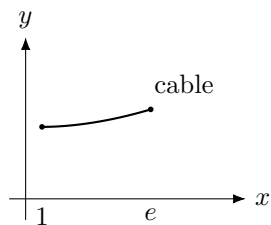


Question 1

A cable follows

$$y = \frac{x^2}{4} - \frac{1}{2} \ln x, \quad 1 \leq x \leq e.$$

Find its exact length.



Question 1 – Solution**Step 1: Differentiate.**

$$y' = \frac{x}{2} - \frac{1}{2x}.$$

Step 2: Simplify the arc-length integrand.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2 \\ &= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \\ &= \left(\frac{x}{2} + \frac{1}{2x} \right)^2. \end{aligned}$$

Since $x > 0$,

$$\sqrt{1 + (y')^2} = \frac{x}{2} + \frac{1}{2x}.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} L &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x} \right) dx \\ &= \left[\frac{x^2}{4} + \frac{1}{2} \ln x \right]_1^e \\ &= \left(\frac{e^2}{4} + \frac{1}{2} \right) - \frac{1}{4} = \frac{e^2 + 1}{4}. \end{aligned}$$

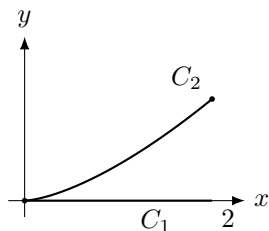
$$\boxed{L = \frac{e^2 + 1}{4}}$$

Question 2

For $0 \leq x \leq 2$, compare

$$C_1 : y = 0, \quad C_2 : y = \frac{2}{3}x^{3/2}.$$

Find $L_2 - L_1$.



Question 2 – Solution

Step 1: Find the length of C_1 . Since $y' = 0$,

$$L_1 = \int_0^2 \sqrt{1 + 0^2} \, dx = \int_0^2 1 \, dx = 2.$$

Step 2: Differentiate C_2 .

$$y' = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = \sqrt{x}.$$

Step 3: Find the length of C_2 .

$$\begin{aligned} L_2 &= \int_0^2 \sqrt{1 + (\sqrt{x})^2} \, dx \\ &= \int_0^2 \sqrt{1 + x} \, dx \\ &= \left[\frac{2}{3} (1 + x)^{3/2} \right]_0^2 \\ &= \frac{2}{3} (3\sqrt{3} - 1) = 2\sqrt{3} - \frac{2}{3}. \end{aligned}$$

Step 4: Subtract.

$$L_2 - L_1 = 2\sqrt{3} - \frac{2}{3} - 2 = 2\sqrt{3} - \frac{8}{3}.$$

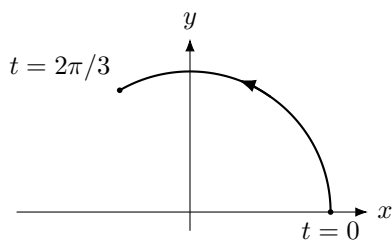
$$L_2 - L_1 = 2\sqrt{3} - \frac{8}{3}$$

Question 3

A bead moves along

$$x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \frac{2\pi}{3}.$$

Find the distance traveled.



Question 3 – Solution

Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = -3 \sin t, \quad \frac{dy}{dt} = 3 \cos t.$$

Step 2: Compute the speed.

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{9 \sin^2 t + 9 \cos^2 t} \\ &= \sqrt{9} = 3. \end{aligned}$$

Step 3: Integrate the speed.

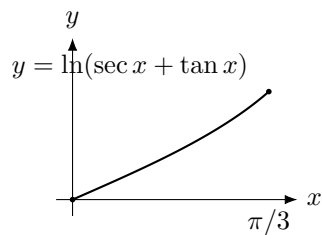
$$\begin{aligned} L &= \int_0^{2\pi/3} 3 \, dt \\ &= [3t]_0^{2\pi/3} = 2\pi. \end{aligned}$$

$L = 2\pi$

Question 4

Find the exact length of

$$y = \ln(\sec x + \tan x), \quad 0 \leq x \leq \frac{\pi}{3}.$$



Question 4 – Solution**Step 1: Differentiate.**

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Thus

$$L = \int_0^{\pi/3} \sqrt{1 + \sec^2 x} \, dx.$$

Step 2: Substitute $u = \tan x$. Then $du = \sec^2 x \, dx$ and the bounds become 0 and $\sqrt{3}$:

$$L = \int_0^{\sqrt{3}} \frac{\sqrt{u^2 + 2}}{1 + u^2} \, du.$$

Step 3: Substitute $u = \sqrt{2} \sinh t$. Then

$$du = \sqrt{2} \cosh t \, dt, \quad \sqrt{u^2 + 2} = \sqrt{2} \cosh t.$$

The upper bound is $T = \operatorname{arsinh} \sqrt{3/2}$, and

$$\begin{aligned} L &= \int_0^T \frac{2 \cosh^2 t}{1 + 2 \sinh^2 t} \, dt \\ &= \int_0^T (1 + \operatorname{sech}(2t)) \, dt \\ &= \left[t + \frac{1}{2} \arctan(\sinh 2t) \right]_0^T. \end{aligned}$$

Since $\sinh(2T) = \sqrt{15}$,

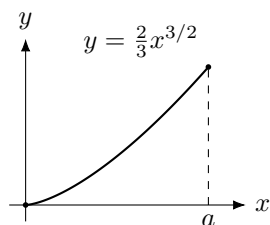
$$L = \operatorname{arsinh} \sqrt{\frac{3}{2}} + \frac{1}{2} \arctan \sqrt{15}$$

Question 5

Find $a > 0$ so that

$$y = \frac{2}{3}x^{3/2}, \quad 0 \leq x \leq a,$$

has length $14/3$.



Question 5 – Solution**Step 1: Differentiate.**

$$y' = \sqrt{x}.$$

Step 2: Write and evaluate the length.

$$\begin{aligned} L &= \int_0^a \sqrt{1+x} \, dx \\ &= \left[\frac{2}{3} (1+x)^{3/2} \right]_0^a \\ &= \frac{2}{3} \left((1+a)^{3/2} - 1 \right). \end{aligned}$$

Step 3: Set the length equal to 14/3 and solve.

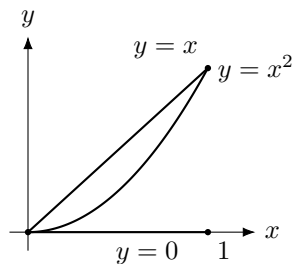
$$\begin{aligned} \frac{2}{3} \left((1+a)^{3/2} - 1 \right) &= \frac{14}{3}, \\ (1+a)^{3/2} - 1 &= 7, \\ (1+a)^{3/2} &= 8, \\ 1+a &= 8^{2/3} = 4, \\ a &= 3. \end{aligned}$$

$$\boxed{a = 3}$$

Question 6

Rank the arc lengths on $0 \leq x \leq 1$:

$$y = 0, \quad y = x, \quad y = x^2.$$



Question 6 – Solution

Step 1: Find the first two lengths.

$$L_0 = \int_0^1 \sqrt{1 + 0^2} \, dx = 1,$$

$$L_x = \int_0^1 \sqrt{1 + 1^2} \, dx = \sqrt{2} \approx 1.414.$$

Step 2: Set up the third length. For $y = x^2$, $y' = 2x$, so

$$L_{x^2} = \int_0^1 \sqrt{1 + 4x^2} \, dx.$$

Use the standard antiderivative

$$\int \sqrt{1 + 4x^2} \, dx = \frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \operatorname{arsinh}(2x).$$

Therefore,

$$\begin{aligned} L_{x^2} &= \left[\frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \operatorname{arsinh}(2x) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \operatorname{arsinh} 2 \\ &\approx 1.479. \end{aligned}$$

Step 3: Rank the values.

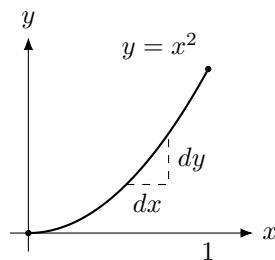
$$\boxed{L_0 < L_x < L_{x^2}}$$

Question 7

A student writes

$$L = \int_0^1 (1 + y') \, dx$$

for $y = x^2$. Explain the error and find the correct length.



Question 7 – Solution

Step 1: Correct the geometric formula. A small arc segment is the hypotenuse of a right triangle:

$$ds^2 = dx^2 + dy^2.$$

Since $dy = y' dx$,

$$ds = \sqrt{dx^2 + (y' dx)^2} = \sqrt{1 + (y')^2} dx.$$

The student's expression adds horizontal and vertical changes instead of using the Pythagorean theorem.

Step 2: Differentiate and set up the length.

$$y' = 2x, \quad L = \int_0^1 \sqrt{1 + 4x^2} dx.$$

Step 3: Evaluate.

$$\begin{aligned} L &= \left[\frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \ln(2x + \sqrt{1 + 4x^2}) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}). \end{aligned}$$

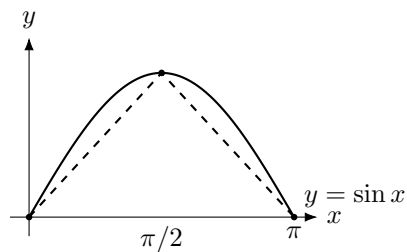
$$L = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5})$$

Question 8

Use two straight chords through $x = 0, \pi/2, \pi$ to estimate the length of

$$y = \sin x, \quad 0 \leq x \leq \pi.$$

Explain why the estimate is too small.



Question 8 – Solution

Step 1: List the three points.

$$(0, 0), \quad \left(\frac{\pi}{2}, 1\right), \quad (\pi, 0).$$

Step 2: Find the first chord length.

$$\begin{aligned} d_1 &= \sqrt{\left(\frac{\pi}{2} - 0\right)^2 + (1 - 0)^2} \\ &= \sqrt{\frac{\pi^2}{4} + 1}. \end{aligned}$$

By symmetry, $d_2 = d_1$. Thus

$$L_{\text{chord}} = 2d_1 = 2\sqrt{\frac{\pi^2}{4} + 1} = \sqrt{\pi^2 + 4}.$$

Step 3: Compare with the true length. The true arc length is

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx.$$

A curved arc between two points is longer than its straight chord. Since the sine curve is not straight on either half,

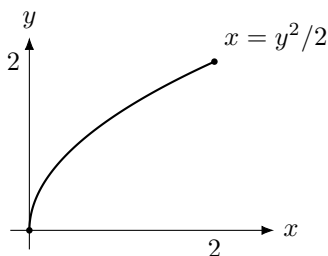
$L_{\text{chord}} = \sqrt{\pi^2 + 4}, \quad L > L_{\text{chord}}$

Question 9

For

$$x = \frac{y^2}{2}, \quad 0 \leq y \leq 2,$$

find the length using y . Then write the equivalent x -integral without evaluating it.



Question 9 – Solution

Step 1: Differentiate with respect to y .

$$\frac{dx}{dy} = y.$$

Step 2: Set up and evaluate the y -integral.

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^2 \sqrt{1 + y^2} dy \\ &= \left[\frac{y}{2} \sqrt{1 + y^2} + \frac{1}{2} \operatorname{arsinh} y \right]_0^2 \\ &= \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2. \end{aligned}$$

$$L = \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2$$

Step 3: Rewrite using x . Since $y = \sqrt{2x}$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}.$$

The x -values run from 0 to 2, so

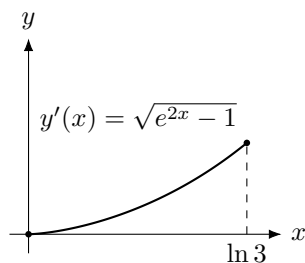
$$L = \int_0^2 \sqrt{1 + \frac{1}{2x}} dx .$$

Question 10

A curve satisfies

$$y'(x) = \sqrt{e^{2x} - 1}, \quad 0 \leq x \leq \ln 3.$$

Find its length. A formula for y is not needed.



Question 10 – Solution

Step 1: Substitute the derivative into the arc-length formula.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\sqrt{e^{2x} - 1}\right)^2 \\ &= 1 + e^{2x} - 1 = e^{2x}. \end{aligned}$$

Step 2: Simplify the square root. Since $e^x > 0$,

$$\sqrt{1 + (y')^2} = \sqrt{e^{2x}} = e^x.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} L &= \int_0^{\ln 3} e^x \, dx \\ &= [e^x]_0^{\ln 3} \\ &= e^{\ln 3} - e^0 = 3 - 1 = 2. \end{aligned}$$

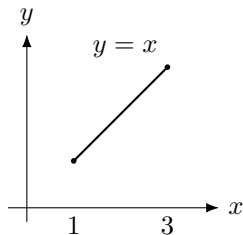
$$\boxed{L = 2}$$

Question 1

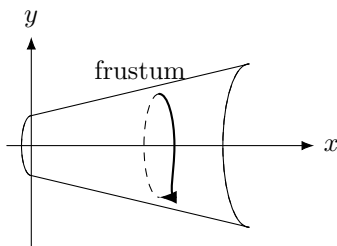
Curve: $y = x$ on $1 \leq x \leq 3$.

Axis of rotation: the x -axis.

Task: Find the lateral surface area, then verify the result with the frustum formula.



Question 1 – Solution



Step 1: Differentiate and find ds .

$$y' = 1, \quad ds = \sqrt{1 + (y')^2} dx = \sqrt{2} dx.$$

Step 2: Apply the surface-area formula.

$$\begin{aligned} S &= 2\pi \int_1^3 y ds \\ &= 2\pi \int_1^3 x\sqrt{2} dx \\ &= 2\pi\sqrt{2} \left[\frac{x^2}{2} \right]_1^3 \\ &= \pi\sqrt{2}(9 - 1) = 8\pi\sqrt{2}. \end{aligned}$$

Step 3: Verify with the frustum formula. The radii are $r_1 = 1$ and $r_2 = 3$. The slant height is

$$s = \sqrt{(3 - 1)^2 + (3 - 1)^2} = 2\sqrt{2}.$$

Therefore,

$$S = \pi(r_1 + r_2)s = \pi(1 + 3)(2\sqrt{2}) = 8\pi\sqrt{2}.$$

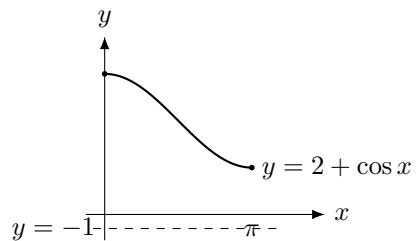
$$\boxed{S = 8\pi\sqrt{2}}$$

Question 2

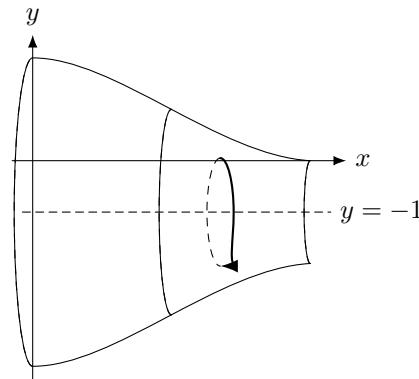
Curve: $y = 2 + \cos x$ on $0 \leq x \leq \pi$.

Axis of rotation: $y = -1$.

Task: Set up, but do not evaluate, the surface-area integral.



Question 2 – Solution



Step 1: Find the radius. The radius is the vertical distance from the curve to the axis $y = -1$:

$$r = (2 + \cos x) - (-1) = 3 + \cos x.$$

Step 2: Find the arc-length factor.

$$y' = -\sin x, \quad ds = \sqrt{1 + \sin^2 x} \, dx.$$

Step 3: Substitute into $S = 2\pi \int r \, ds$.

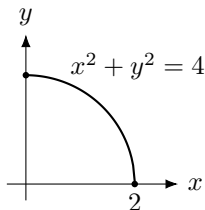
$$S = 2\pi \int_0^\pi (3 + \cos x) \sqrt{1 + \sin^2 x} \, dx$$

Question 3

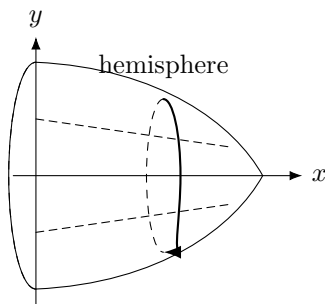
Curve: $y = \sqrt{4 - x^2}$ on $0 \leq x \leq 2$.

Axis of rotation: the x -axis.

Task: Identify the resulting surface and find its area.



Question 3 – Solution



Step 1: Identify the surface. The curve is the upper-right quarter of the circle $x^2 + y^2 = 4$. Rotating it about the x -axis produces a hemisphere of radius (2).

Step 2: Differentiate.

$$y' = -\frac{x}{\sqrt{4-x^2}}.$$

Step 3: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1 + \frac{x^2}{4-x^2}} \\ &= \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}.\end{aligned}$$

Step 4: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_0^2 y \sqrt{1+(y')^2} dx \\ &= 2\pi \int_0^2 \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx \\ &= 4\pi \int_0^2 dx = 8\pi.\end{aligned}$$

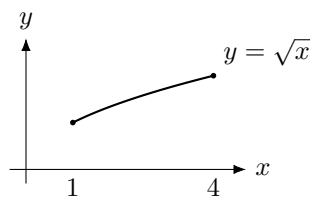
hemisphere, $S = 8\pi$

Question 4

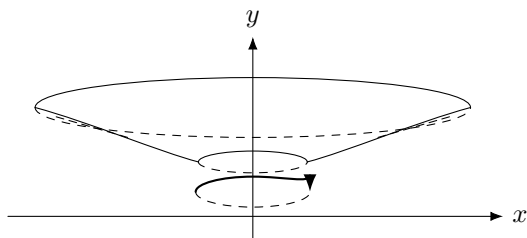
Curve: $y = \sqrt{x}$ on $1 \leq x \leq 4$.

Axis of rotation: the y -axis.

Task: Find the exact surface area.



Question 4 – Solution



Step 1: Find the radius and derivative. Rotation about the y -axis gives radius $r = x$, and

$$y' = \frac{1}{2\sqrt{x}}.$$

Step 2: Set up and simplify.

$$\begin{aligned} S &= 2\pi \int_1^4 x \sqrt{1 + \frac{1}{4x}} \, dx \\ &= \pi \int_1^4 \sqrt{x(4x+1)} \, dx. \end{aligned}$$

Step 3: Substitute $u = 8x + 1$. Then

$$dx = \frac{du}{8}, \quad \sqrt{x(4x+1)} = \frac{1}{4} \sqrt{u^2 - 1}.$$

Thus

$$S = \frac{\pi}{32} \int_9^{33} \sqrt{u^2 - 1} \, du.$$

Step 4: Integrate and return to x .

$$\int \sqrt{u^2 - 1} \, du = \frac{1}{2} \left(u \sqrt{u^2 - 1} - \ln(u + \sqrt{u^2 - 1}) \right).$$

Therefore,

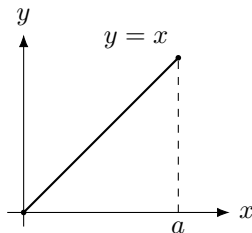
$$S = \pi \left[\frac{(8x+1)\sqrt{x(4x+1)}}{16} - \frac{1}{64} \ln(8x+1+4\sqrt{x(4x+1)}) \right]_1^4$$

Question 5

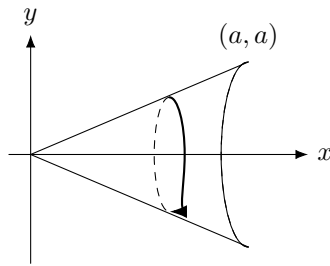
Curve: $y = x$ on $0 \leq x \leq a$, where $a > 0$.

Axis of rotation: the x -axis.

Task: Find a so that the generated surface area is $9\pi\sqrt{2}$.



Question 5 – Solution



Step 1: Find the arc-length factor.

$$y' = 1, \quad \sqrt{1 + (y')^2} = \sqrt{2}.$$

Step 2: Write and evaluate the area.

$$\begin{aligned} S &= 2\pi \int_0^a y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_0^a x \sqrt{2} dx \\ &= 2\pi \sqrt{2} \left[\frac{x^2}{2} \right]_0^a = \pi \sqrt{2} a^2. \end{aligned}$$

Step 3: Set the area equal to $9\pi\sqrt{2}$ and solve.

$$\begin{aligned} \pi \sqrt{2} a^2 &= 9\pi \sqrt{2}, \\ a^2 &= 9, \\ a &= 3 \end{aligned}$$

because $a > 0$.

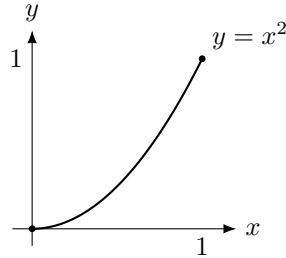
$$\boxed{a = 3}$$

Question 6

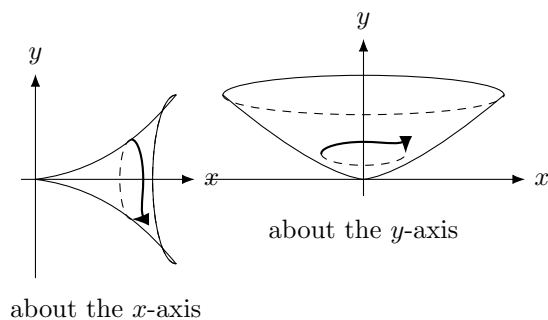
Curve: $y = x^2$ on $0 \leq x \leq 1$.

Axes of rotation: first the x -axis, then the y -axis.

Task: Write both surface-area integrals and determine which is larger without evaluating either integral.



Question 6 – Solution



Step 1: Find the common arc-length factor.

$$y' = 2x, \quad ds = \sqrt{1 + 4x^2} \, dx.$$

Step 2: Rotate about the x -axis. The radius is $y = x^2$:

$$S_x = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx.$$

Step 3: Rotate about the y -axis. The radius is x :

$$S_y = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx.$$

Step 4: Compare the integrands. For $0 < x < 1$,

$$x > x^2.$$

Multiplying by the positive common factor $2\pi\sqrt{1 + 4x^2}$ preserves the inequality. Therefore,

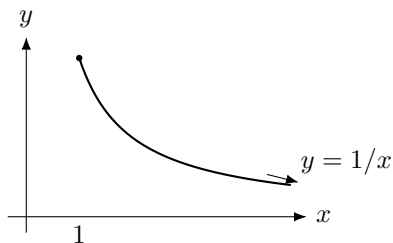
$$\boxed{S_y > S_x}.$$

Question 7

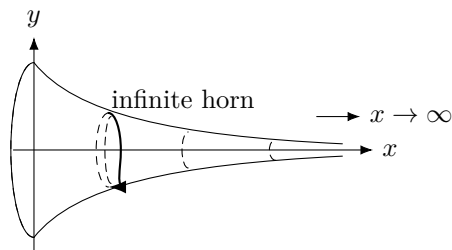
Curve: $y = 1/x$ on $x \geq 1$.

Axis of rotation: the x -axis.

Task: Determine whether the generated surface has finite area.



Question 7 – Solution



Step 1: Differentiate and set up the improper integral.

$$y' = -\frac{1}{x^2}.$$

Thus

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Step 2: Find a lower comparison. Since

$$\sqrt{1 + x^{-4}} > 1,$$

we have

$$\frac{1}{x} \sqrt{1 + x^{-4}} > \frac{1}{x}.$$

Step 3: Test the comparison integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

By direct comparison, the surface-area integral diverges.

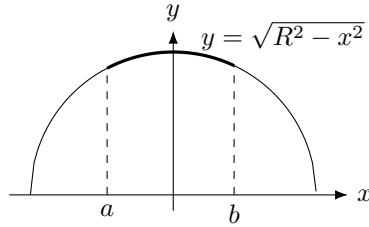
$S = \infty$ (infinite surface area)

Question 8

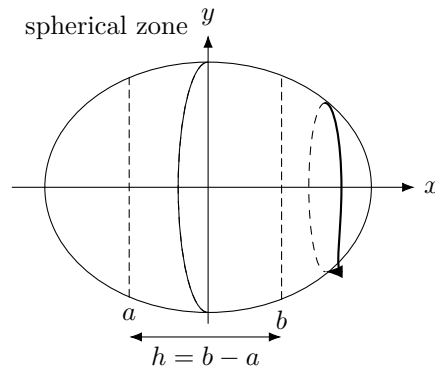
Curve: $y = \sqrt{R^2 - x^2}$ on $a \leq x \leq b$, where $R > 0$ and $-R \leq a < b \leq R$.

Axis of rotation: the x -axis.

Task: For fixed R , show that the surface area depends only on the zone height $h = b - a$.



Question 8 – Solution



Step 1: Differentiate.

$$y' = -\frac{x}{\sqrt{R^2 - x^2}}.$$

Step 2: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{R^2 - x^2}} \\ &= \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.\end{aligned}$$

Step 3: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_a^b y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_a^b \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}} dx \\ &= 2\pi R \int_a^b dx \\ &= 2\pi R(b - a) = 2\pi Rh.\end{aligned}$$

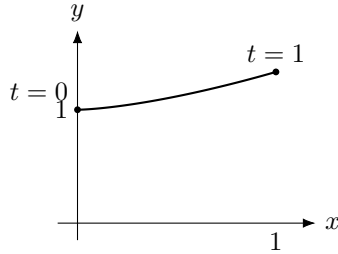
$$\boxed{S = 2\pi R(b - a) = 2\pi Rh}$$

Question 9

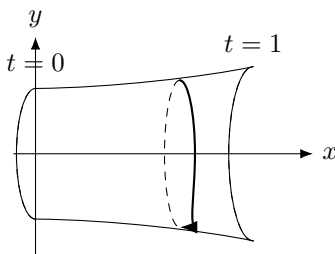
Parametric curve: $x = t^2$, $y = 1 + t^3/3$, for $0 \leq t \leq 1$.

Axis of rotation: the x -axis.

Task: Set up, but do not evaluate, the surface-area integral.



Question 9 – Solution



Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2.$$

Step 2: Find the parametric arc-length factor.

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \sqrt{4t^2 + t^4} dt \\ &= t\sqrt{4 + t^2} dt \end{aligned}$$

because $t \geq 0$.

Step 3: Use radius $y = 1 + t^3/3$.

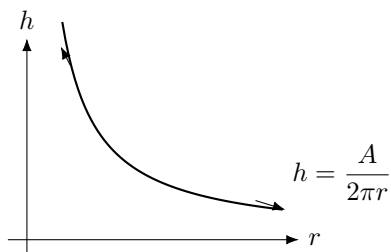
$$\begin{aligned} S &= 2\pi \int_0^1 y \, ds \\ &= 2\pi \int_0^1 \left(1 + \frac{t^3}{3}\right) t\sqrt{4 + t^2} \, dt \\ &= 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt. \end{aligned}$$

$$\boxed{S = 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt}$$

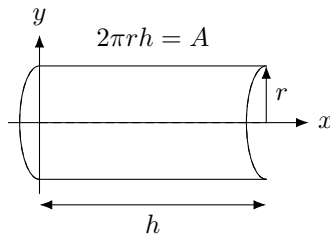
Question 10

Given: A cylinder has radius r , height h , and fixed lateral area A .

Tasks: Solve for h , find the total area including both circular ends, and then analyze the limit as $r \rightarrow 0^+$.



Question 10 – Solution



Step 1: Solve the lateral-area equation for h .

$$2\pi r h = A,$$

$$h = \frac{A}{2\pi r}.$$

Step 2: Add the two circular ends. Each end has area πr^2 , so

$$S_{\text{total}} = A + 2\pi r^2.$$

Step 3: Take the limit as $r \rightarrow 0^+$.

$$\lim_{r \rightarrow 0^+} S_{\text{total}} = \lim_{r \rightarrow 0^+} (A + 2\pi r^2) = A,$$

$$\lim_{r \rightarrow 0^+} h = \lim_{r \rightarrow 0^+} \frac{A}{2\pi r} = \infty.$$

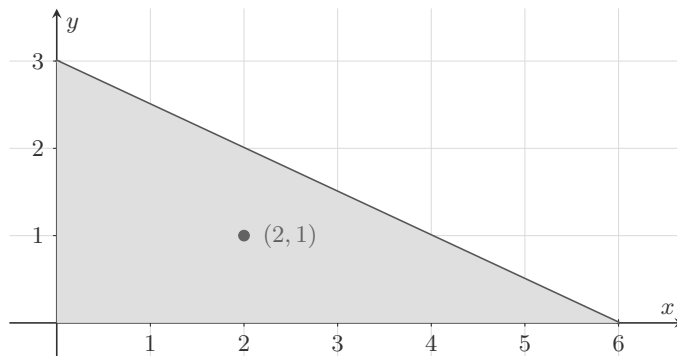
$S_{\text{total}} \rightarrow A, \quad h \rightarrow \infty$
--

Question 1

A triangular sign occupies $x \geq 0, y \geq 0, x + 2y \leq 6$. Find its centroid using integrals.



Question 1 – Solution



Step 1: Write the upper boundary as a function of x . From $x + 2y = 6$,

$$y = f(x) = 3 - \frac{x}{2}, \quad 0 \leq x \leq 6.$$

Step 2: Find the area (and hence the mass for unit density).

$$\begin{aligned} A &= \int_0^6 f(x) dx = \int_0^6 \left(3 - \frac{x}{2}\right) dx \\ &= \left[3x - \frac{x^2}{4}\right]_0^6 = 18 - 9 = 9. \end{aligned}$$

Step 3: Find the moment about the y -axis.

$$\begin{aligned} M_y &= \int_0^6 x f(x) dx = \int_0^6 \left(3x - \frac{x^2}{2}\right) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{6}\right]_0^6 = 54 - 36 = 18. \end{aligned}$$

Step 4: Find the moment about the x -axis. Each vertical strip has centroid height $f(x)/2$, so

$$M_x = \frac{1}{2} \int_0^6 f(x)^2 dx = \frac{1}{2} \int_0^6 \left(3 - \frac{x}{2}\right)^2 dx = 9.$$

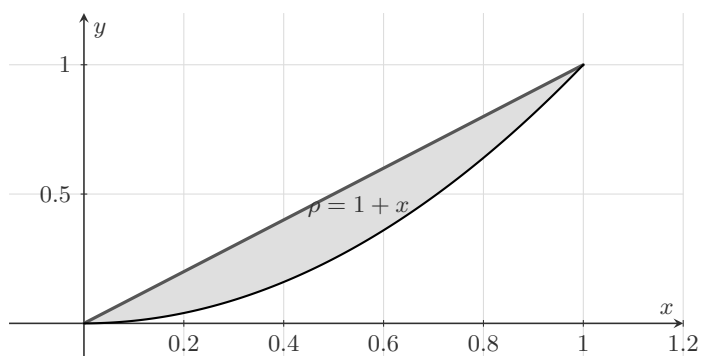
Step 5: Divide each moment by the area.

$$\bar{x} = \frac{M_y}{A} = \frac{18}{9} = 2, \quad \bar{y} = \frac{M_x}{A} = \frac{9}{9} = 1.$$

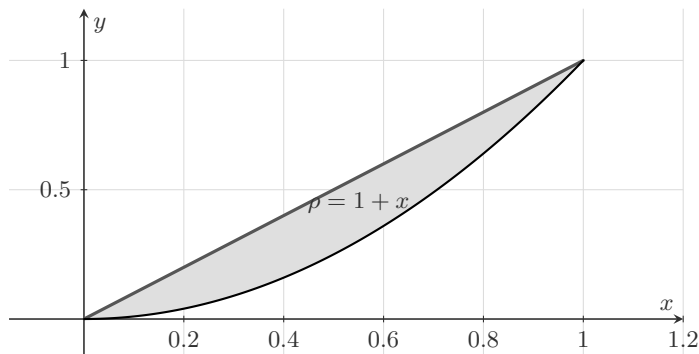
$$\boxed{(\bar{x}, \bar{y}) = (2, 1)}$$

Question 2

A lamina lies between $y = x$ and $y = x^2$, $0 \leq x \leq 1$, with density $\rho(x) = 1 + x$. Find its mass and $\bar{x}$.



Question 2 – Solution



Step 1: Identify the strip dimensions. On $0 \leq x \leq 1$, the upper curve is $y = x$ and the lower curve is $y = x^2$. Thus

$$dA = (x - x^2) dx, \quad dm = \rho(x) dA = (1 + x)(x - x^2) dx.$$

Step 2: Find the mass. Since $(1 + x)(x - x^2) = x - x^3$,

$$m = \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}.$$

Step 3: Find the moment about the y -axis. The moment arm is x , so

$$\begin{aligned} M_y &= \int_0^1 x dm = \int_0^1 x(x - x^3) dx \\ &= \int_0^1 (x^2 - x^4) dx = \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{2}{15}. \end{aligned}$$

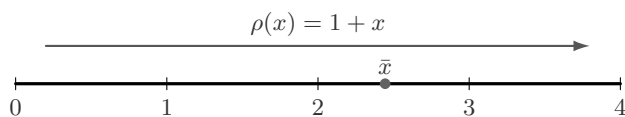
Step 4: Compute the x -coordinate of the center of mass.

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{2}{15}}{\frac{1}{4}} = \frac{8}{15}.$$

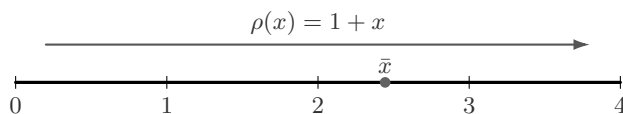
$m = \frac{1}{4}, \quad \bar{x} = \frac{8}{15}$

Question 3

A 4-m rod on $0 \leq x \leq 4$ has density $\rho(x) = k(1 + x)$. Its mass is (12) kg. Find k and its balance point.



Question 3 – Solution



Step 1: Use the total mass to determine k .

$$\begin{aligned} 12 &= \int_0^4 \rho(x) dx = k \int_0^4 (1 + x) dx \\ &= k \left[x + \frac{x^2}{2} \right]_0^4 = k(4 + 8) = 12k. \end{aligned}$$

Therefore, $k = 1$, so $\rho(x) = 1 + x$.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_0 &= \int_0^4 x\rho(x) dx = \int_0^4 (x + x^2) dx \\ &= \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_0^4 = 8 + \frac{64}{3} = \frac{88}{3}. \end{aligned}$$

Step 3: Divide the moment by the mass.

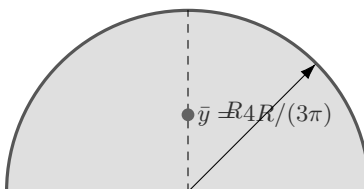
$$\bar{x} = \frac{M_0}{m} = \frac{\frac{88}{3}}{12} = \frac{22}{9} \text{ m}.$$

The balance point lies to the right of the midpoint because the density increases with x .

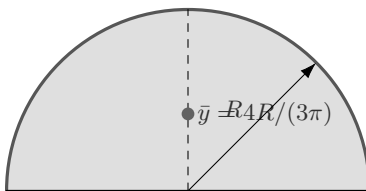
$k = 1, \quad \bar{x} = \frac{22}{9} \text{ m}$

Question 4

A uniform semicircular lamina $x^2 + y^2 \leq R^2, y \geq 0$ has $\bar{x} = 0$. Derive $\bar{y}$.



Question 4 – Solution



Step 1: Use symmetry. The lamina is symmetric about the y -axis, so

$$\bar{x} = 0.$$

Step 2: Find the area. The region is one-half of a disk of radius R :

$$A = \frac{1}{2}\pi R^2.$$

Step 3: Write the upper boundary. For $-R \leq x \leq R$,

$$f(x) = \sqrt{R^2 - x^2}.$$

A vertical strip of height $f(x)$ has centroid height $f(x)/2$.

Step 4: Find the moment about the x -axis.

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-R}^R f(x)^2 dx = \frac{1}{2} \int_{-R}^R (R^2 - x^2) dx \\ &= \frac{1}{2} \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R = \frac{2R^3}{3}. \end{aligned}$$

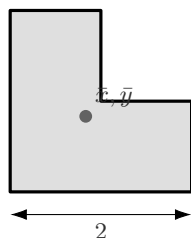
Step 5: Divide the moment by the area.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2} = \frac{4R}{3\pi}.$$

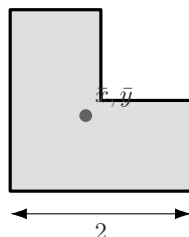
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4R}{3\pi}\right)}$$

Question 5

A uniform square $0 \leq x, y \leq 2$ has the upper-right unit square removed. Find the centroid of the remaining L-shape.



Question 5 – Solution



Step 1: Treat the removed square as negative area. The full 2×2 square has

$$A_1 = 4, \quad (x_1, y_1) = (1, 1).$$

The removed 1×1 square has

$$A_2 = 1, \quad (x_2, y_2) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Step 2: Find the remaining area.

$$A = A_1 - A_2 = 4 - 1 = 3.$$

Step 3: Subtract the moments about the y -axis.

$$M_y = A_1x_1 - A_2x_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

Step 4: Subtract the moments about the x -axis.

$$M_x = A_1y_1 - A_2y_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

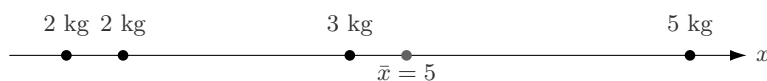
Step 5: Divide by the remaining area.

$$\bar{x} = \frac{M_y}{A} = \frac{5}{6}, \quad \bar{y} = \frac{M_x}{A} = \frac{5}{6}.$$

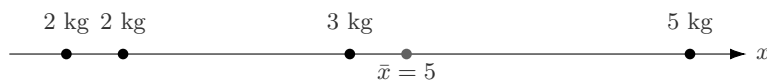
$$(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{6}\right)$$

Question 6

Masses (2,3,5) kg sit at $x = 0, 4, 10$. Where must a (2)-kg mass be placed so the center is $x = 5$?



Question 6 – Solution



Step 1: Let a be the location of the new mass. The four masses are 2, 3, 5, and 2 kg, so the total mass is

$$m = 2 + 3 + 5 + 2 = 12 \text{ kg.}$$

Step 2: Write the center-of-mass equation. For point masses on a line,

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}.$$

Requiring $\bar{x} = 5$ gives

$$5 = \frac{2(0) + 3(4) + 5(10) + 2a}{12}.$$

Step 3: Solve for a .

$$60 = 0 + 12 + 50 + 2a,$$

$$60 = 62 + 2a,$$

$$2a = -2,$$

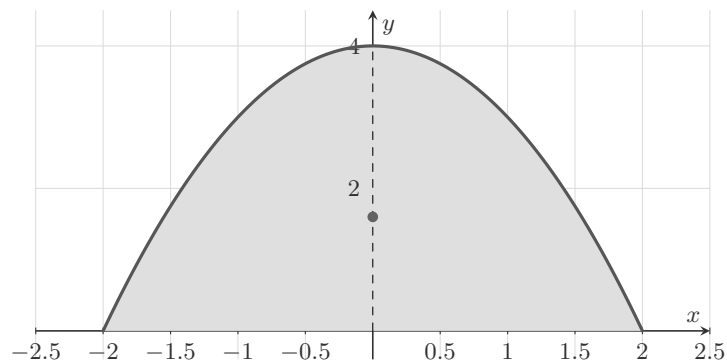
$$a = -1.$$

Step 4: Check. The total moment is $62 + 2(-1) = 60$, and $60/12 = 5$, as required.

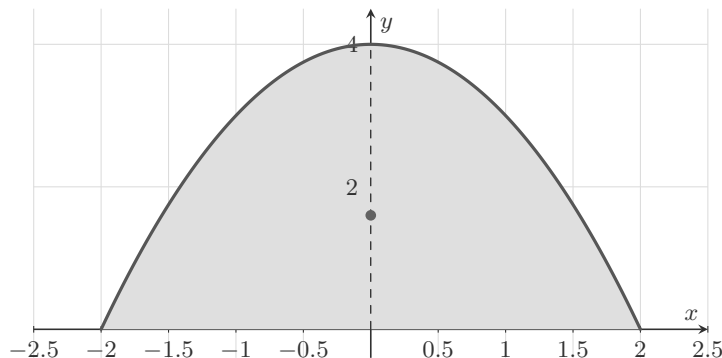
Place the 2-kg mass at $x = -1$.

Question 7

A plate under $y = 4 - x^2$, $-2 \leq x \leq 2$, has uniform density. Use symmetry before integration; find its centroid.



Question 7 – Solution



Step 1: Use symmetry before integrating. Both the region and its uniform density are symmetric about the y -axis. Therefore,

$$\bar{x} = 0.$$

Step 2: Find the area.

$$\begin{aligned} A &= \int_{-2}^2 (4 - x^2) dx = 2 \int_0^2 (4 - x^2) dx \\ &= 2 \left[4x - \frac{x^3}{3} \right]_0^2 = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}. \end{aligned}$$

Step 3: Find the moment about the x -axis. For a vertical strip under $f(x) = 4 - x^2$,

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-2}^2 f(x)^2 dx = \int_0^2 (4 - x^2)^2 dx \\ &= \int_0^2 (16 - 8x^2 + x^4) dx \\ &= \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2 = \frac{256}{15}. \end{aligned}$$

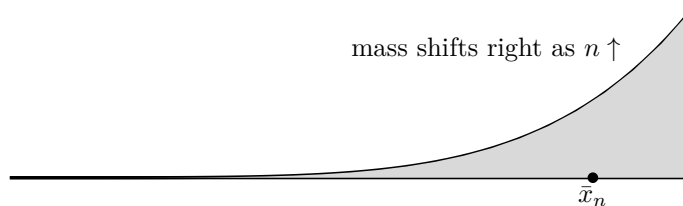
Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{256}{15}}{\frac{32}{3}} = \frac{8}{5}.$$

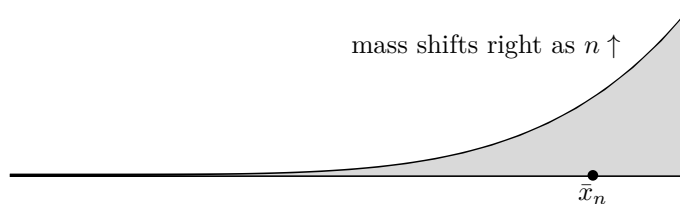
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)}$$

Question 8

A rod on $[0, 1]$ has density $\rho_n(x) = (n+1)x^n$, for integers $n \geq 0$. Find $\bar{x}_n$ and its limiting location as $n \rightarrow \infty$.



Question 8 – Solution



Step 1: Find the total mass. For $n \geq 0$,

$$\begin{aligned} m_n &= \int_0^1 \rho_n(x) dx = \int_0^1 (n+1)x^n dx \\ &= (n+1) \left[\frac{x^{n+1}}{n+1} \right]_0^1 = 1. \end{aligned}$$

Thus the density is normalized to have total mass 1.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_n &= \int_0^1 x \rho_n(x) dx = (n+1) \int_0^1 x^{n+1} dx \\ &= (n+1) \left[\frac{x^{n+2}}{n+2} \right]_0^1 = \frac{n+1}{n+2}. \end{aligned}$$

Step 3: Compute the center of mass.

$$\bar{x}_n = \frac{M_n}{m_n} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2}.$$

Step 4: Take the limit.

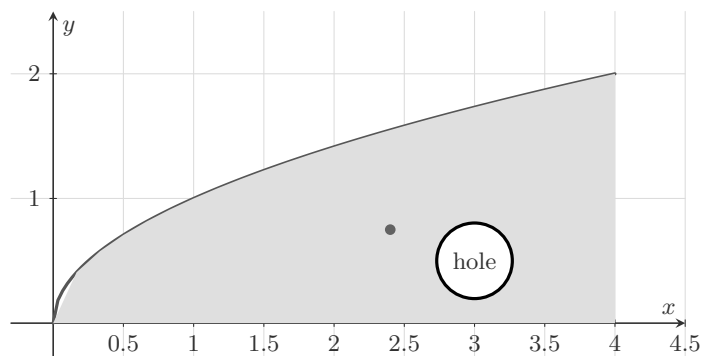
$$\lim_{n \rightarrow \infty} \bar{x}_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+2} = 1.$$

The density becomes increasingly concentrated near the right endpoint.

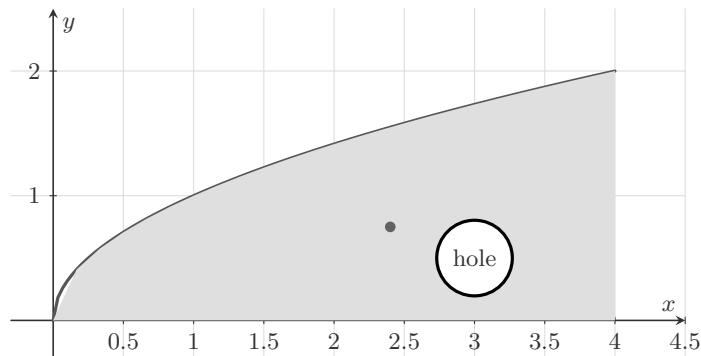
$\bar{x}_n = \frac{n+1}{n+2}, \quad \bar{x}_n \rightarrow 1$
--

Question 9

A lamina is bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$. A circular hole of radius $(1/2)$, centered at $((3, 1/2))$, is removed. Find the new centroid using subtraction.



Question 9 – Solution



Step 1: Find the area of the original region.

$$A_0 = \int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{16}{3}.$$

Step 2: Find the original moments.

$$M_{y,0} = \int_0^4 x \sqrt{x} \, dx = \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{64}{5},$$

$$M_{x,0} = \frac{1}{2} \int_0^4 (\sqrt{x})^2 \, dx = \frac{1}{2} \int_0^4 x \, dx = 4.$$

Hence the original centroid is

$$(\bar{x}_0, \bar{y}_0) = \left(\frac{M_{y,0}}{A_0}, \frac{M_{x,0}}{A_0} \right) = \left(\frac{12}{5}, \frac{3}{4} \right).$$

Step 3: Record the hole's area and moments. The hole has radius $1/2$ and centroid $(3, 1/2)$, so

$$A_h = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4}, \quad M_{y,h} = 3 \left(\frac{\pi}{4} \right) = \frac{3\pi}{4}, \quad M_{x,h} = \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}.$$

Step 4: Subtract the hole.

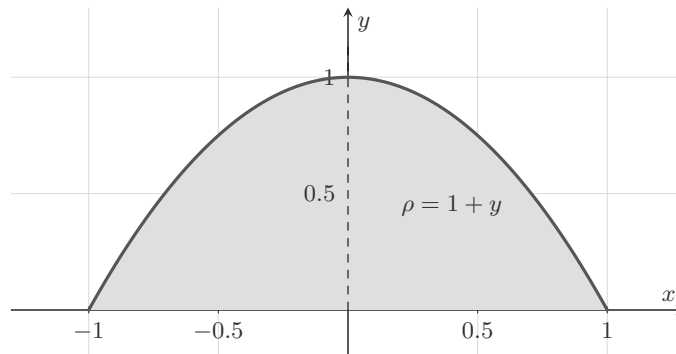
$$A = \frac{16}{3} - \frac{\pi}{4}, \quad M_y = \frac{64}{5} - \frac{3\pi}{4}, \quad M_x = 4 - \frac{\pi}{8}.$$

Step 5: Divide the remaining moments by the remaining area.

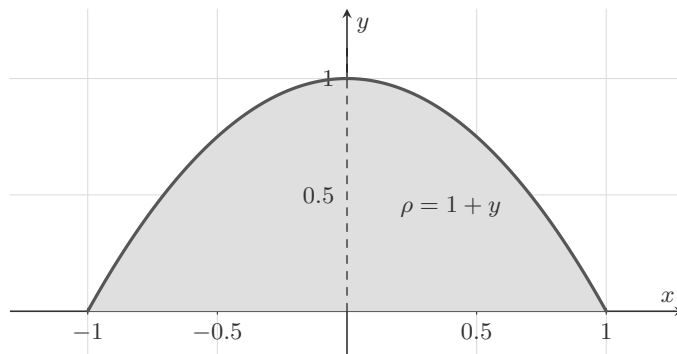
$$\boxed{\bar{x} = \frac{\frac{64}{5} - \frac{3\pi}{4}}{\frac{16}{3} - \frac{\pi}{4}}, \quad \bar{y} = \frac{4 - \frac{\pi}{8}}{\frac{16}{3} - \frac{\pi}{4}}}$$

Question 10

A symmetric lamina has density $\rho(x, y) = 1 + y$ on $-1 \leq x \leq 1, 0 \leq y \leq 1 - x^2$. State $\bar{x}$, then set up exact integrals for m and $\bar{y}$.



Question 10 – Solution



Step 1: Use symmetry. The region is symmetric about the y -axis, and $\rho(x, y) = 1 + y$ is unchanged when x is replaced by $-x$. Therefore,

$$\boxed{\bar{x} = 0}.$$

Step 2: Set up and evaluate the mass. Let $h = 1 - x^2$. Then

$$\begin{aligned} m &= \int_{-1}^1 \int_0^{1-x^2} (1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[h + \frac{h^2}{2} \right] dx \\ &= \int_{-1}^1 \left[(1-x^2) + \frac{1}{2}(1-x^2)^2 \right] dx = \frac{28}{15}. \end{aligned}$$

Step 3: Set up and evaluate the moment about the x -axis.

$$\begin{aligned} M_x &= \int_{-1}^1 \int_0^{1-x^2} y(1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[\frac{h^2}{2} + \frac{h^3}{3} \right] dx \\ &= \frac{1}{2} \int_{-1}^1 (1-x^2)^2 dx + \frac{1}{3} \int_{-1}^1 (1-x^2)^3 dx \\ &= \frac{8}{15} + \frac{32}{105} = \frac{88}{105}. \end{aligned}$$

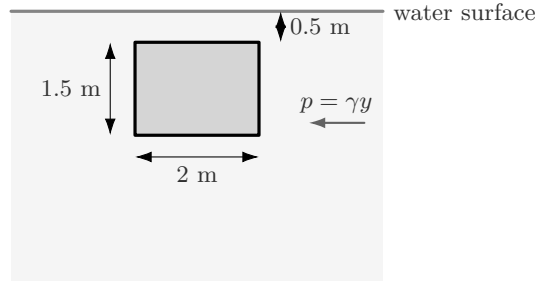
Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{88}{105}}{\frac{28}{15}} = \frac{22}{49}.$$

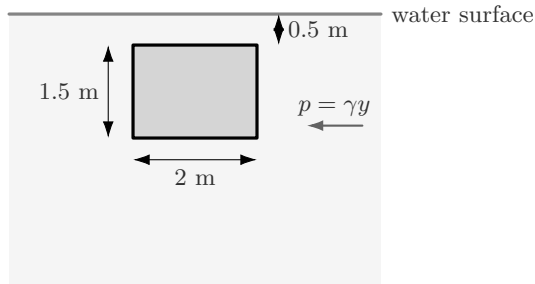
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{22}{49} \right)}$$

Question 1

A vertical aquarium window is (2) m wide and (1.5) m high; its top is (0.5) m below freshwater. Find the force ($\gamma = 9800 \text{ N/m}^3$).



Question 1 – Solution



Step 1: Choose the depth variable. Let y be the depth below the water surface, measured in meters. The window extends from

$$y = 0.5 \quad \text{to} \quad y = 0.5 + 1.5 = 2.$$

Step 2: Write the pressure at depth y . Freshwater has weight density $\gamma = 9800 \text{ N/m}^3$, so

$$p(y) = \gamma y = 9800y \quad \text{N/m}^2.$$

Step 3: Find the area of a horizontal strip. The window is 2 m wide. A strip of thickness dy has area

$$dA = 2 \, dy.$$

Therefore,

$$dF = p(y) \, dA = (9800y)(2 \, dy).$$

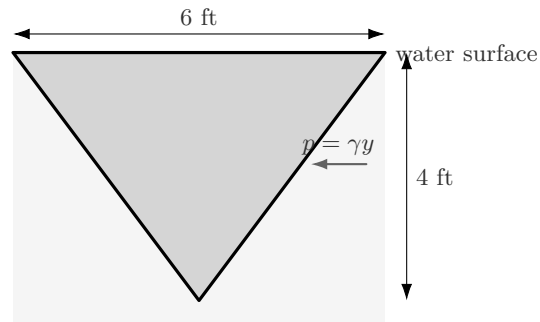
Step 4: Integrate over the window.

$$\begin{aligned} F &= 9800 \int_{0.5}^2 2y \, dy = 9800 [y^2]_{0.5}^2 \\ &= 9800(4 - 0.25) = 9800(3.75) = 36,750 \text{ N}. \end{aligned}$$

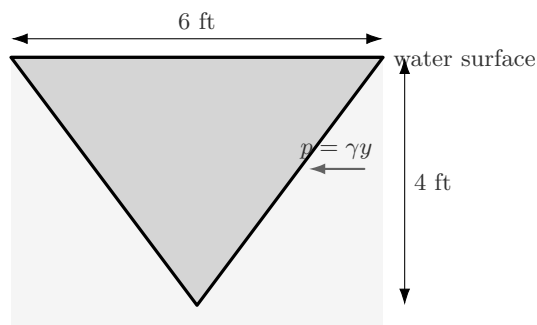
$$\boxed{F = 36,750 \text{ N}}$$

Question 2

An inverted triangular gate has top width (6) ft at the surface and vertex (4) ft deep. Find the force ($\gamma = 62.4 \text{ lb/ft}^3$).



Question 2 – Solution



Step 1: Choose the depth variable. Let y be depth below the water surface. The gate extends from $y = 0$ to $y = 4$ ft.

Step 2: Find the width of the gate at depth y . The width decreases linearly from 6 ft at $y = 0$ to 0 ft at $y = 4$. Its slope is

$$\frac{0 - 6}{4 - 0} = -\frac{3}{2},$$

so

$$w(y) = 6 - \frac{3}{2}y.$$

Step 3: Write the force on a horizontal strip. At depth y ,

$$p(y) = 62.4y \text{ lb/ft}^2, \quad dA = w(y) dy.$$

Thus

$$dF = 62.4y \left(6 - \frac{3}{2}y \right) dy.$$

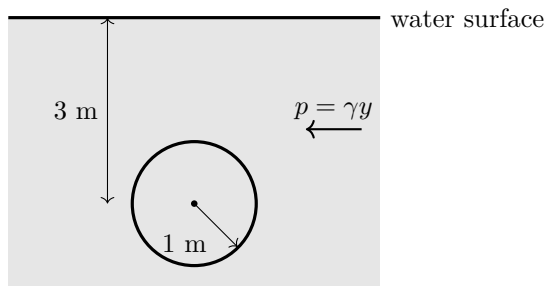
Step 4: Integrate and evaluate.

$$\begin{aligned} F &= 62.4 \int_0^4 y \left(6 - \frac{3}{2}y \right) dy \\ &= 62.4 \int_0^4 \left(6y - \frac{3}{2}y^2 \right) dy \\ &= 62.4 \left[3y^2 - \frac{1}{2}y^3 \right]_0^4 \\ &= 62.4(48 - 32) = 62.4(16) = 998.4 \text{ lb.} \end{aligned}$$

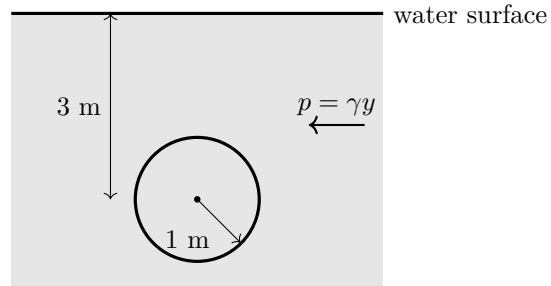
$$\boxed{F = 998.4 \text{ lb}}$$

Question 3

A vertical circular porthole of radius 1 m has center 3 m below the water surface. Use water weight density 9800 N/m^3 . Find the force without integrating strips.



Question 3 – Solution



Step 1: Use the centroid form of the hydrostatic-force formula. For a submerged vertical plane region,

$$F = \gamma A \bar{y},$$

where A is its area and $\bar{y}$ is the depth of its centroid.

Step 2: Find the area of the porthole. The radius is 1 m, so

$$A = \pi r^2 = \pi(1)^2 = \pi \text{ m}^2.$$

Step 3: Identify the centroid depth. The centroid of a circle is its center, which is 3 m below the surface. Therefore,

$$\bar{y} = 3 \text{ m}.$$

Step 4: Substitute.

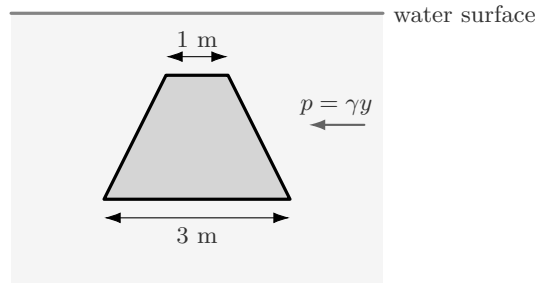
$$\begin{aligned} F &= (9800)(\pi)(3) \\ &= 29,400\pi \text{ N}. \end{aligned}$$

$$\boxed{F = 29,400\pi \text{ N}}$$

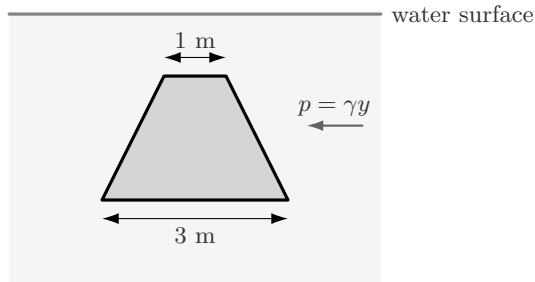
No strip integration is needed because the area and centroid depth are known.

Question 4

Oil has weight density 8000 N/m^3 . A trapezoidal plate is (1) m wide at its top, (3) m wide at its bottom, (2) m high, with top depth (1) m. Find the force.



Question 4 – Solution



Step 1: Choose the depth variable. Let y be depth below the oil surface. The plate begins at $y = 1$ and ends at $y = 3$ m.

Step 2: Find the width at depth y . The width grows linearly from 1 m at $y = 1$ to 3 m at $y = 3$. The slope is

$$\frac{3 - 1}{3 - 1} = 1,$$

so the linear width function is

$$w(y) = y, \quad 1 \leq y \leq 3.$$

Step 3: Write the force on a horizontal strip.

$$p(y) = 8000y, \quad dA = w(y) dy = y dy.$$

Hence

$$dF = p(y) dA = 8000y^2 dy.$$

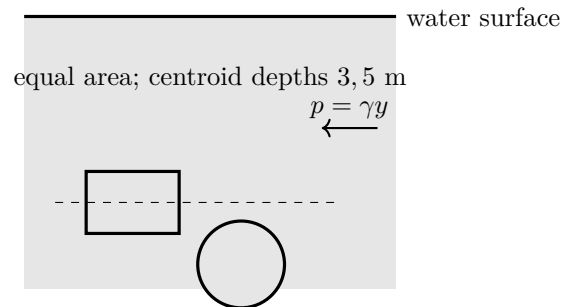
Step 4: Integrate and evaluate.

$$\begin{aligned} F &= 8000 \int_1^3 y^2 dy = 8000 \left[\frac{y^3}{3} \right]_1^3 \\ &= \frac{8000}{3} (27 - 1) = \frac{208000}{3} \text{ N}. \end{aligned}$$

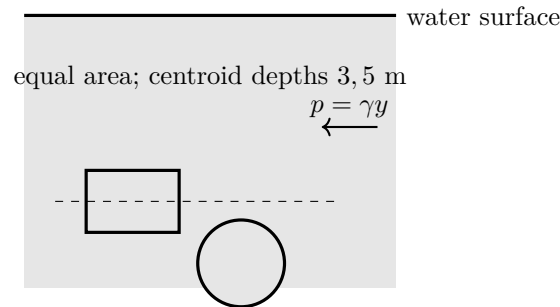
$$F = \frac{208000}{3} \text{ N} \approx 69,333.3 \text{ N}$$

Question 5

Two equal-area vertical plates are fully submerged in the same liquid of constant weight density, with centroid depths 3 m and 5 m. Compare their hydrostatic forces without knowing their shapes.



Question 5 – Solution



Step 1: Use the centroid formula. For any vertical plane region submerged in a liquid of constant weight density,

$$F = \gamma A \bar{y}.$$

Step 2: Write the force on each plate. Let the common area be A . If Plate 1 has centroid depth 3 m and Plate 2 has centroid depth 5 m, then

$$F_1 = \gamma A(3), \quad F_2 = \gamma A(5).$$

Step 3: Compare the forces.

$$\frac{F_2}{F_1} = \frac{\gamma A(5)}{\gamma A(3)} = \frac{5}{3}.$$

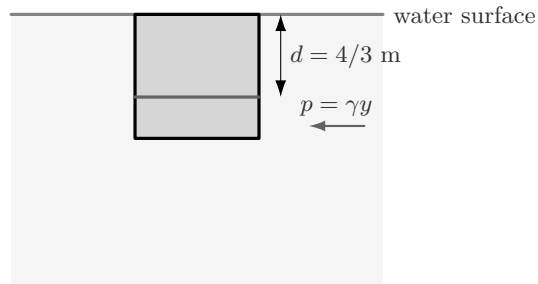
The shape does not affect this comparison because both plates have the same area; only the centroid depth differs.

$$F_2 = \frac{5}{3} F_1$$

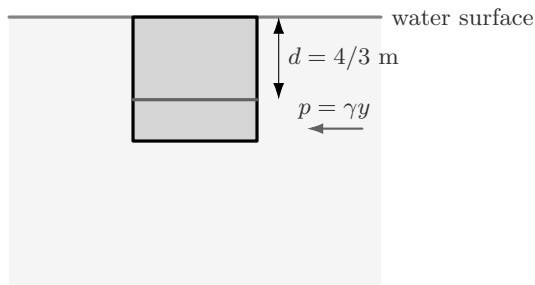
Thus the deeper plate experiences $66\frac{2}{3}\%$ more force.

Question 6

A 2×2 m square gate has its top at the surface. Find the depth d of the center of pressure, where $d = M/F$.



Question 6 – Solution



Step 1: Set up the hydrostatic force. Let y be depth below the surface. The gate has constant width 2 m and extends from $y = 0$ to $y = 2$. Thus

$$dF = (9800y)(2 dy).$$

Therefore,

$$\begin{aligned} F &= 9800 \int_0^2 2y \, dy = 19600 \left[\frac{y^2}{2} \right]_0^2 \\ &= 39,200 \text{ N.} \end{aligned}$$

Step 2: Find the moment of the pressure force about the surface. The moment arm of the strip force is y , so

$$\begin{aligned} M &= \int_0^2 y \, dF = 9800 \int_0^2 2y^2 \, dy \\ &= 19600 \left[\frac{y^3}{3} \right]_0^2 = \frac{156,800}{3} \text{ N m.} \end{aligned}$$

Step 3: Divide moment by total force.

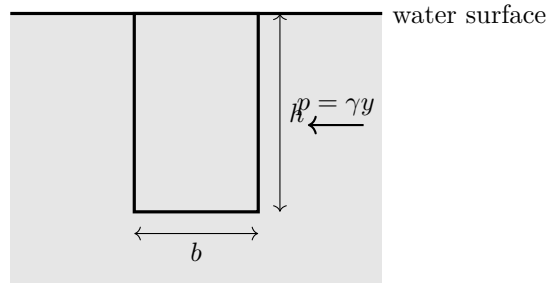
$$d = \frac{M}{F} = \frac{\frac{156800}{3}}{39200} = \frac{4}{3} \text{ m.}$$

$$d = \frac{4}{3} \text{ m below the surface}$$

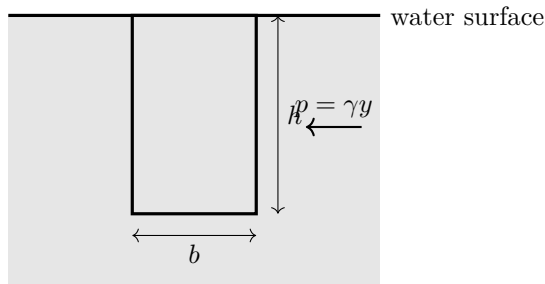
The center of pressure lies below the geometric center at depth 1 m because pressure increases with depth.

Question 7

A rectangular dam face is b meters wide and extends from the surface to depth h . Derive $F(h)$. If h doubles, by what factor does F change?



Question 7 – Solution



Step 1: Choose the depth variable. Let y measure depth below the surface. The dam face extends from $y = 0$ to $y = h$.

Step 2: Write the force on a horizontal strip. The pressure is $p(y) = \gamma y$, and a strip of width b and thickness dy has area $dA = b dy$. Therefore,

$$dF = p(y) dA = \gamma b y dy.$$

Step 3: Integrate from the surface to depth h .

$$\begin{aligned} F(h) &= \gamma b \int_0^h y dy \\ &= \gamma b \left[\frac{y^2}{2} \right]_0^h = \frac{1}{2} \gamma b h^2. \end{aligned}$$

$$F(h) = \frac{1}{2} \gamma b h^2$$

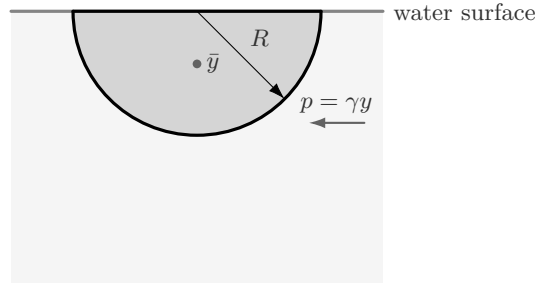
Step 4: Determine the effect of doubling the depth.

$$F(2h) = \frac{1}{2} \gamma b (2h)^2 = 4 \left(\frac{1}{2} \gamma b h^2 \right) = 4F(h).$$

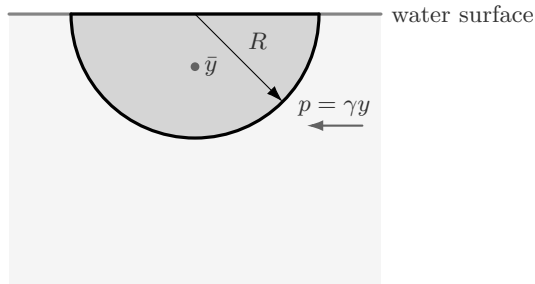
Doubling h multiplies the force by 4.

Question 8

A semicircular gate of radius R has its diameter along the water surface and lies below it. Find the force using centroid depth $4R/(3\pi)$.



Question 8 – Solution



Step 1: Use the centroid form of hydrostatic force. For a submerged vertical plane region,

$$F = \gamma A \bar{y}.$$

Step 2: Find the area of the semicircle.

$$A = \frac{1}{2}\pi R^2.$$

Step 3: Use the given centroid depth. Because the diameter lies on the water surface and the semicircle lies below it,

$$\bar{y} = \frac{4R}{3\pi}.$$

Step 4: Substitute and simplify.

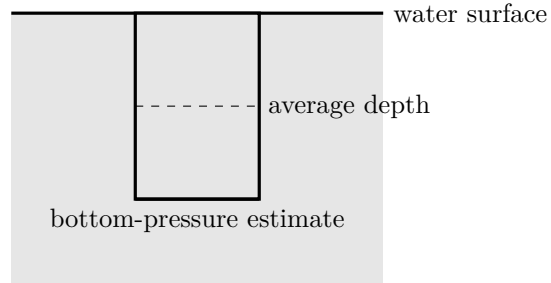
$$\begin{aligned} F &= \gamma \left(\frac{1}{2}\pi R^2 \right) \left(\frac{4R}{3\pi} \right) \\ &= \gamma \left(\frac{4R^3}{6} \right) = \frac{2}{3}\gamma R^3. \end{aligned}$$

$$\boxed{F = \frac{2}{3}\gamma R^3}$$

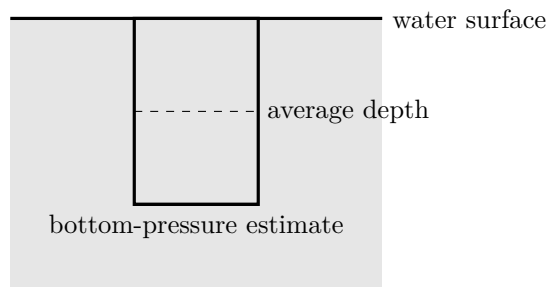
The factors of π cancel because the semicircle's area contains π while its centroid depth contains $1/\pi$.

Question 9

A student uses constant pressure at the bottom of a 3-m-tall vertical rectangular gate. Explain the resulting bias and find the ratio of this estimate to the true force when the top is at the surface.



Question 9 – Solution



Step 1: Write the student's estimate. The student applies the bottom pressure $p = \gamma(3)$ uniformly over the entire gate of area A . Thus

$$F_{\text{est}} = 3\gamma A.$$

Step 2: Find the true force. For a rectangular gate extending from the surface to depth 3 m, the centroid is at the average depth

$$\bar{y} = \frac{0 + 3}{2} = \frac{3}{2} \text{ m.}$$

Therefore,

$$F_{\text{true}} = \gamma A \bar{y} = \frac{3}{2} \gamma A.$$

Step 3: Compare the estimate with the true force.

$$\frac{F_{\text{est}}}{F_{\text{true}}} = \frac{3\gamma A}{\frac{3}{2}\gamma A} = 2.$$

Thus the estimate is twice the true value:

$$F_{\text{est}} = 2F_{\text{true}}.$$

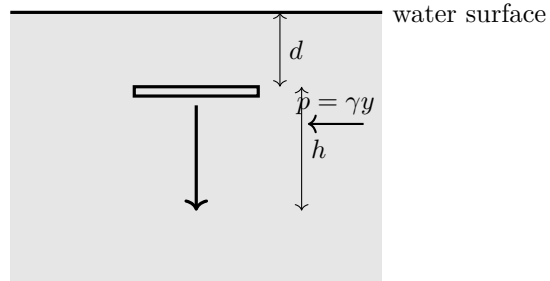
Step 4: Express the bias as a percent error.

$$\frac{F_{\text{est}} - F_{\text{true}}}{F_{\text{true}}} \times 100\% = (2 - 1)100\% = 100\%.$$

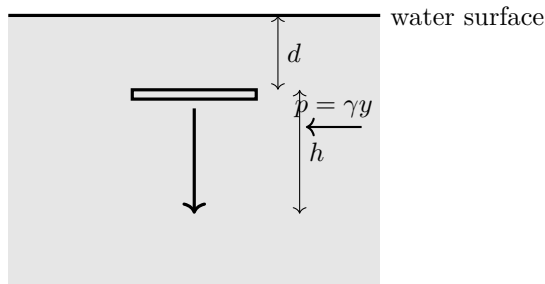
The bottom-pressure estimate is 100% too high.

Question 10

A horizontal hatch of constant area $A > 0$ is slowly lowered from depth d to $d + h$, where $d, h > 0$. Assume an upward resisting hydrostatic force $F(y) = \gamma Ay$ at depth y , with constant weight density $\gamma > 0$. Find the work done against this force, ignoring other forces.



Question 10 – Solution



Step 1: State the force model. Assume the hydrostatic force resisting the downward motion has magnitude

$$F(y) = p(y)A = \gamma yA,$$

where y is the hatch depth. The area A is constant.

Step 2: Write a differential amount of work. Moving the hatch through a small downward distance dy requires

$$dW = F(y) dy = \gamma Ay dy.$$

Step 3: Integrate over the change in depth. The hatch moves from $y = d$ to $y = d + h$, so

$$\begin{aligned} W &= \int_d^{d+h} \gamma Ay dy \\ &= \gamma A \left[\frac{y^2}{2} \right]_d^{d+h} \\ &= \frac{\gamma A}{2} ((d+h)^2 - d^2). \end{aligned}$$

Step 4: Simplify.

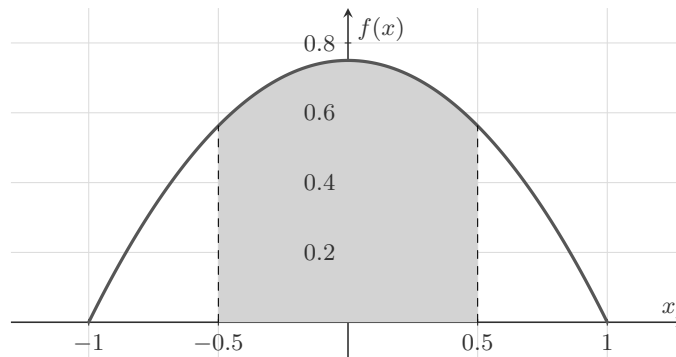
$$\begin{aligned} W &= \frac{\gamma A}{2} (d^2 + 2dh + h^2 - d^2) \\ &= \gamma A \left(dh + \frac{h^2}{2} \right). \end{aligned}$$

$$\boxed{W = \gamma A \left(dh + \frac{h^2}{2} \right)}$$

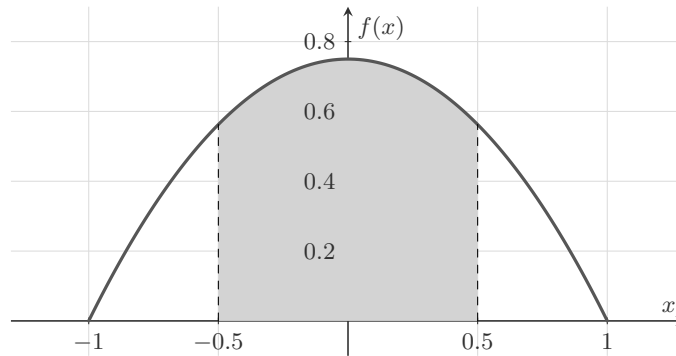
This is the external work under the stated resisting-force assumption.

Question 1

A sensor error has density $f(x) = c(1 - x^2)$, $-1 \leq x \leq 1$. Find c and $P(|X| < 1/2)$.



Question 1 – Solution



Step 1: Use the normalization condition. A probability density must have total area 1:

$$1 = c \int_{-1}^1 (1 - x^2) dx.$$

Step 2: Evaluate the normalization integral.

$$\begin{aligned} \int_{-1}^1 (1 - x^2) dx &= \left[x - \frac{x^3}{3} \right]_{-1}^1 \\ &= \frac{2}{3} - \left(-\frac{2}{3} \right) = \frac{4}{3}. \end{aligned}$$

Thus

$$1 = c \left(\frac{4}{3} \right), \quad \boxed{c = \frac{3}{4}}.$$

Step 3: Translate the event.

$$|X| < \frac{1}{2} \iff -\frac{1}{2} < X < \frac{1}{2}.$$

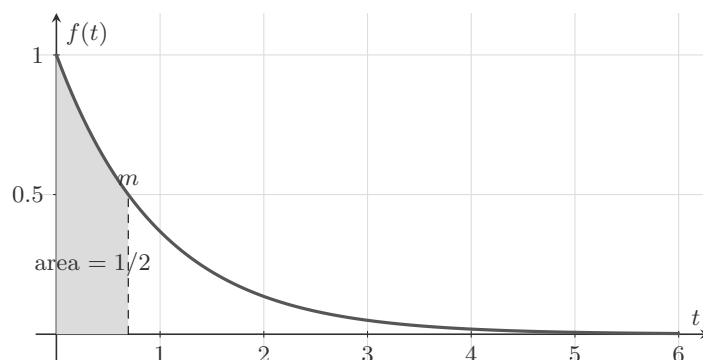
Step 4: Integrate the density over this interval.

$$\begin{aligned} P\left(|X| < \frac{1}{2}\right) &= \frac{3}{4} \int_{-1/2}^{1/2} (1 - x^2) dx \\ &= \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{-1/2}^{1/2} \\ &= \frac{3}{4} \left(\frac{11}{12} \right) = \frac{11}{16}. \end{aligned}$$

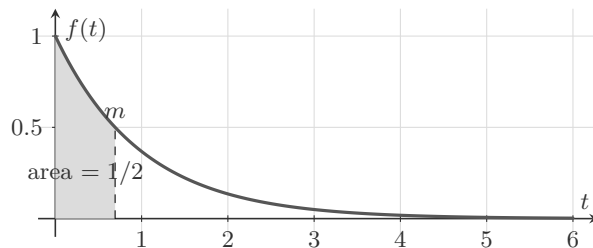
$$\boxed{P\left(|X| < \frac{1}{2}\right) = \frac{11}{16}}$$

Question 2

A waiting time has $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$. Find the median in terms of λ .



Question 2 – Solution



Step 1: Use the definition of a median. The median m satisfies

$$P(T \leq m) = \frac{1}{2}.$$

Because $T \geq 0$,

$$\int_0^m \lambda e^{-\lambda t} dt = \frac{1}{2}.$$

Step 2: Evaluate the integral.

$$\begin{aligned} \int_0^m \lambda e^{-\lambda t} dt &= [-e^{-\lambda t}]_0^m \\ &= 1 - e^{-\lambda m}. \end{aligned}$$

Therefore,

$$1 - e^{-\lambda m} = \frac{1}{2}.$$

Step 3: Solve for m .

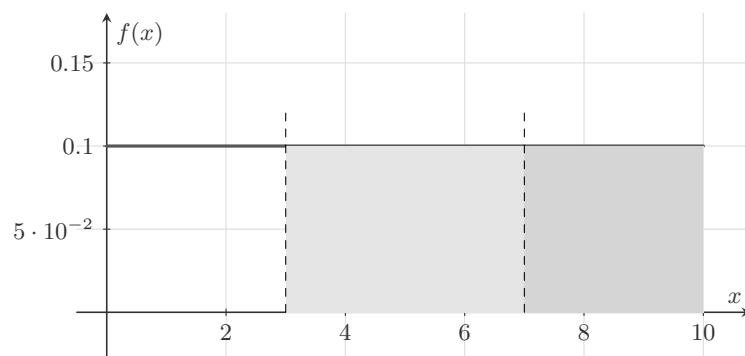
$$\begin{aligned} e^{-\lambda m} &= \frac{1}{2}, \\ -\lambda m &= \ln\left(\frac{1}{2}\right) = -\ln 2, \\ m &= \frac{\ln 2}{\lambda}. \end{aligned}$$

$$\boxed{m = \frac{\ln 2}{\lambda}}$$

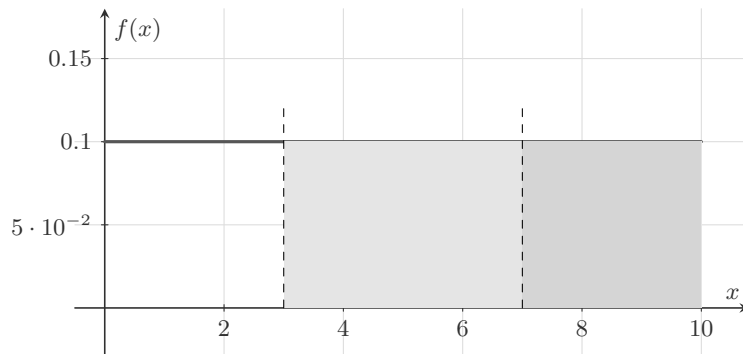
Since $\lambda > 0$, the median is positive as expected.

Question 3

A random point is chosen uniformly on $[0, 10]$. Given $X > 3$, find $P(X > 7 \mid X > 3)$.



Question 3 – Solution



Step 1: Identify the two events. Let

$$A = \{X > 7\}, \quad B = \{X > 3\}.$$

Because $A \subset B$, we have $A \cap B = A$.

Step 2: Apply conditional probability.

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

Step 3: Use uniform interval lengths. On $[0, 10]$, probability equals interval length divided by 10:

$$P(X > 7) = \frac{10 - 7}{10} = \frac{3}{10}, \quad P(X > 3) = \frac{10 - 3}{10} = \frac{7}{10}.$$

Step 4: Form the ratio.

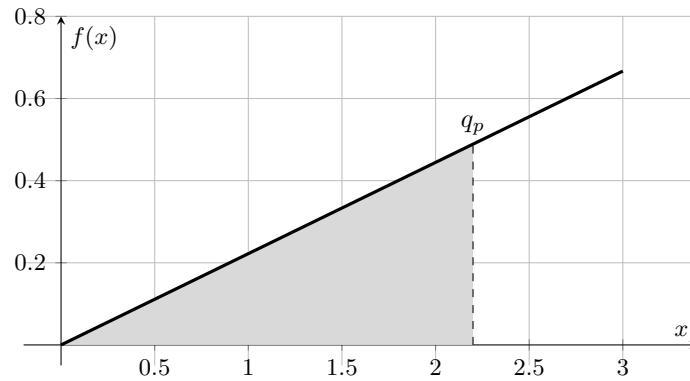
$$P(X > 7 | X > 3) = \frac{\frac{3}{10}}{\frac{7}{10}} = \frac{3}{7}.$$

$$\boxed{P(X > 7 | X > 3) = \frac{3}{7}}$$

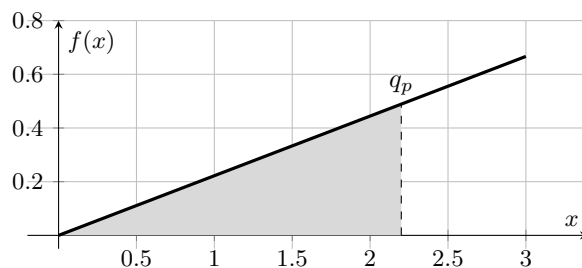
Equivalently, after conditioning on $X > 3$, the available interval is $(3, 10]$, and the favorable portion $(7, 10]$ has length 3 out of 7.

Question 4

A density is $f(x) = kx$ on $[0, a]$, where $a > 0$, and is zero elsewhere. Determine k , the CDF, and the p -quantile.



Question 4 – Solution



Step 1: Normalize the density.

$$1 = \int_0^a kx \, dx = k \left[\frac{x^2}{2} \right]_0^a = \frac{ka^2}{2}.$$

Hence

$$\boxed{k = \frac{2}{a^2}}.$$

Step 2: Build the CDF on the support. For $0 \leq x \leq a$,

$$\begin{aligned} F(x) &= \int_0^x \frac{2t}{a^2} \, dt \\ &= \frac{2}{a^2} \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{a^2}. \end{aligned}$$

Thus the complete CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{a^2}, & 0 \leq x \leq a, \\ 1, & x > a. \end{cases}$$

Step 3: Solve the quantile equation. The p -quantile q_p satisfies $F(q_p) = p$:

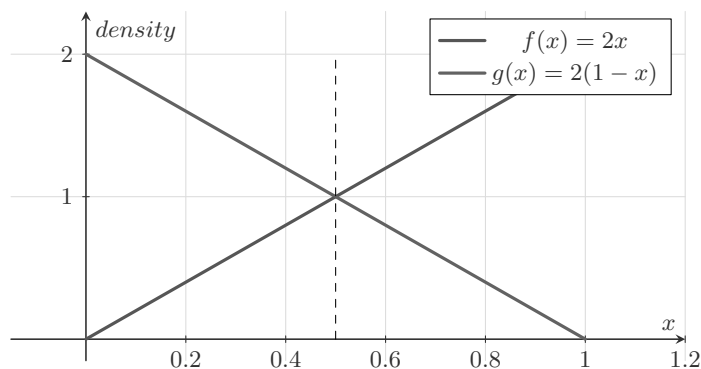
$$\frac{q_p^2}{a^2} = p.$$

Since $q_p \geq 0$,

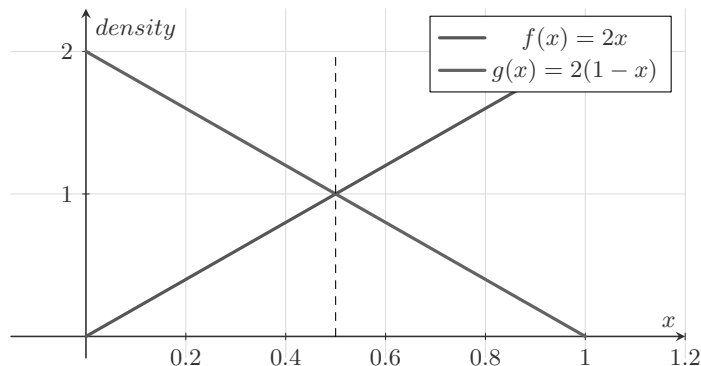
$$\boxed{q_p = a\sqrt{p}}, \quad 0 \leq p \leq 1.$$

Question 5

Two densities on $[0, 1]$ are $f(x) = 2x$ and $g(x) = 2(1 - x)$. Compare their means without integration by identifying a reflection.



Question 5 – Solution



Step 1: Identify the reflection.

$$f(1-x) = 2(1-x) = g(x).$$

Therefore, the graph of g is the reflection of the graph of f across $x = 1/2$.

Step 2: Find the mean of f geometrically. The area under $f(x) = 2x$ is a right triangle with vertices

$$(0,0), \quad (1,0), \quad (1,2).$$

The x -coordinate of a triangle's centroid is the average of its vertex x -coordinates:

$$\mu_f = \frac{0 + 1 + 1}{3} = \frac{2}{3}.$$

Step 3: Reflect the mean. Reflection across $x = 1/2$ sends a location x to $1-x$. Hence

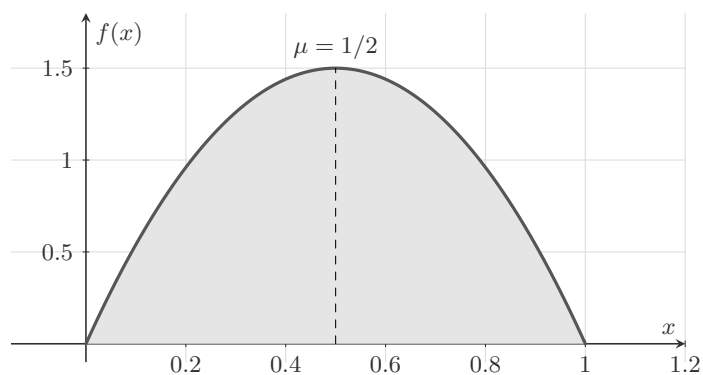
$$\mu_g = 1 - \mu_f = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\boxed{\mu_f = \frac{2}{3}, \quad \mu_g = \frac{1}{3}}$$

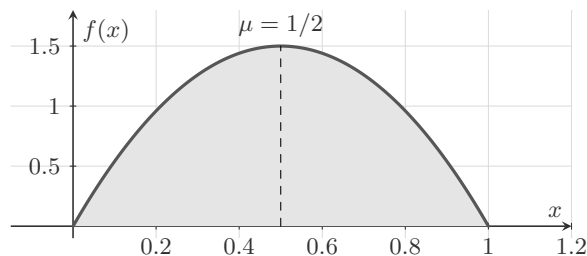
As a check, the two reflected means add to 1.

Question 6

For $f(x) = 6x(1 - x)$, $0 \leq x \leq 1$, find $\text{Var}(X)$. Use symmetry for the mean.



Question 6 – Solution



Step 1: Use symmetry to find the mean. The density satisfies

$$f(1-x) = 6(1-x)x = f(x),$$

so it is symmetric about $x = 1/2$. Therefore,

$$\mu = E[X] = \frac{1}{2}.$$

Step 2: Compute the second moment.

$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx \\ &= \int_0^1 x^2 (6x - 6x^2) dx \\ &= \int_0^1 (6x^3 - 6x^4) dx \\ &= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5 \right]_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}. \end{aligned}$$

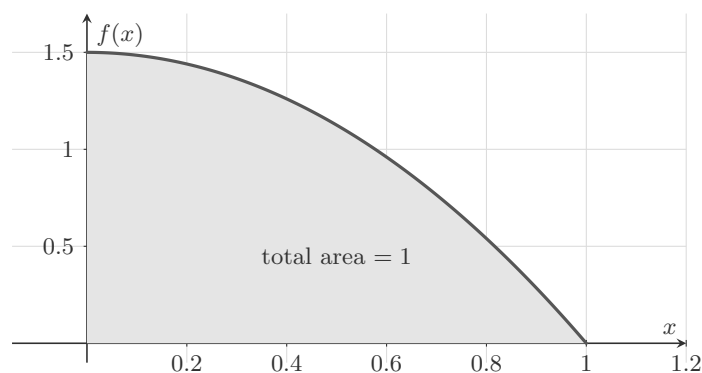
Step 3: Apply the variance formula.

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{3}{10} - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{10} - \frac{1}{4} = \frac{1}{20}. \end{aligned}$$

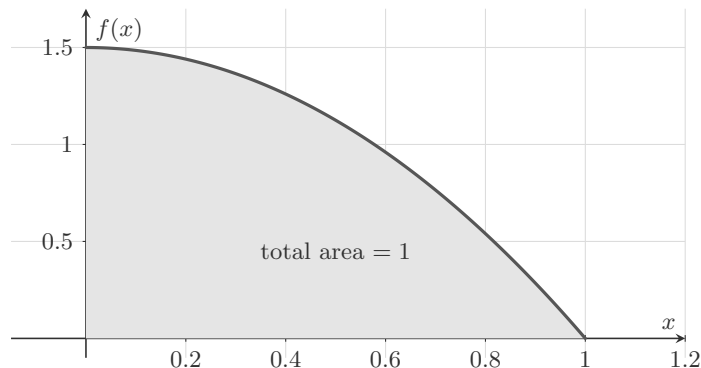
$\text{Var}(X) = \frac{1}{20}$

Question 7

A proposed density is $f(x) = \frac{3}{2}(1 - x^2)$ on $[0, 1]$. Is it valid? If not, repair only the constant.



Question 7 – Solution



Step 1: Check nonnegativity. For $0 \leq x \leq 1$,

$$1 - x^2 \geq 0,$$

so

$$f(x) = \frac{3}{2}(1 - x^2) \geq 0.$$

Step 2: Check the total area.

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{3}{2} \int_0^1 (1 - x^2) dx \\ &= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_0^1 \\ &= \frac{3}{2} \left(1 - \frac{1}{3} \right) = \frac{3}{2} \left(\frac{2}{3} \right) = 1. \end{aligned}$$

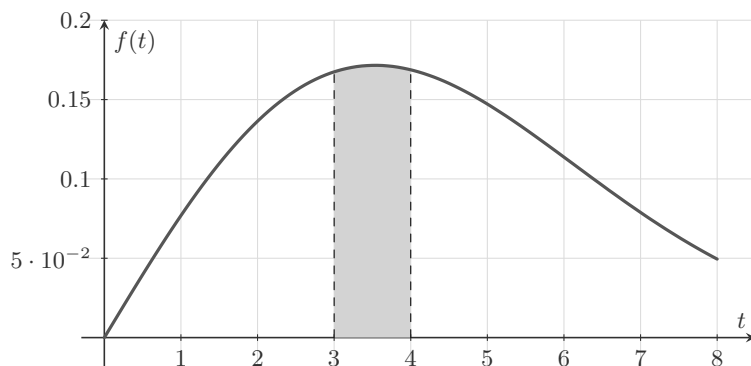
Step 3: State the conclusion. The function is nonnegative on its support and integrates to 1. Therefore it already satisfies both requirements for a probability density.

The proposed density is valid; no repair is needed.

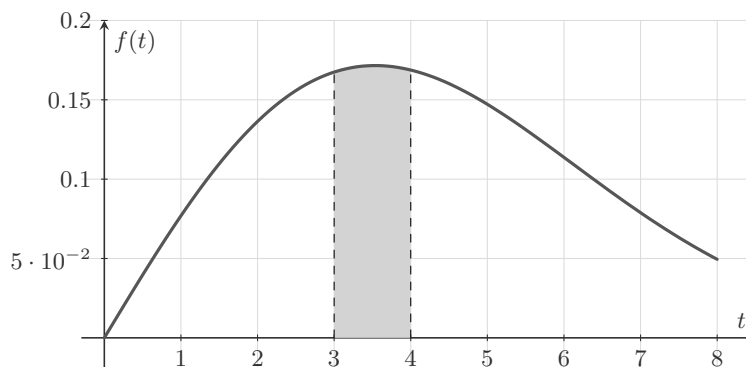
If the constant had been unknown, normalization would give $c(2/3) = 1$, hence $c = 3/2$.

Question 8

A lifetime satisfies $P(X > t) = e^{-(t/5)^2}$, $t \geq 0$. Find its density and $P(3 < X < 4)$.



Question 8 – Solution



Step 1: Convert the survival function to the CDF. For $t \geq 0$,

$$P(X > t) = e^{-(t/5)^2} = e^{-t^2/25}.$$

Therefore,

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-t^2/25}.$$

Step 2: Differentiate to obtain the density.

$$\begin{aligned} f(t) &= F'(t) \\ &= -\frac{d}{dt} \left(e^{-t^2/25} \right) \\ &= \frac{2t}{25} e^{-t^2/25}, \quad t \geq 0. \end{aligned}$$

Also, $f(t) = 0$ for $t < 0$.

Step 3: Compute the interval probability. Using the CDF,

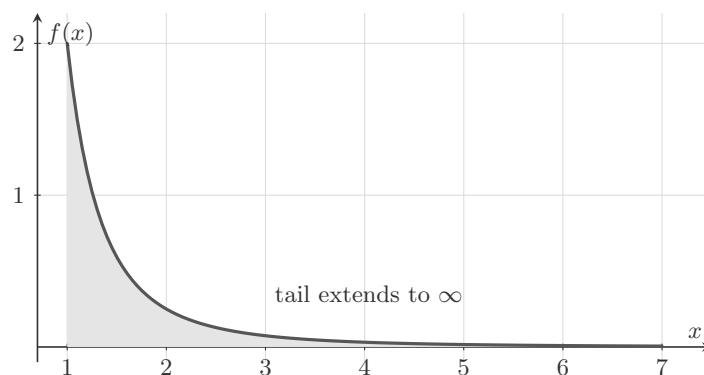
$$\begin{aligned} P(3 < X < 4) &= F(4) - F(3) \\ &= \left(1 - e^{-16/25} \right) - \left(1 - e^{-9/25} \right) \\ &= e^{-9/25} - e^{-16/25}. \end{aligned}$$

$$P(3 < X < 4) = e^{-9/25} - e^{-16/25}$$

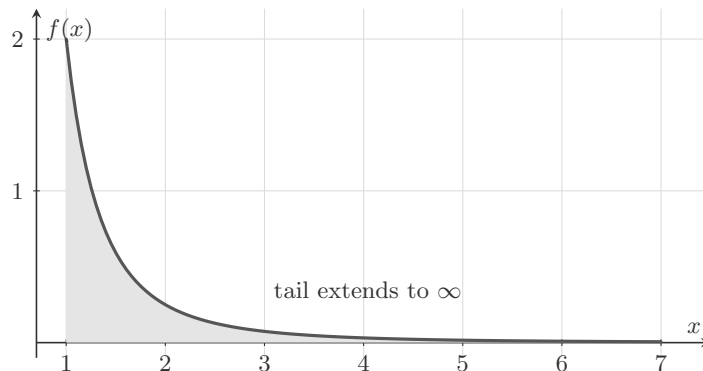
The same result follows by integrating $f(t)$ from 3 to 4.

Question 9

Choose c so $f(x) = c/x^3$, $x \geq 1$, is a density. Determine whether $E[X]$ and $E[X^2]$ are finite.



Question 9 – Solution



Step 1: Normalize the density.

$$\begin{aligned} 1 &= c \int_1^{\infty} x^{-3} dx \\ &= c \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{c}{2}. \end{aligned}$$

Thus

$$\boxed{c = 2}.$$

Step 2: Test the first moment.

$$\begin{aligned} E[X] &= \int_1^{\infty} x \frac{2}{x^3} dx = 2 \int_1^{\infty} x^{-2} dx \\ &= 2 \left[-\frac{1}{x} \right]_1^{\infty} = 2. \end{aligned}$$

Therefore, the mean is finite.

Step 3: Test the second moment.

$$E[X^2] = \int_1^{\infty} x^2 \frac{2}{x^3} dx = 2 \int_1^{\infty} \frac{dx}{x}.$$

For an upper cutoff b ,

$$2 \int_1^b \frac{dx}{x} = 2 \ln b \rightarrow \infty.$$

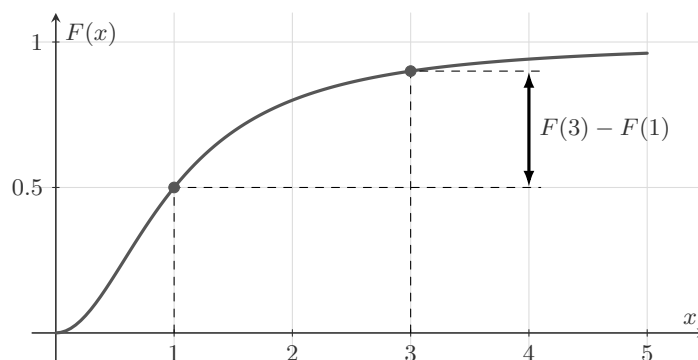
Thus the second moment diverges.

$$\boxed{E[X] = 2, \quad E[X^2] = \infty}$$

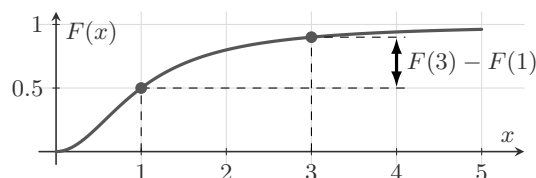
Consequently, $\text{Var}(X)$ is also infinite.

Question 10

A continuous random variable has CDF $F(x) = 0$ for $x < 0$, $F(x) = x^2/(1+x^2)$ for $x \geq 0$. Find the density and $P(1 < X < 3)$.



Question 10 – Solution



Step 1: Differentiate the CDF on each interval. For $x < 0$, $F(x) = 0$, so $f(x) = 0$. For $x > 0$,

$$\begin{aligned} f(x) &= \frac{d}{dx} \left(\frac{x^2}{1+x^2} \right) \\ &= \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} \\ &= \frac{2x}{(1+x^2)^2}. \end{aligned}$$

The CDF is continuous at $x = 0$, so there is no point mass there. Hence

$$f(x) = \begin{cases} 0, & x < 0, \\ \frac{2x}{(1+x^2)^2}, & x \geq 0. \end{cases}$$

Step 2: Use the CDF to compute the interval probability. For a continuous random variable,

$$P(1 < X < 3) = F(3) - F(1).$$

Now

$$F(3) = \frac{3^2}{1+3^2} = \frac{9}{10}, \quad F(1) = \frac{1^2}{1+1^2} = \frac{1}{2}.$$

Step 3: Subtract.

$$P(1 < X < 3) = \frac{9}{10} - \frac{1}{2} = \frac{9}{10} - \frac{5}{10} = \frac{4}{10} = \frac{2}{5}.$$

$$P(1 < X < 3) = \frac{2}{5}$$