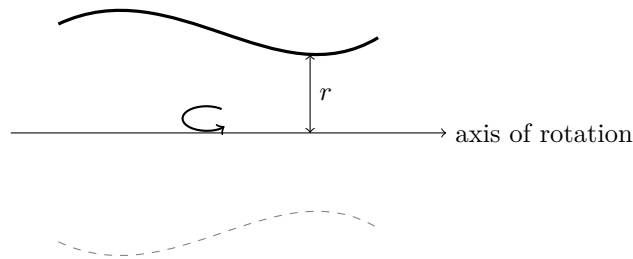
Problem

A cylinder is generated parametrically by rotating the horizontal segment $x = t, y = h, 0 \leq t \leq L$, about the x -axis, where $h, L > 0$. Derive its lateral area.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 7 – Solution



Solution

1. Differentiate the horizontal segment:

$$x'(t) = 1, \quad y'(t) = 0.$$

Hence

$$ds = \sqrt{1^2 + 0^2} dt = dt.$$

2. Every point of the segment is the constant distance h from the x -axis, assuming $h \geq 0$. Thus the radius of rotation is h .
3. Apply the surface-area formula:

$$\begin{aligned} S &= 2\pi \int_0^L h dt \\ &= 2\pi h[t]_0^L = \boxed{2\pi hL}. \end{aligned}$$

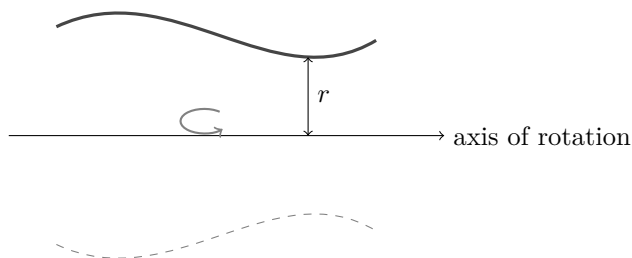
4. This is circumference times length: $(2\pi h)L$, which is precisely the lateral area of a cylinder of radius h and length L .

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 8

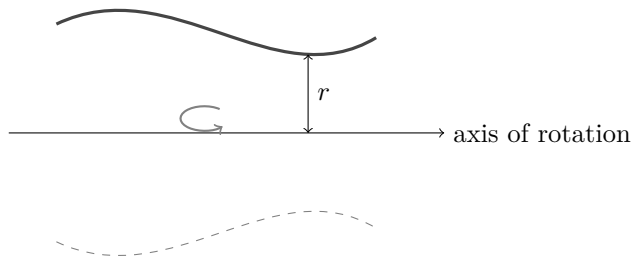
Problem

A curve crosses the axis of rotation. What precaution is needed in $S = 2\pi \int y \, ds$?



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 8 – Solution



Solution

1. In $S = 2\pi \int y \, ds$, the factor multiplying 2π must be the radius of rotation. A radius is a distance and therefore cannot be negative.
2. When the axis is the x -axis, the distance from $(x(t), y(t))$ to that axis is

$$r(t) = |y(t)|,$$

not signed $y(t)$.

3. The correct general formula is therefore

$$S = 2\pi \int_a^b |y(t)| \sqrt{x'(t)^2 + y'(t)^2} \, dt.$$

Equivalently, split the integral at every parameter value where $y(t) = 0$ and use the appropriate sign on each subinterval.

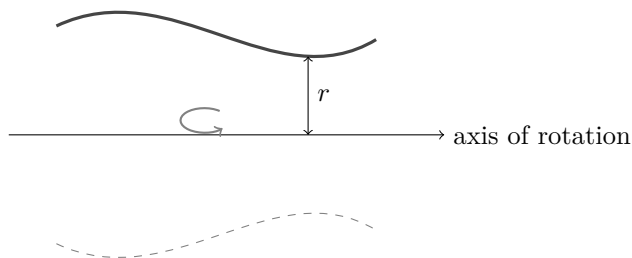
4. Without the absolute value, contributions below the axis would be negative and could cancel contributions above the axis. Also check whether portions on opposite sides of the axis generate the same physical surface; if they do, restrict the interval to prevent double-counting.

Calculus II Section 3-5: Surface Area with Parametric Equations

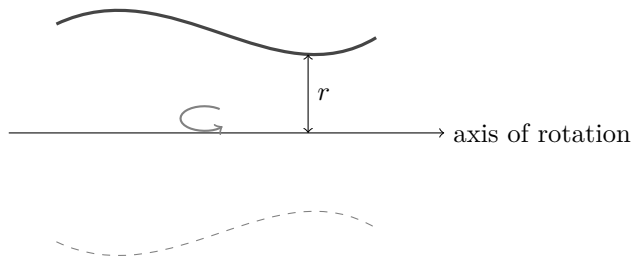
Question 9

Problem

Rotate the astroid quadrant $x = \cos^3 t, y = \sin^3 t, 0 \leq t \leq \pi/2$, about the x -axis. Set up a simplified integral.



Question 9 – Solution



Solution

1. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

2. As in the astroid arc-length calculation,

$$ds = 3 |\sin t \cos t| dt.$$

Both factors are nonnegative on $0 \leq t \leq \pi/2$, so

$$ds = 3 \sin t \cos t dt.$$

3. The radius about the x -axis is $y = \sin^3 t$. Therefore,

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} y ds \\ &= 6\pi \int_0^{\pi/2} \sin^4 t \cos t dt. \end{aligned}$$

4. Let $u = \sin t$, so $du = \cos t dt$ and $0 \leq u \leq 1$:

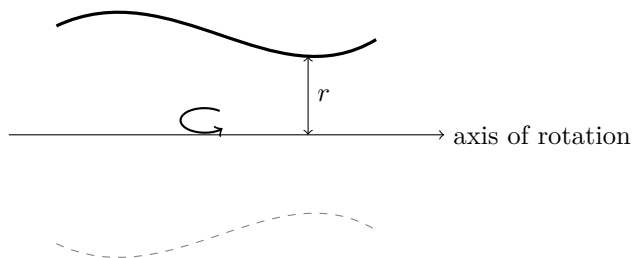
$$S = 6\pi \int_0^1 u^4 du = 6\pi \left[\frac{u^5}{5} \right]_0^1 = \boxed{\frac{6\pi}{5}}.$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 10

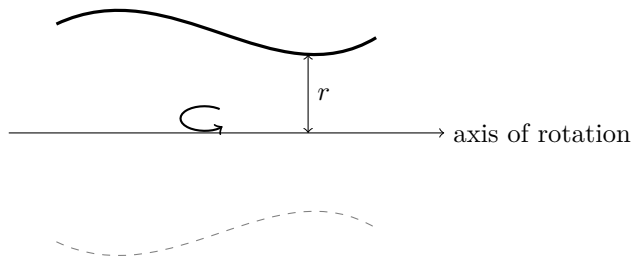
Problem

Two curves have the same length and the same arc-length-weighted average distance from an axis, and each sweeps its surface exactly once. Must their surfaces of revolution have equal area?



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 10 – Solution



Solution

1. Let $r(s)$ denote the nonnegative distance from a point on a generating curve to the axis, where s is arc length. Its arc-length-weighted average distance is

$$r_{\text{avg}} = \frac{1}{L} \int_C r \, ds.$$

2. Multiply by L :

$$\int_C r \, ds = L r_{\text{avg}}.$$

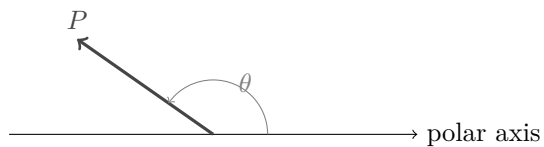
The surface area of revolution is therefore

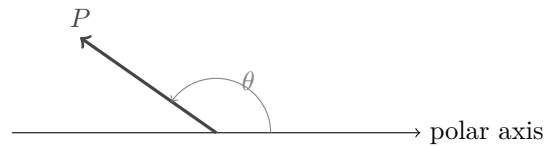
$$S = 2\pi \int_C r \, ds = 2\pi L r_{\text{avg}}.$$

3. If two curves have the same length L and the same arc-length-weighted average radius, this formula gives the same value of S for both.
4. Thus the answer is , provided each generating curve sweeps its surface exactly once. If a parametrization retraces the curve or distinct pieces generate the same surface, the integral counts multiplicity and the comparison requires removing that double-counting first.

Question 1**Problem**

The point has polar coordinates $(-4, \pi/3)$. Give two equivalent representations with positive radius and locate its quadrant.



Question 1 – Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

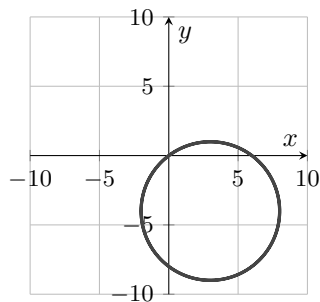
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

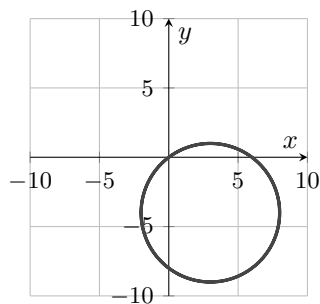
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Adding π while changing the radius sign gives $(4, 4\pi/3)$; also $(4, -2\pi/3)$.
4. It lies in Quadrant III.

Question 2**Problem**

Convert $r = 6 \cos \theta - 8 \sin \theta$ to Cartesian form and identify the curve.



Question 2 – Solution

**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

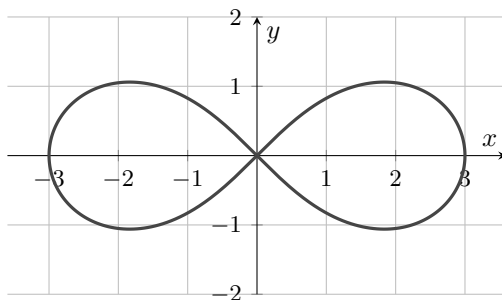
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

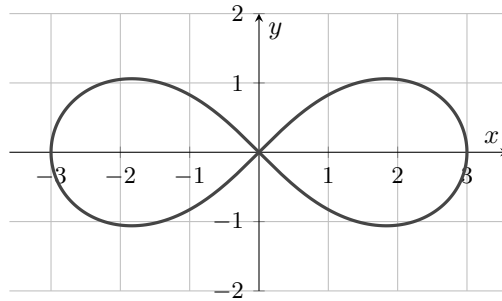
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. $r^2 = 6x - 8y$, so $(x - 3)^2 + (y + 4)^2 = 25$: a circle centered at $(3, -4)$ with radius 5.

Question 3**Problem**

Convert $r^2 = 9 \cos 2\theta$ to Cartesian form without using inverse trigonometric functions.



Question 3 – Solution



Solution

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

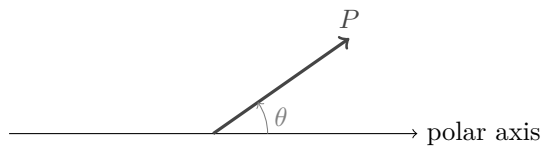
Equivalent polar coordinates satisfy

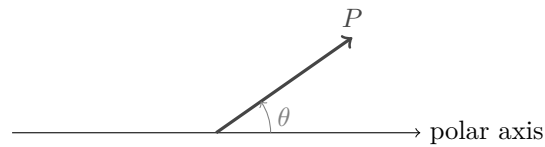
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Since $\cos 2\theta = (x^2 - y^2)/r^2$, $r^4 = 9(x^2 - y^2)$.
4. Thus $\boxed{(x^2 + y^2)^2 = 9(x^2 - y^2)}$.

Question 4**Problem**

Determine all intersections of $r = 2$ and $r = 4 \cos \theta$.



Question 4 – Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

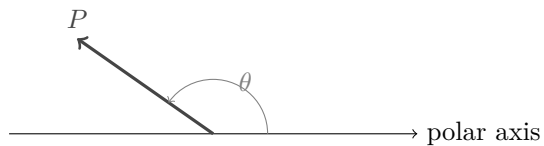
Equivalent polar coordinates satisfy

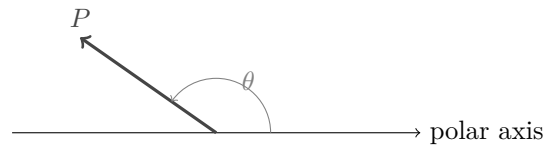
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. $2 = 4 \cos \theta$ gives $\theta = \pm\pi/3$.
4. The points are $(1, \pm\sqrt{3})$.

Question 5**Problem**

A polar curve is unchanged when θ is replaced by $-\theta$. What symmetry does this reveal, and why?



Question 5 – Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

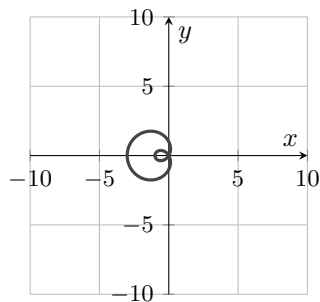
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

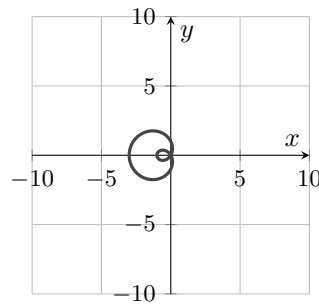
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The points at angles θ and $-\theta$ reflect across the polar axis, so the curve is symmetric about the x -axis.

Question 6**Problem**

Sketch mentally $r = 1 - 2 \cos \theta$. Why does it have an inner loop?



Question 6 – Solution

**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

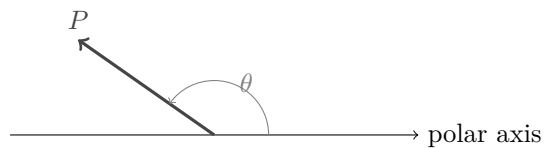
Equivalent polar coordinates satisfy

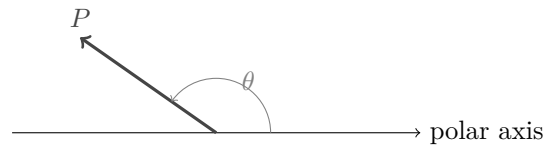
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The radius becomes negative when $\cos \theta > 1/2$, i.e. near $\theta = 0$.
4. Negative radii plot in the opposite direction, producing the inner loop.

Question 7**Problem**

Find the Cartesian equation corresponding to $\theta = 2\pi/3$ and explain why r must be unrestricted.



Question 7 – Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

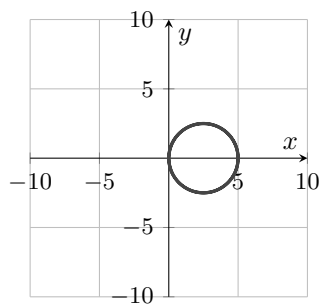
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

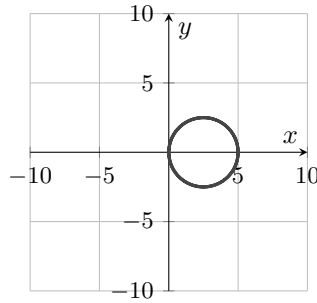
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The line has slope $\tan(2\pi/3) = -\sqrt{3}$, so $y = -\sqrt{3}x$.
4. Allowing negative r supplies the opposite ray and therefore the entire line.

Question 8**Problem**

Which description better reveals the geometry of $x^2 + y^2 = 5x$: Cartesian or polar? Convert it and explain.



Question 8 – Solution

**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

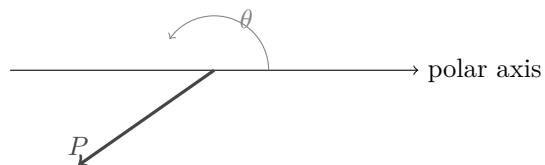
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

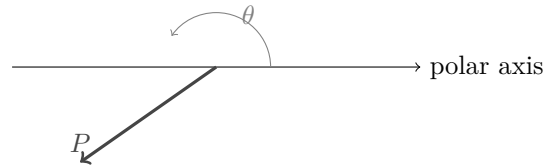
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Polar form is $r = 5 \cos \theta$, immediately revealing a circle through the pole with diameter 5 along the polar axis.

Question 9**Problem**

Find the smallest positive angle between the rays representing $(3, 17\pi/12)$ and $(2, -\pi/4)$.



Question 9 – Solution



Solution

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

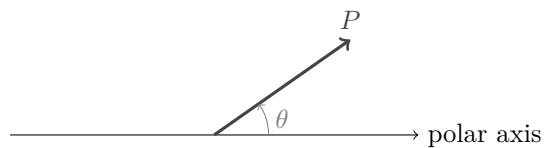
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

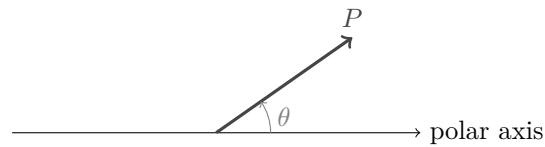
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Reduce angles to $17\pi/12$ and $21\pi/12$.
4. Their smaller difference is $\boxed{\pi/3}$.

Question 10**Problem**

A navigation system reports $(r, \theta) = (5, \text{atan2}(4, -3))$. Convert to Cartesian coordinates.



Question 10 – Solution

**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

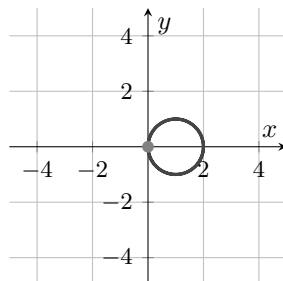
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

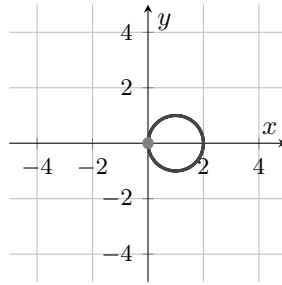
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The angle is in Quadrant II with cosine $-3/5$ and sine $4/5$.
4. Hence $(x, y) = \boxed{(-3, 4)}$.

Question 1**Problem**

Find the tangent line to $r = 2 \cos \theta$ at $\theta = \pi/4$.



Question 1 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

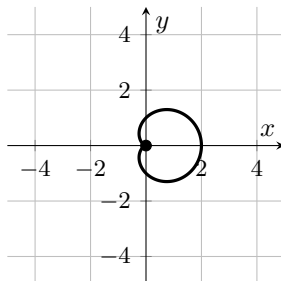
3. The point is $(1, 1)$.

4. With $r' = -2 \sin \theta$, the polar slope formula gives 0, so the tangent is $y = 1$.

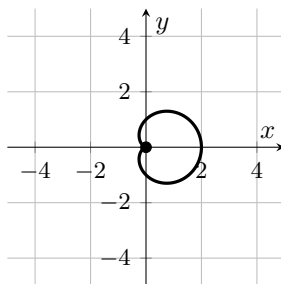
Question 2

Problem

Find all horizontal tangents of the cardioid $r = 1 + \cos \theta$ away from the pole.



Question 2 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

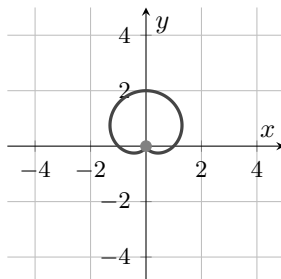
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

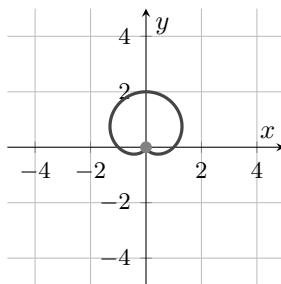
3. The numerator $r' \sin \theta + r \cos \theta = \cos \theta + \cos 2\theta$ vanishes.
4. This gives $\cos \theta = 1/2$ or -1 , and excluding the pole leaves $\theta = \pi/3, 5\pi/3$. At these angles $dx/d\theta \neq 0$, and the points are $(3/4, \pm 3\sqrt{3}/4)$. The tangent lines are $y = \pm 3\sqrt{3}/4$.

Question 3**Problem**

A student claims that $\theta = 0$ gives the rightmost point and a vertical tangent of $r = 1 + \sin \theta$. Test both parts of the claim.



Question 3 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

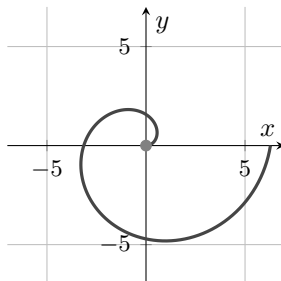
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

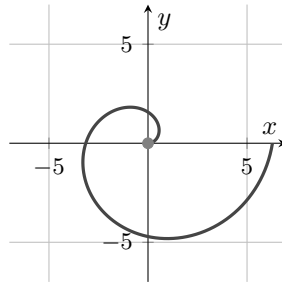
3. At $\theta = 0$, the point is $(1, 0)$ and the slope is 1, so the tangent is not vertical.
4. Also $x'(0) = 1 > 0$, so x is still increasing; the point is not rightmost.
5. Both parts are false.

Question 4**Problem**

The spiral $r = \theta$ reaches $\theta = \pi$. Find its tangent slope there.



Question 4 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

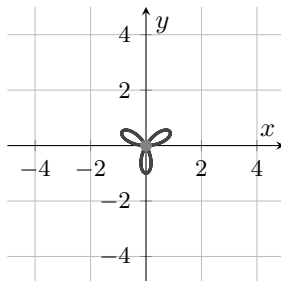
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

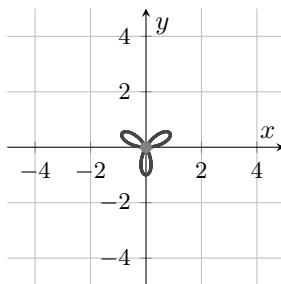
3. With $r' = 1$, the slope is $(\sin \pi + \pi \cos \pi)/(\cos \pi - \pi \sin \pi) = \boxed{\pi}$.

Question 5**Problem**

Find the tangent directions at the pole for the rose $r = \sin 3\theta$.



Question 5 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

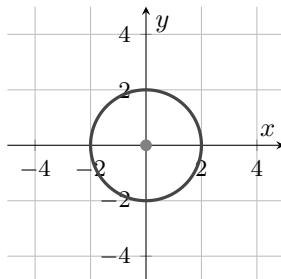
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The pole occurs when $3\theta = k\pi$, so the tangent rays satisfy $\theta = k\pi/3$.
4. The distinct unoriented lines are

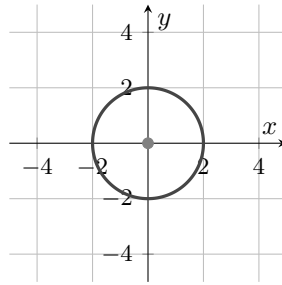
$$\theta = 0, \quad \theta = \pi/3, \quad \theta = 2\pi/3.$$

Question 6**Problem**

At an intersection of $r = 2$ and $r = 4 \cos \theta$ in the upper half-plane, find the acute angle between the two curves.



Question 6 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

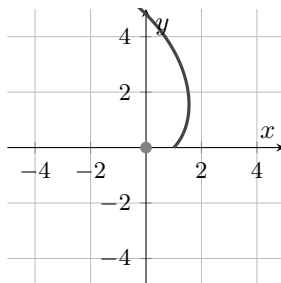
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $\theta = \pi/3$, the circle $r = 2$ has tangent direction $5\pi/6$.
4. The second circle has slope $1/\sqrt{3}$, hence tangent direction $\pi/6$.
5. The smaller angle between the tangent lines is $\boxed{\pi/3}$.

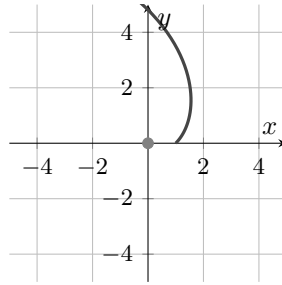
Question 7

Problem

For $r = e^\theta$, show that the angle between the radius vector and tangent is constant.



Question 7 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

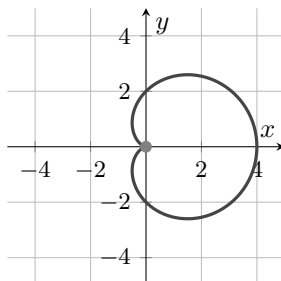
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. In polar components, tangent has radial component $r' = r$ and transverse component r .
4. Thus it makes angle $\arctan(r/r) = \boxed{\pi/4}$ with the radius vector.

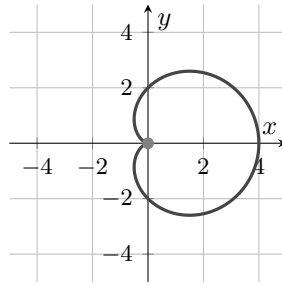
Question 8

Problem

Find a value of a so that $r = a + 2 \cos \theta$ has tangent slope 1 at $\theta = \pi/2$.



Question 8 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

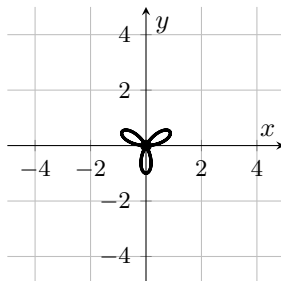
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $\pi/2$, the slope is $2/a$.

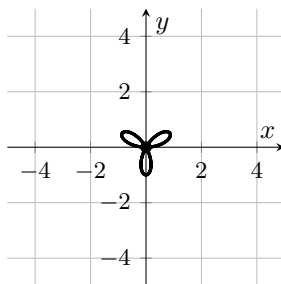
4. Setting $2/a = 1$ gives $\boxed{a = 2}$.

Question 9**Problem**

A student cancels r before using the polar slope formula at a pole. Explain the danger.



Question 9 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

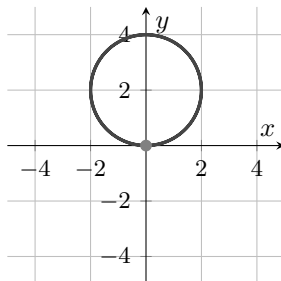
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

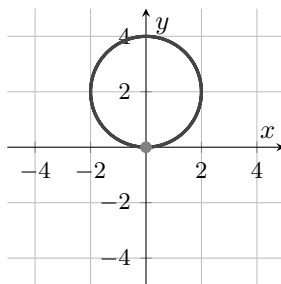
3. At $r = 0$, direct division by r is undefined. Valid cancellation where $r \neq 0$ can help compute a limit, but does not by itself justify substitution at the pole.
4. One must use a limit or inspect the parameter values producing the pole.

Question 10**Problem**

For the circle $r = 4 \sin \theta$, find the tangent at its highest point using polar calculus.



Question 10 – Solution

**Solution**

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

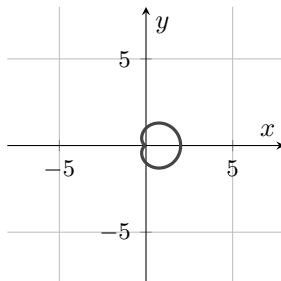
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

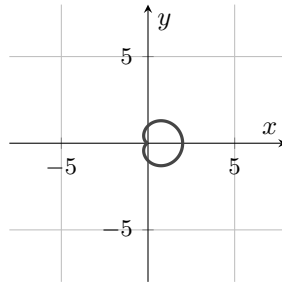
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The highest point occurs at $\theta = \pi/2$, giving $(0, 4)$.
4. The slope numerator is 0 and denominator -4 , so the tangent is horizontal: $y = 4$.

Question 1**Problem**

Find the area enclosed by the cardioid $r = 1 + \cos \theta$.



Question 1 – Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

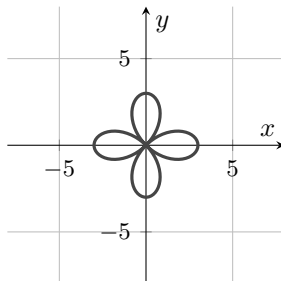
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

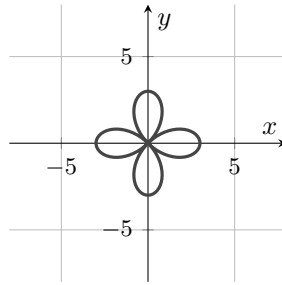
3. $A = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \boxed{3\pi/2}.$

Question 2**Problem**

Find the area of one petal of $r = 3 \cos 2\theta$.



Question 2 – Solution

**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

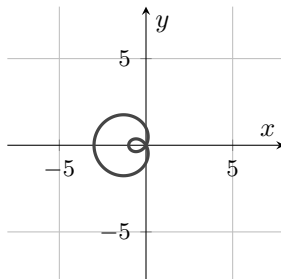
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

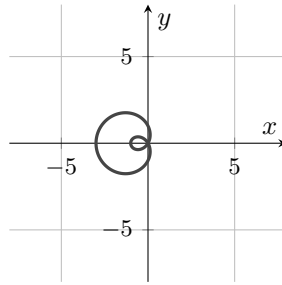
3. One petal is traced on $[-\pi/4, \pi/4]$.
4. Thus $A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} 9 \cos^2 2\theta d\theta = \boxed{9\pi/8}$.

Question 3**Problem**

Find the area of the inner loop of $r = 1 - 2 \cos \theta$.



Question 3 – Solution

**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

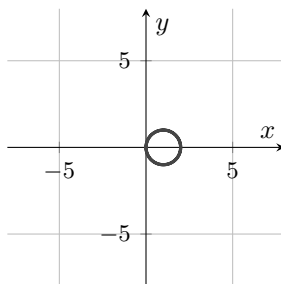
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

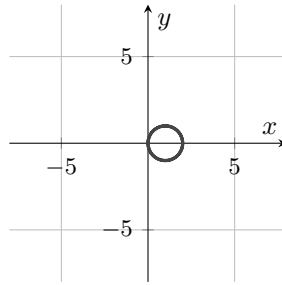
3. The loop is traced for $-\pi/3 \leq \theta \leq \pi/3$.
4. Hence $A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (1 - 2 \cos \theta)^2 d\theta = \boxed{\pi - 3\sqrt{3}/2}$.

Question 4**Problem**

Find the common area of the circles $r = 2 \cos \theta$ and $r = 2 \sin \theta$.



Question 4 – Solution



Solution

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

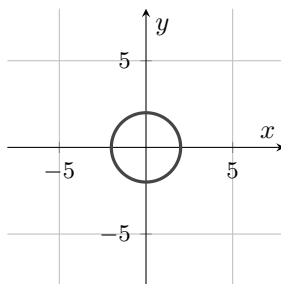
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

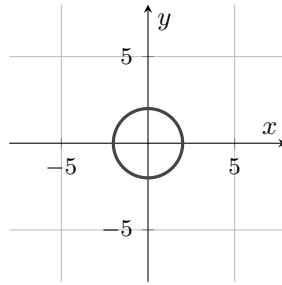
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. In the first quadrant the smaller radius switches at $\pi/4$.
4. Symmetry gives $A = \int_0^{\pi/4} (2 \sin \theta)^2 d\theta = \boxed{\pi/2 - 1}$.

Question 5**Problem**

Find the area inside $r = 2$ but outside $r = 2 \cos \theta$.



Question 5 – Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

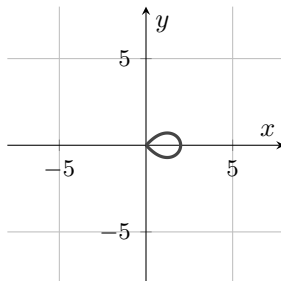
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

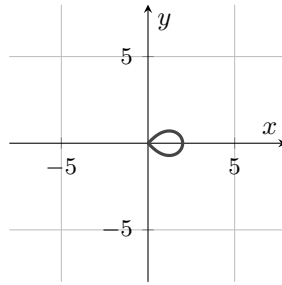
3. The circle $r = 2 \cos \theta$ lies entirely inside $r = 2$.
4. Subtract areas: $4\pi - \pi = \boxed{3\pi}$.

Question 6**Problem**

The curve $r^2 = 4 \cos 2\theta$ has two loops. Find its total area.



Question 6 – Solution

**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

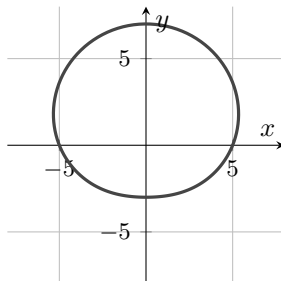
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

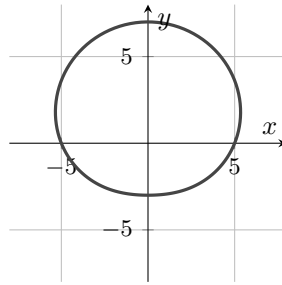
3. One loop uses $-\pi/4 \leq \theta \leq \pi/4$: $\text{area} = \frac{1}{2} \int 4 \cos 2\theta d\theta = 2$.
4. Doubling gives 4.

Question 7**Problem**

A radar beam reaches distance $r = 5 + 2 \sin \theta$. Find the area swept while $0 \leq \theta \leq \pi$.



Question 7 – Solution



Solution

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

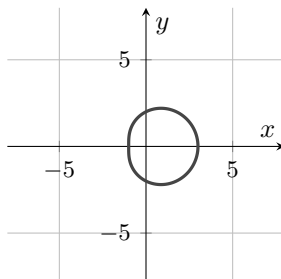
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

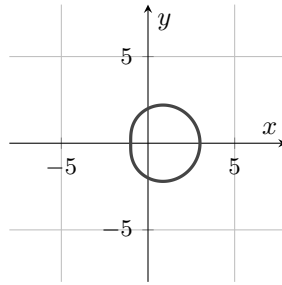
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. $A = \frac{1}{2} \int_0^\pi (5 + 2 \sin \theta)^2 d\theta = \boxed{27\pi/2 + 20}.$

Question 8**Problem**

Two polar curves satisfy $0 \leq r_1(\theta) \leq r_2(\theta)$. Explain why the area between them involves squares, not merely $r_2 - r_1$.



Question 8 – Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

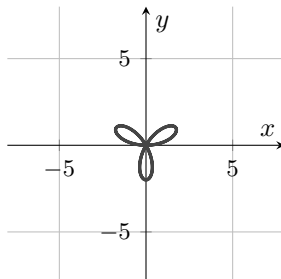
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

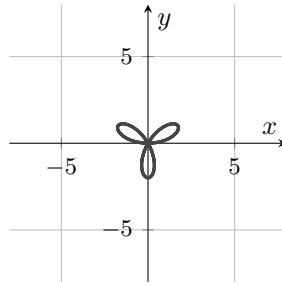
3. A thin polar sector has area $\frac{1}{2}r^2 d\theta$.
4. Subtracting sector areas yields $\frac{1}{2}(r_2^2 - r_1^2)d\theta$, reflecting increasing width with radius.

Question 9**Problem**

Find the area enclosed by one loop of $r = 2 \sin 3\theta$ and explain why multiplying by six is wrong for total area.



Question 9 – Solution



Solution

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

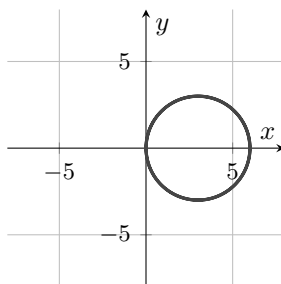
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

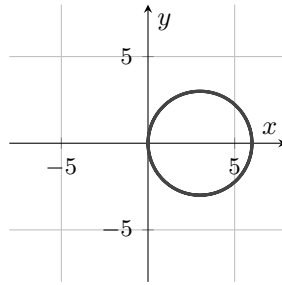
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. One loop area is $\frac{1}{2} \int_0^{\pi/3} 4 \sin^2 3\theta d\theta = \pi/3$.
4. The rose has three petals, not six, because negative radii retrace opposite petals; total area is $\boxed{\pi}$.

Question 10**Problem**

Choose $a > 0$ so that the circle $r = a \cos \theta$ has area 9π .



Question 10 – Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

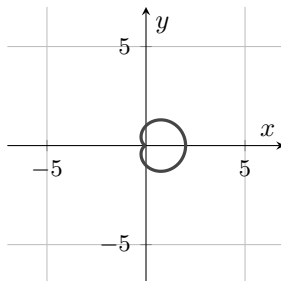
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

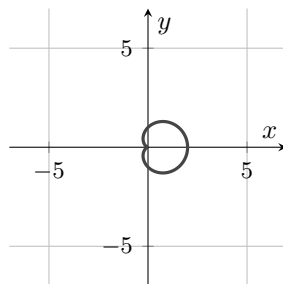
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. It is a circle of radius $a/2$, so area is $\pi a^2/4$.
4. Setting this to 9π gives $\boxed{a = 6}$.

Question 1**Problem**

Find the length of the cardioid $r = 1 + \cos \theta$.



Question 1 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

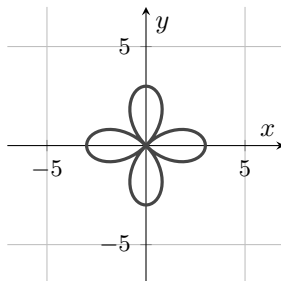
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

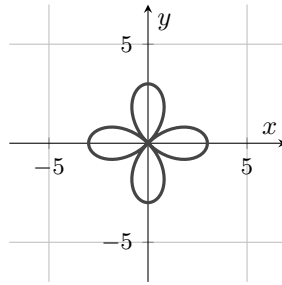
3. The speed simplifies to $2|\cos(\theta/2)|$.
4. Integrating from 0 to 2π gives $\boxed{8}$.

Question 2**Problem**

Find the length of one petal of $r = 3 \cos 2\theta$.



Question 2 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

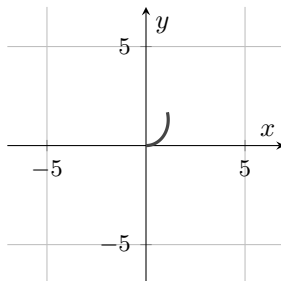
Simplify the expression under the square root before evaluating or reporting the integral.

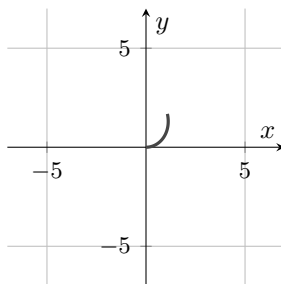
3. On $[-\pi/4, \pi/4]$, speed is $3\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}$.

4. Thus $L = 3 \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta} d\theta$; it is an elliptic integral.

Question 3**Problem**

Find the length of the spiral $r = 2\theta$, $0 \leq \theta \leq 1$.



Question 3 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

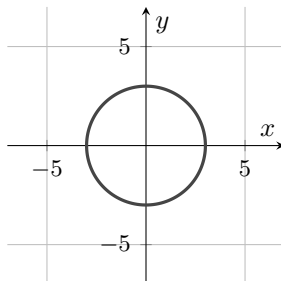
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

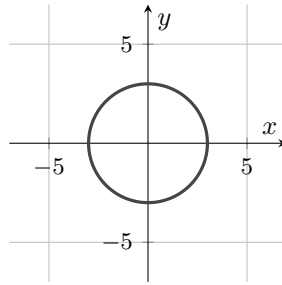
Simplify the expression under the square root before evaluating or reporting the integral.

3. $L = 2 \int_0^1 \sqrt{1 + \theta^2} d\theta = \boxed{\sqrt{2} + \operatorname{arsinh} 1}.$

Question 4**Problem**

Use the polar arc-length formula to recover the circumference of $r = R$.



Question 4 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

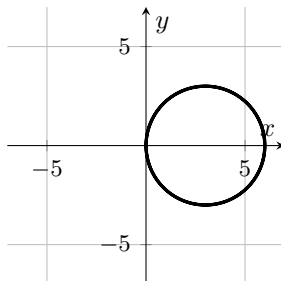
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

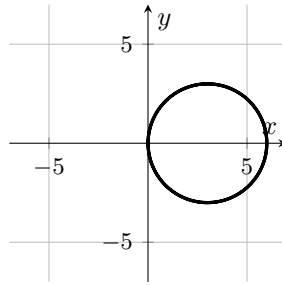
Simplify the expression under the square root before evaluating or reporting the integral.

3. Since $r' = 0$, speed is R .
4. Hence $L = \int_0^{2\pi} R d\theta = \boxed{2\pi R}$.

Question 5**Problem**

A circle of the same radius is $r = 2R \cos \theta$. Which interval traces it once, and what length results?



Question 5 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

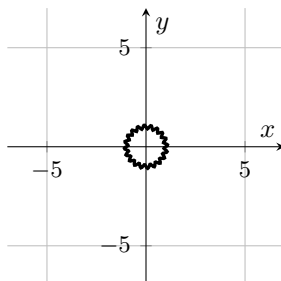
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

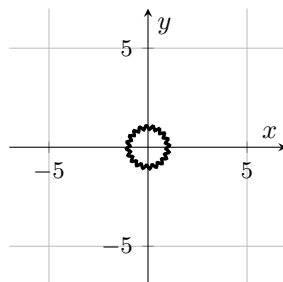
3. $-\pi/2 \leq \theta \leq \pi/2$ traces it once.
4. The speed simplifies to $2R$, so $L = 2R\pi = \boxed{2\pi R}$.

Question 6**Problem**

Without evaluating, compare the lengths of $r = 1$ and $r = 1 + 0.1 \sin(20\theta)$ over $0 \leq \theta \leq 2\pi$.



Question 6 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

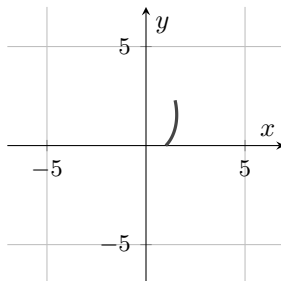
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

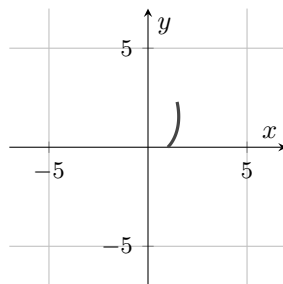
Simplify the expression under the square root before evaluating or reporting the integral.

3. Here $r > 0$ and $r' = 2 \cos 2\theta$. Thus $\sqrt{r^2 + (r')^2} \geq r$, with strict inequality except at finitely many angles. Consequently $L > \int_0^{2\pi} (1 + 0.1 \sin 2\theta) d\theta = 2\pi$, the unit circle's length.

Question 7**Problem**

Find the length of $r = e^\theta$, $0 \leq \theta \leq \ln 2$.



Question 7 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

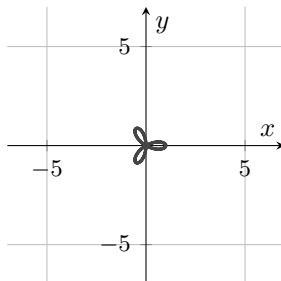
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

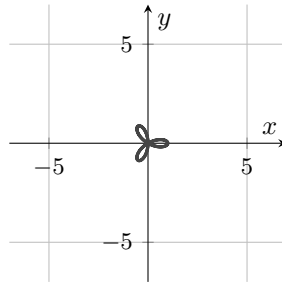
Simplify the expression under the square root before evaluating or reporting the integral.

3. Since $r' = r$, speed is $\sqrt{2}e^\theta$.
4. Thus $L = \sqrt{2}[e^\theta]_0^{\ln 2} = \boxed{\sqrt{2}}$.

Question 8**Problem**

Explain why integrating the rose $r = \cos 3\theta$ from 0 to 2π overcounts its geometric length.



Question 8 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

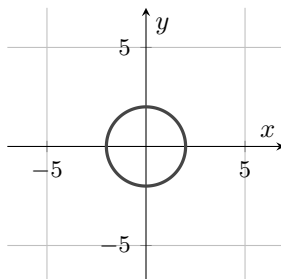
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

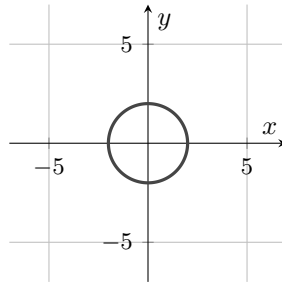
Simplify the expression under the square root before evaluating or reporting the integral.

3. For odd 3, the full curve is completed on $0 \leq \theta \leq \pi$; the next π retraces it because negative radius duplicates opposite points.
4. The 0 to 2π integral counts every petal twice.

Question 9**Problem**

A polar path has constant arc-length speed 5 and runs for an angular interval of width $2\pi/3$. What is its length?



Question 9 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

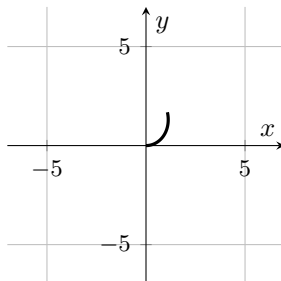
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

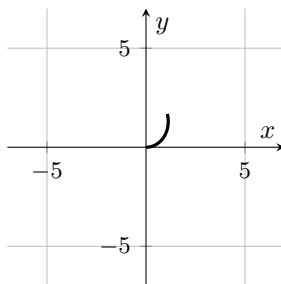
Simplify the expression under the square root before evaluating or reporting the integral.

3. Arc length is the integral of speed: $L = 5(2\pi/3) = \boxed{10\pi/3}$.

Question 10**Problem**

Find a so that $r = a\theta$, $0 \leq \theta \leq 1$, has length $\sqrt{2} + \operatorname{arsinh} 1$.



Question 10 – Solution**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

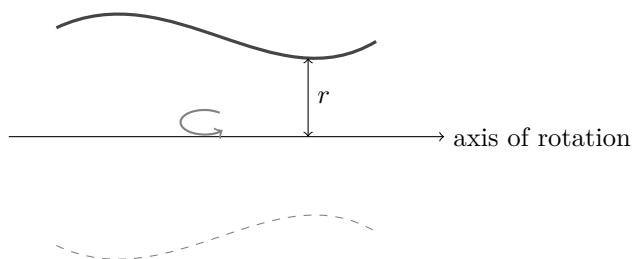
Simplify the expression under the square root before evaluating or reporting the integral.

3. Its length is $|a| \int_0^1 \sqrt{1 + \theta^2} d\theta = \frac{|a|}{2}(\sqrt{2} + \operatorname{arsinh} 1)$.
4. Thus $\boxed{a = \pm 2}$.

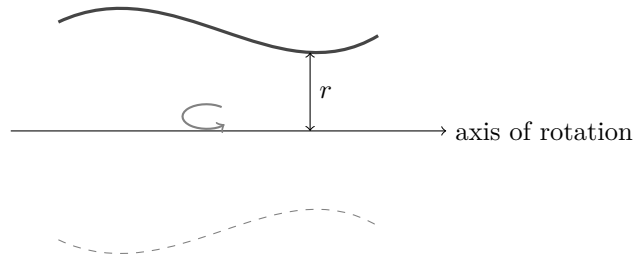
Question 1

Problem

The upper semicircle $r = 2R \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the polar axis. Find the generated surface area.



Question 1 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

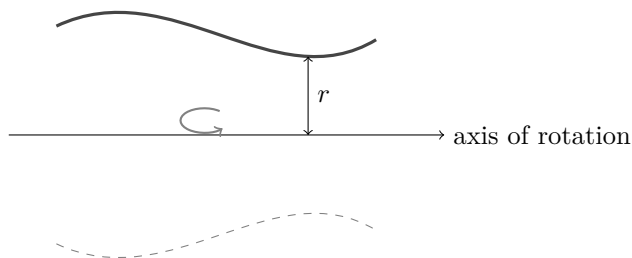
3. Here $y = 2R \sin \theta \cos \theta$ and $ds = 2R d\theta$.

4. Thus $S = 8\pi R^2 \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \boxed{4\pi R^2}$.

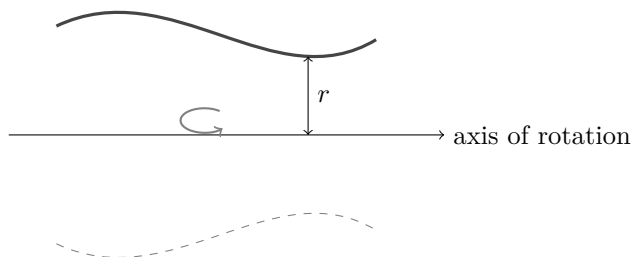
Question 2

Problem

Set up the surface area formed by rotating the cardioid $r = 1 + \cos \theta$, $0 \leq \theta \leq \pi$, about the polar axis.



Question 2 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

3. Use $y = r \sin \theta$ and $ds = 2 \cos(\theta/2) d\theta$ on this interval.

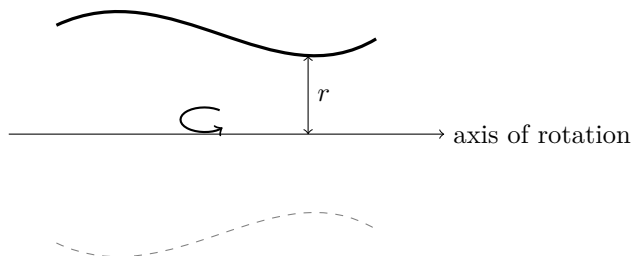
4. Therefore

$$S = 4\pi \int_0^\pi (1 + \cos \theta) \sin \theta \cos(\theta/2) d\theta.$$

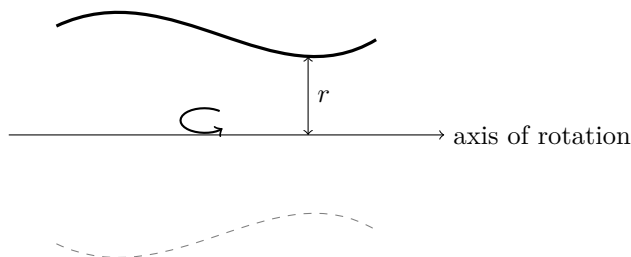
Question 3

Problem

Rotate the spiral arc $r = \theta$, $0 \leq \theta \leq \pi/2$, about the y -axis. Write the exact surface-area integral.



Question 3 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

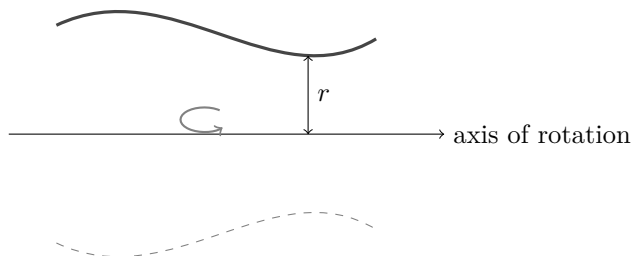
3. Distance to the y -axis is $x = \theta \cos \theta$, and $ds = \sqrt{1 + \theta^2} d\theta$.

4. Thus
$$S = 2\pi \int_0^{\pi/2} \theta \cos \theta \sqrt{1 + \theta^2} d\theta.$$

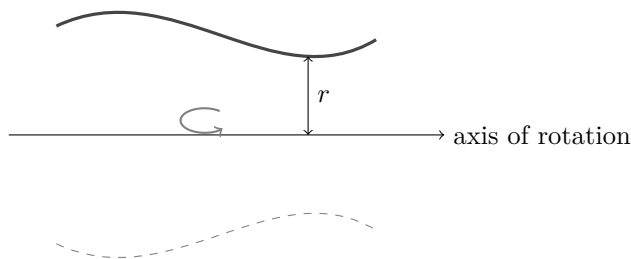
Question 4

Problem

Why must $|r \sin \theta|$ replace $r \sin \theta$ when a polar curve crosses the x -axis during rotation about that axis?



Question 4 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

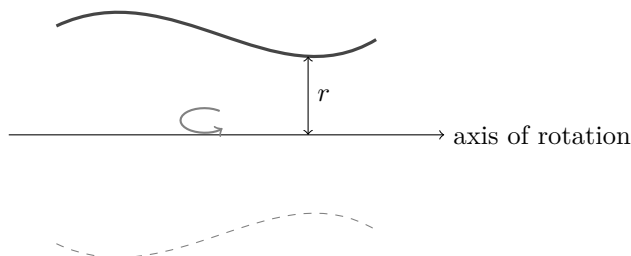
over an interval that generates the surface exactly once.

3. Radius to the axis is the nonnegative distance $|y| = |r \sin \theta|$.
4. Without the absolute value, areas on opposite sides would cancel.

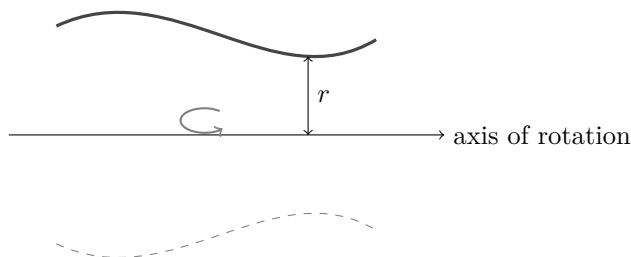
Question 5

Problem

A student proposes using only $0 \leq \theta \leq \pi/2$ when the circle $r = 4 \sin \theta$ rotates about the x -axis. Explain why that omits part of the surface and give a correct interval.



Question 5 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

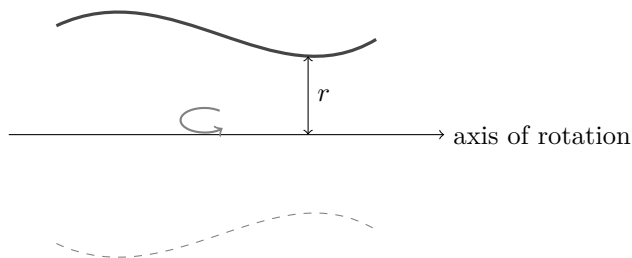
over an interval that generates the surface exactly once.

3. The interval $0 \leq \theta \leq \pi/2$ traces only the right semicircle; rotation preserves the x -coordinate, so the left half is not duplicated.
4. Use $\boxed{0 \leq \theta \leq \pi}$ to trace the full generating circle once.

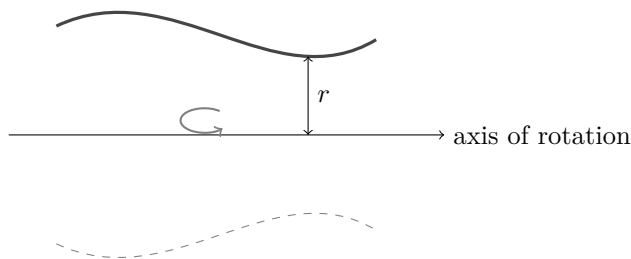
Question 6

Problem

For $r = R$, $0 \leq \theta \leq \pi$, rotate the semicircle about the x -axis and recover the sphere area.



Question 6 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

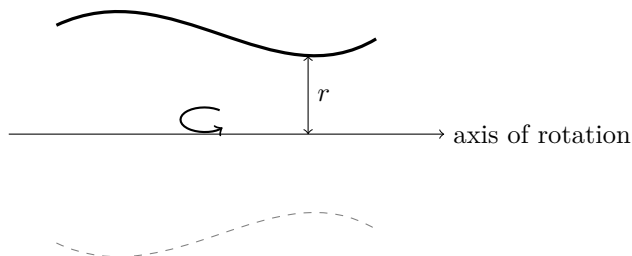
3. $y = R \sin \theta$ and $ds = R d\theta$.

4. Thus $S = 2\pi R^2 \int_0^\pi \sin \theta d\theta = \boxed{4\pi R^2}$.

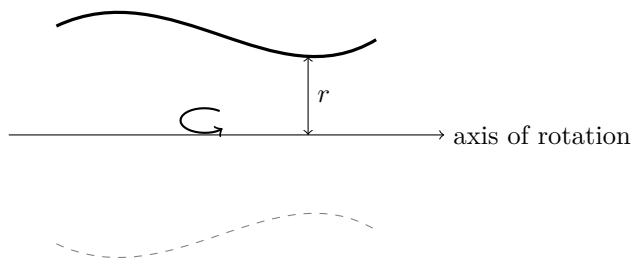
Question 7

Problem

Set up the surface area obtained by rotating one petal of $r = \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$, about the y -axis.



Question 7 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

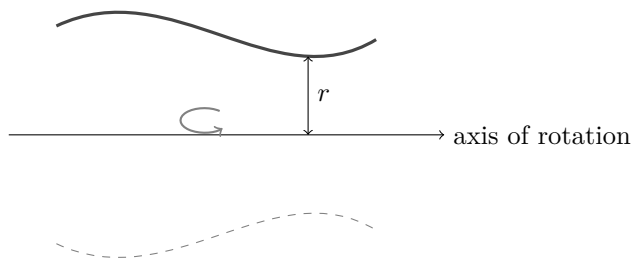
3. Use radius $|x| = |r \cos \theta|$ and $ds = \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta$.
4. On this interval $r \cos \theta \geq 0$, so the requested setup is

$$S = 2\pi \int_{-\pi/4}^{\pi/4} \cos 2\theta \cos \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta.$$

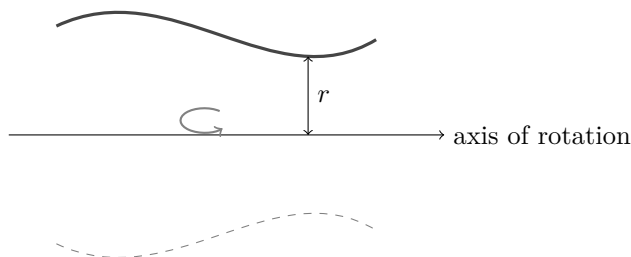
Question 8

Problem

A curve is entirely in the first quadrant. Compare its areas of revolution about the x - and y -axes using only weighted average coordinates.



Question 8 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

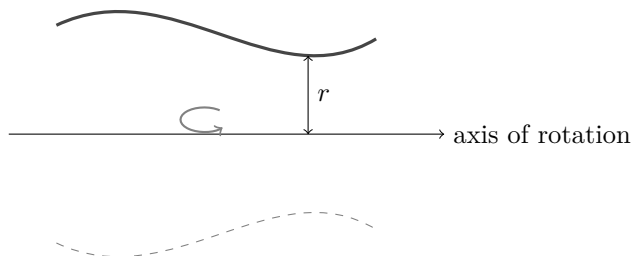
over an interval that generates the surface exactly once.

3. $S_x = 2\pi L \bar{y}_s$ and $S_y = 2\pi L \bar{x}_s$, where the bars are arc-length-weighted means.
4. Therefore $S_x/S_y = \bar{y}_s/\bar{x}_s$.

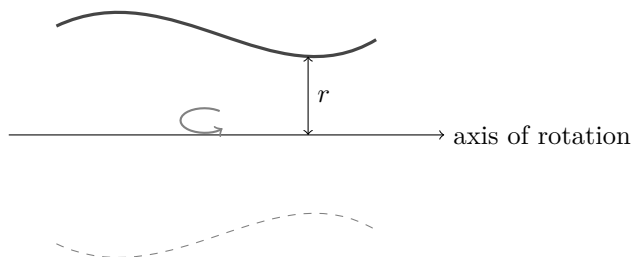
Question 9

Problem

Find the area generated when $r = 2 \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the line $y = -1$.



Question 9 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

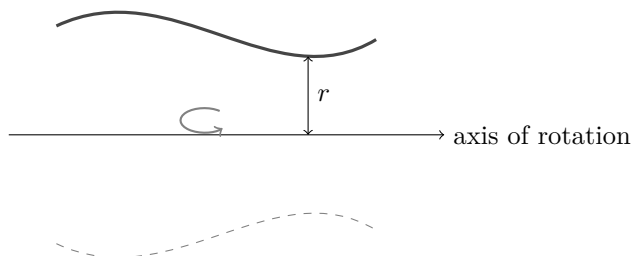
3. Here $ds = 2d\theta$ and radius is $y + 1 = 2 \sin \theta \cos \theta + 1$.

4. Hence $S = 4\pi \int_0^{\pi/2} (1 + 2 \sin \theta \cos \theta) d\theta = \boxed{2\pi^2 + 4\pi}$.

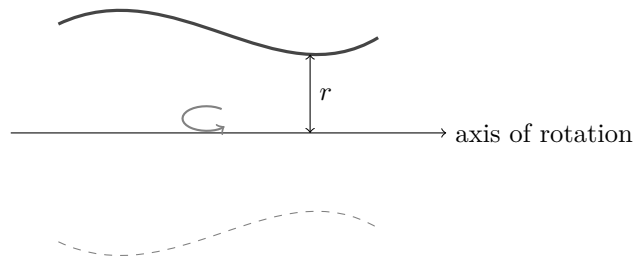
Question 10

Problem

A student integrates a closed polar curve over $0 \leq \theta \leq 2\pi$ and gets twice the expected surface area. Name two likely causes.



Question 10 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

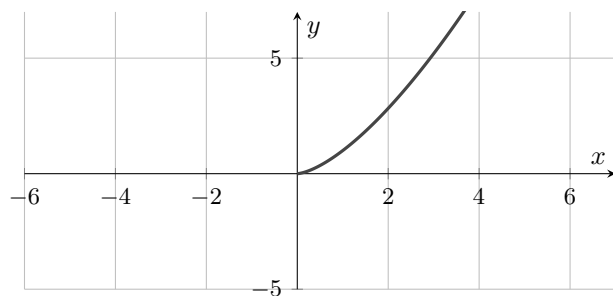
3. The curve may itself be traced twice on that interval, or two different portions of the generating curve may sweep the same surface after rotation.
4. Both are geometric double-counting.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 1

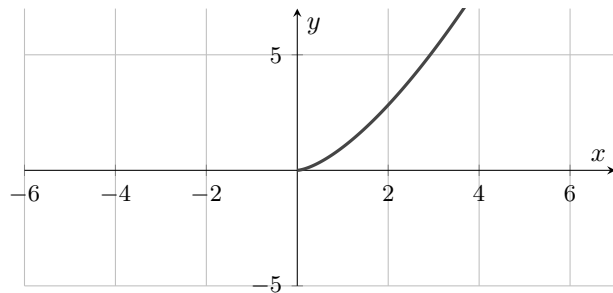
Problem

For $y = x^{3/2}$, $0 \leq x \leq 4$, compare the Cartesian and parametric arc-length setups using $x = t^2$, $y = t^3$.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 1 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

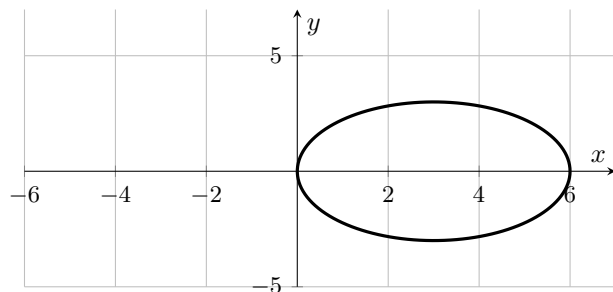
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian: $\int_0^4 \sqrt{1 + 9x/4} dx$.
4. Parametric with $0 \leq t \leq 2$: $\int_0^2 t\sqrt{4 + 9t^2} dt$.
5. Substitution $x = t^2$ makes them identical.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

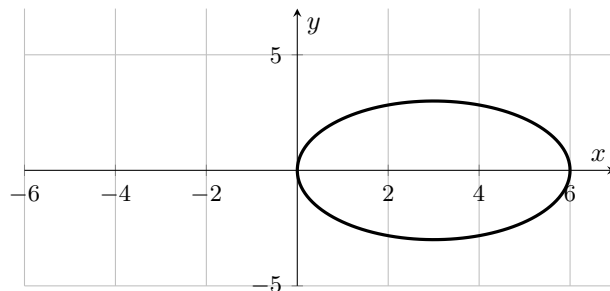
Question 2

Problem

A circle is given parametrically and polarly. Which representation makes its circumference easiest: $x = 3 + 3 \cos t$, $y = 3 \sin t$ or $r = 6 \cos \theta$? Explain.



Question 2 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

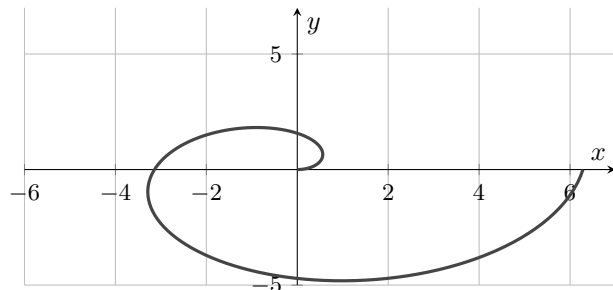
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametric speed is 3 on $0 \leq t \leq 2\pi$. The polar arc-length factor is 6 on $-\pi/2 \leq \theta \leq \pi/2$. Each interval traces the circle once, so both forms give $L = 6\pi$.
4. The best choice is the one whose single-tracing interval is clearest.

Question 3

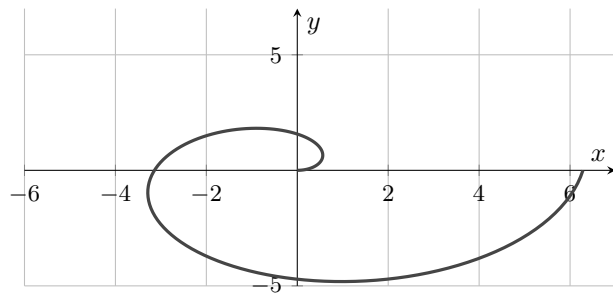
Problem

The spiral $r = \theta$ can be parametrized by $x = \theta \cos \theta, y = \theta \sin \theta$. Verify that both arc-length formulas give the same integrand.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 3 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

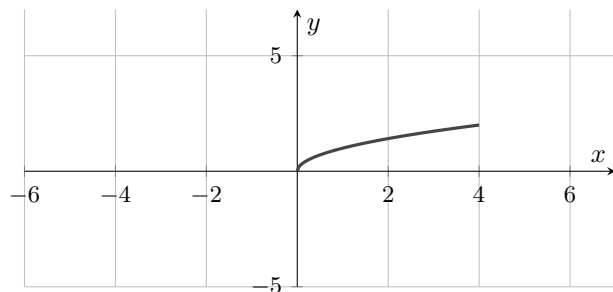
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Polar gives $\sqrt{r^2 + (r')^2} = \sqrt{\theta^2 + 1}$.
4. Squaring the Cartesian derivatives cancels cross terms and gives the same $\sqrt{\theta^2 + 1}$.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 4

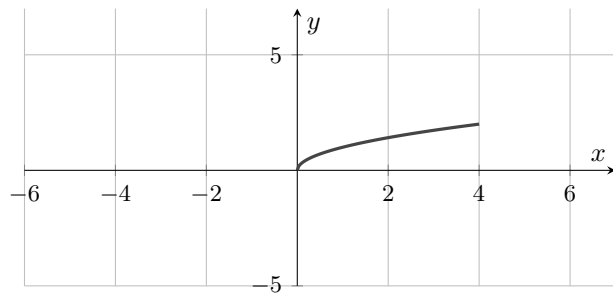
Problem

A surface is generated from $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x -axis. Decide whether Cartesian or the parametrization $x = t^2, y = t$ is cleaner.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 4 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametrization removes the singular derivative at $x = 0$:

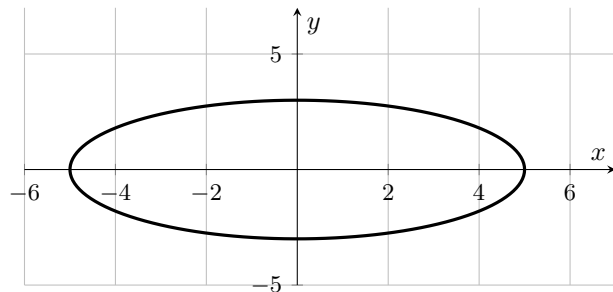
$$ds = \sqrt{4t^2 + 1} dt, \quad S = 2\pi \int_0^2 t\sqrt{4t^2 + 1} dt.$$

Therefore the parametric form is cleaner.

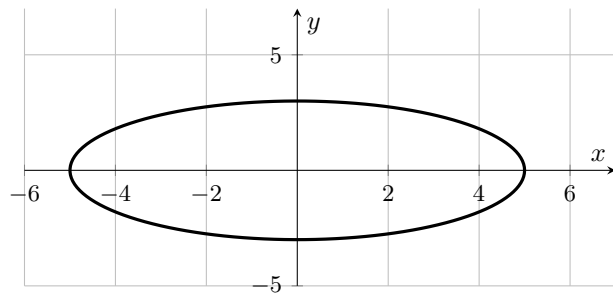
Question 5

Problem

A noncircular ellipse lacks an elementary perimeter formula. Explain why switching among Cartesian, parametric, and polar forms does not remove the essential difficulty.



Question 5 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

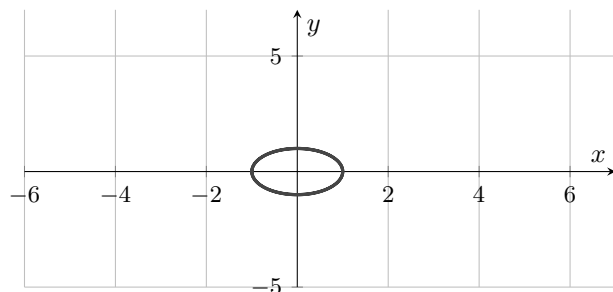
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Each representation reduces to an elliptic integral.
4. Reparametrization changes appearance, not the intrinsic arc-length integral or its non-elementary nature.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

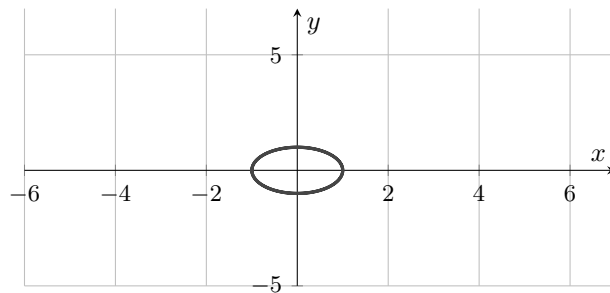
Question 6

Problem

A curve is traced twice by a parametrization but once by a polar interval. Which arc length is the geometric length, and which is the traveled distance?



Question 6 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

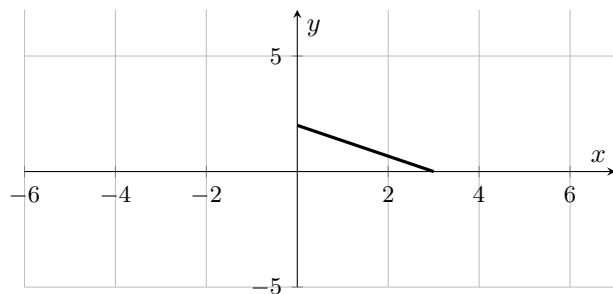
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The once-traced polar integral gives geometric curve length.
4. The twice-traced parametric integral gives total traveled distance, twice the geometric length.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

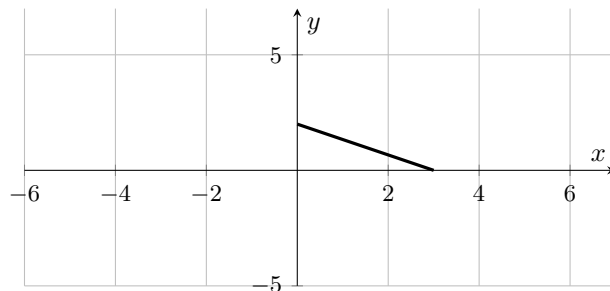
Question 7

Problem

Recover the cone lateral-area formula by parametrizing its slanted edge, then compare it with the slant-height mnemonic.



Question 7 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

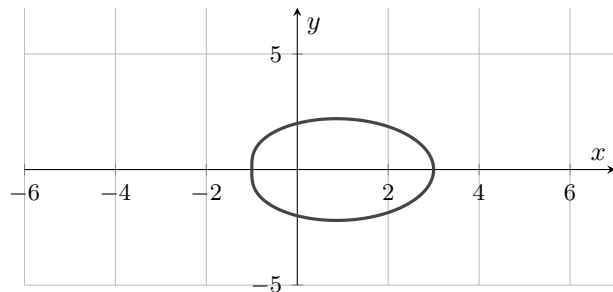
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. For radius $R > 0$ and height $h > 0$, rotate $(x, y) = (Rt/h, h - t)$, $0 \leq t \leq h$, about the y -axis.
4. Then $ds = (\sqrt{R^2 + h^2}/h)dt$ and $S = \pi R\sqrt{R^2 + h^2} = \pi R\ell$, matching the mnemonic.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 8

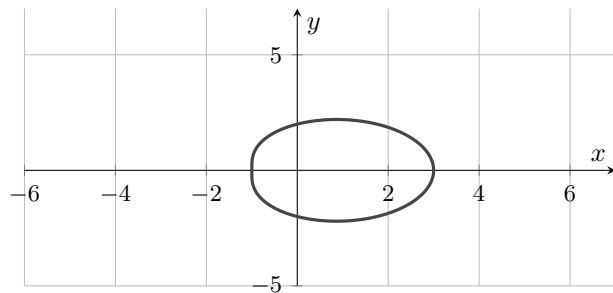
Problem

A curve has polar form $r = f(\theta)$ but is rotated about the y -axis. State when converting to Cartesian form might be safer.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 8 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

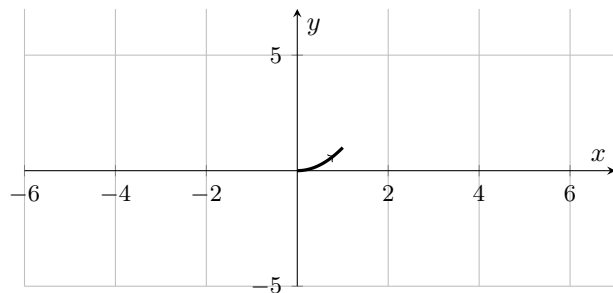
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Conversion is safer when $x = r \cos \theta$ changes sign or the polar interval retraces points, because the radius to the y -axis is $|x|$ and double-covering is easier to see geometrically.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 9

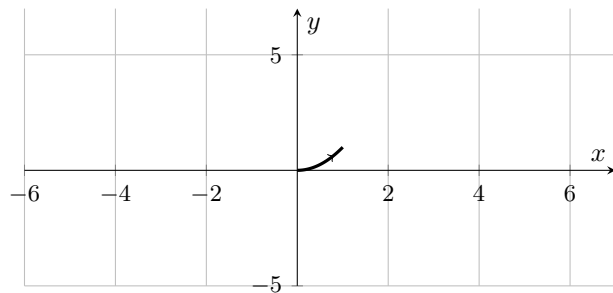
Problem

Design a consistency check for any computed surface area of revolution using dimensions and scaling.



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Question 9 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

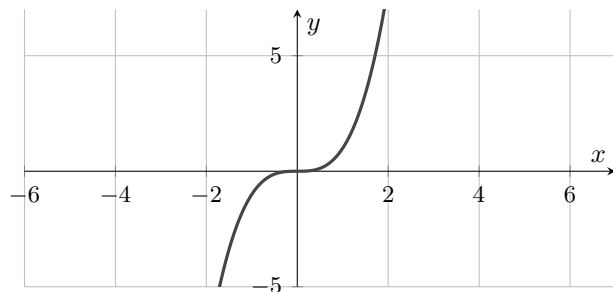
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Area must have length-squared units.
4. If every coordinate is scaled by $k > 0$, both radius and arc length scale by k , so the answer must scale by k^2 .

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Question 10

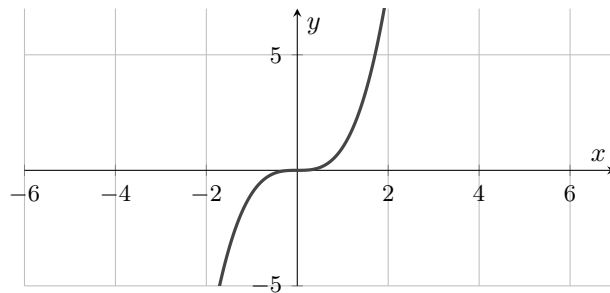
Problem

A numerical system can evaluate either a Cartesian or parametric arc-length integral. Give one reason to prefer each form.



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Question 10 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian is convenient for a single-valued smooth graph with simple derivative.
4. Parametric form is preferable for loops, vertical tangents, motion data, or when it avoids an infinite derivative.