

Question 1

Evaluate by tabular integration:

$$\int x^3 e^{2x} dx.$$

Question 1 – Solution

We use tabular integration because repeatedly differentiating the polynomial x^3 eventually gives zero. Differentiate in the D column and integrate in the I column. The signs alternate, beginning with a plus sign.

Sign	D	I
+	x^3	$\frac{e^{2x}}{2}$
–	$3x^2$	$\frac{e^{2x}}{4}$
+	$6x$	$\frac{e^{2x}}{8}$
–	6	$\frac{e^{2x}}{16}$
	0	

For example, each entry in the integration column is obtained using

$$\int e^{2x} dx = \frac{1}{2}e^{2x},$$

so every step down that column introduces another factor of $\frac{1}{2}$.

Now multiply across each row, pairing each entry in the derivative column with the entry in the integration column on the same row, and use the indicated signs:

$$\begin{aligned} \int x^3 e^{2x} dx &= x^3 \left(\frac{e^{2x}}{2} \right) - 3x^2 \left(\frac{e^{2x}}{4} \right) + 6x \left(\frac{e^{2x}}{8} \right) - 6 \left(\frac{e^{2x}}{16} \right) + C \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{6x e^{2x}}{8} - \frac{6e^{2x}}{16} + C \\ &= e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C. \end{aligned}$$

Therefore,

$$\boxed{\int x^3 e^{2x} dx = e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C}$$

As a check, differentiating the answer gives

$$\frac{d}{dx} \left[e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) \right] = x^3 e^{2x},$$

which is the original integrand.

Question 2

Find the unique constant a such that

$$\int_0^1 x e^{ax} dx = 1.$$

Give the defining equation and a numerical value.

Question 2 – Solution

First consider the special case $a = 0$. In that case,

$$\int_0^1 x e^{0x} dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \neq 1.$$

Therefore, $a = 0$ is not a solution, so we may assume that $a \neq 0$.

To evaluate the integral, use integration by parts with

$$u = x, \quad dv = e^{ax} dx.$$

Then

$$du = dx, \quad v = \frac{e^{ax}}{a}.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_0^1 x e^{ax} dx &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{1}{a} \int_0^1 e^{ax} dx \\ &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{e^{ax}}{a^2} \Big|_0^1 \\ &= \frac{e^a}{a} - \frac{e^a - 1}{a^2} \\ &= \frac{a e^a - e^a + 1}{a^2} \\ &= \frac{e^a(a - 1) + 1}{a^2}. \end{aligned}$$

We want this integral to equal 1, so

$$\begin{aligned} \frac{e^a(a - 1) + 1}{a^2} &= 1, \\ e^a(a - 1) + 1 &= a^2. \end{aligned}$$

Thus, the defining equation is

$$\boxed{e^a(a - 1) + 1 = a^2}.$$

Substituting $a = 1$ verifies that it satisfies the equation:

$$e^1(1 - 1) + 1 = 1 = 1^2.$$

Hence, $a = 1$ is a solution.

To see that it is the only solution, define

$$F(a) = \int_0^1 x e^{ax} dx.$$

Differentiating with respect to a gives

$$F'(a) = \int_0^1 x^2 e^{ax} dx > 0$$

for every real a . Therefore, F is strictly increasing and can equal 1 for at most one value of a . Since $F(1) = 1$, the unique constant is

$$\boxed{a = 1.0000}.$$

Question 3

Consider the integral

$$\int x^2 e^x dx.$$

A student makes the integration-by-parts choice

$$u = e^x, \quad dv = x^2 dx.$$

Explain why this choice does not simplify the integral. Then evaluate the integral efficiently using a better choice of u and dv .

Question 3 – Solution

Recall the integration-by-parts formula

$$\int u \, dv = uv - \int v \, du.$$

With the student's choice $u = e^x$ and $dv = x^2 \, dx$, we have

$$du = e^x \, dx, \quad v = \frac{x^3}{3}.$$

Therefore,

$$\begin{aligned} \int x^2 e^x \, dx &= e^x \left(\frac{x^3}{3} \right) - \int \frac{x^3}{3} e^x \, dx \\ &= \frac{x^3 e^x}{3} - \frac{1}{3} \int x^3 e^x \, dx. \end{aligned}$$

The polynomial degree has increased from 2 to 3, so the remaining integral is more complicated. Repeating this choice would keep increasing the degree.

Instead, choose

$$u = x^2, \quad dv = e^x \, dx, \quad du = 2x \, dx, \quad v = e^x.$$

Applying integration by parts gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

For the remaining integral, use integration by parts again:

$$u = x, \quad dv = e^x \, dx, \quad du = dx, \quad v = e^x.$$

Thus,

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x. \end{aligned}$$

Substituting this result into the original calculation gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C. \end{aligned}$$

$$\boxed{\int x^2 e^x \, dx = e^x (x^2 - 2x + 2) + C}$$

Check by differentiating:

$$\begin{aligned} \frac{d}{dx} [e^x (x^2 - 2x + 2)] &= e^x (x^2 - 2x + 2) + e^x (2x - 2) \\ &= x^2 e^x, \end{aligned}$$

which is the original integrand.

Question 4

Define

$$I_n = \int x^n e^x dx, \quad n \geq 1.$$

Use integration by parts to derive a recurrence relation connecting I_n and I_{n-1} . Then use the recurrence to evaluate I_4 .

Question 4 – Solution

Start with

$$I_n = \int x^n e^x dx.$$

To reduce the power of x , choose

$$u = x^n, \quad dv = e^x dx, \quad du = nx^{n-1} dx, \quad v = e^x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} I_n &= x^n e^x - \int e^x (nx^{n-1}) dx \\ &= x^n e^x - n \int x^{n-1} e^x dx. \end{aligned}$$

Since $I_{n-1} = \int x^{n-1} e^x dx$, the recurrence relation is

$$\boxed{I_n = x^n e^x - n I_{n-1}}.$$

Here, as usual for indefinite integrals, the constant of integration is added after the recurrence has been applied. The base case is

$$I_0 = \int e^x dx = e^x.$$

Now apply the recurrence repeatedly:

$$\begin{aligned} I_1 &= x e^x - I_0 = x e^x - e^x = e^x(x - 1), \\ I_2 &= x^2 e^x - 2I_1 \\ &= x^2 e^x - 2e^x(x - 1) = e^x(x^2 - 2x + 2), \\ I_3 &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3e^x(x^2 - 2x + 2) \\ &= e^x(x^3 - 3x^2 + 6x - 6), \\ I_4 &= x^4 e^x - 4I_3 \\ &= x^4 e^x - 4e^x(x^3 - 3x^2 + 6x - 6) \\ &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24). \end{aligned}$$

Therefore,

$$\boxed{I_4 = \int x^4 e^x dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C}$$

Indeed, differentiating the result gives

$$\begin{aligned} \frac{d}{dx} [e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)] &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) \\ &\quad + e^x(4x^3 - 12x^2 + 24x - 24) \\ &= x^4 e^x, \end{aligned}$$

which confirms the answer.

Question 5

Evaluate the cyclic integral

$$\int e^{2x} \cos(3x) dx$$

using integration by parts. Do not guess the form of the antiderivative.

Question 5 – Solution

Let

$$I = \int e^{2x} \cos(3x) dx.$$

For the first integration by parts, choose

$$u = \cos(3x), \quad dv = e^{2x} dx, \quad du = -3 \sin(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Using $\int u dv = uv - \int v du$, we get

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) - \int \frac{1}{2} e^{2x} (-3 \sin(3x)) dx \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) dx. \end{aligned} \tag{1}$$

Now evaluate the remaining integral. Let

$$J = \int e^{2x} \sin(3x) dx.$$

Use integration by parts again with

$$u = \sin(3x), \quad dv = e^{2x} dx, \quad du = 3 \cos(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Then

$$\begin{aligned} J &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I. \end{aligned} \tag{2}$$

Substitute equation (2) into equation (1):

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I \right) \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) - \frac{9}{4} I. \end{aligned}$$

Move the I -term to the left and solve:

$$\begin{aligned} I + \frac{9}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ \frac{13}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ I &= \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)). \end{aligned}$$

Therefore,

$$\boxed{\int e^{2x} \cos(3x) dx = \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) + C}$$

To check, differentiate the antiderivative:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) \right] &= \frac{e^{2x}}{13} [2(2 \cos(3x) + 3 \sin(3x)) \\ &\quad - 6 \sin(3x) + 9 \cos(3x)] \\ &= e^{2x} \cos(3x), \end{aligned}$$

which is the original integrand.

Question 6

Use integration by parts to prove that

$$\int_0^1 \ln(x) \, dx = -1.$$

Because $\ln(x)$ is unbounded as $x \rightarrow 0^+$, be sure to treat the lower endpoint correctly as an improper integral.

Question 6 – Solution

The function $\ln(x)$ is not defined at $x = 0$ and satisfies

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty.$$

Therefore, we must interpret the integral as

$$\int_0^1 \ln(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln(x) dx.$$

For $\varepsilon > 0$, evaluate the proper integral by integration by parts. Choose

$$u = \ln(x), \quad dv = dx, \quad du = \frac{1}{x} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 x \left(\frac{1}{x} \right) dx \\ &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 1 dx \\ &= [x \ln(x) - x]_{\varepsilon}^1. \end{aligned}$$

Now substitute the bounds:

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= (1 \ln(1) - 1) - (\varepsilon \ln(\varepsilon) - \varepsilon) \\ &= -1 - \varepsilon \ln(\varepsilon) + \varepsilon. \end{aligned}$$

It remains to evaluate the indeterminate endpoint term. Rewrite it as

$$\varepsilon \ln(\varepsilon) = \frac{\ln(\varepsilon)}{1/\varepsilon}.$$

As $\varepsilon \rightarrow 0^+$, this has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon \ln(\varepsilon) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} \\ &= \lim_{\varepsilon \rightarrow 0^+} (-\varepsilon) = 0. \end{aligned}$$

Also, $\lim_{\varepsilon \rightarrow 0^+} \varepsilon = 0$. Hence,

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln(\varepsilon) + \varepsilon) \\ &= -1. \end{aligned}$$

Therefore, the improper integral converges and

$$\boxed{\int_0^1 \ln(x) dx = -1}.$$

This sign is reasonable because $\ln(x) < 0$ for $0 < x < 1$.

Question 7

Find an antiderivative F of

$$f(x) = \arcsin(x), \quad -1 < x < 1,$$

that satisfies the initial condition

$$F(0) = 2.$$

Question 7 – Solution

We begin by evaluating

$$\int \arcsin(x) \, dx.$$

Since the integrand can be viewed as $\arcsin(x) \cdot 1$, use integration by parts with

$$u = \arcsin(x), \quad dv = dx, \quad du = \frac{1}{\sqrt{1-x^2}} \, dx, \quad v = x.$$

Using $\int u \, dv = uv - \int v \, du$, we obtain

$$\int \arcsin(x) \, dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} \, dx. \quad (1)$$

To evaluate the remaining integral, let

$$w = 1 - x^2.$$

Then

$$dw = -2x \, dx, \quad x \, dx = -\frac{1}{2} \, dw.$$

Therefore,

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} \, dx &= -\frac{1}{2} \int w^{-1/2} \, dw \\ &= -\frac{1}{2} (2w^{1/2}) \\ &= -\sqrt{w} \\ &= -\sqrt{1-x^2}. \end{aligned}$$

Substitute this result into equation (1):

$$\begin{aligned} \int \arcsin(x) \, dx &= x \arcsin(x) - (-\sqrt{1-x^2}) + C \\ &= x \arcsin(x) + \sqrt{1-x^2} + C. \end{aligned}$$

Thus, the general antiderivative is

$$F(x) = x \arcsin(x) + \sqrt{1-x^2} + C.$$

Now apply the condition $F(0) = 2$:

$$\begin{aligned} F(0) &= 0 \cdot \arcsin(0) + \sqrt{1-0^2} + C \\ &= 1 + C. \end{aligned}$$

Hence,

$$1 + C = 2 \quad \implies \quad C = 1.$$

Therefore, the required antiderivative is

$$\boxed{F(x) = x \arcsin(x) + \sqrt{1-x^2} + 1}.$$

To verify, differentiate:

$$\begin{aligned} F'(x) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x). \end{aligned}$$

Also, $F(0) = 0 + 1 + 1 = 2$, so both requirements are satisfied.

Question 8

Reverse engineer an antiderivative by finding a quadratic polynomial $P(x)$ such that

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1).$$

Determine $P(x)$ and verify that it satisfies the equation.

Question 8 – Solution

We want to choose $P(x)$ so that differentiating $e^x P(x)$ produces the given expression. By the product rule,

$$\begin{aligned}\frac{d}{dx} [e^x P(x)] &= e^x P(x) + e^x P'(x) \\ &= e^x (P(x) + P'(x)).\end{aligned}$$

Thus, the required identity

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1)$$

will hold if

$$P(x) + P'(x) = x^2 + 1. \tag{1}$$

Because P is quadratic, write

$$P(x) = ax^2 + bx + c,$$

where a , b , and c are constants. Then

$$P'(x) = 2ax + b.$$

Adding $P(x)$ and $P'(x)$ gives

$$\begin{aligned}P(x) + P'(x) &= (ax^2 + bx + c) + (2ax + b) \\ &= ax^2 + (b + 2a)x + (c + b).\end{aligned}$$

Equation (1) requires this polynomial to equal

$$x^2 + 1 = 1x^2 + 0x + 1.$$

Matching coefficients of like powers of x yields

$$\begin{array}{rcl}a & = & 1, \quad \text{coefficient of } x^2, \\ b + 2a & = & 0, \quad \text{coefficient of } x, \\ c + b & = & 1, \quad \text{constant term.}\end{array}$$

Solve the equations in order. Since $a = 1$,

$$b + 2(1) = 0 \quad \implies \quad b = -2.$$

Then

$$c + (-2) = 1 \quad \implies \quad c = 3.$$

Therefore,

$$\boxed{P(x) = x^2 - 2x + 3}.$$

Finally, verify the result. Since $P'(x) = 2x - 2$,

$$\begin{aligned}\frac{d}{dx} [e^x (x^2 - 2x + 3)] &= e^x (x^2 - 2x + 3) + e^x (2x - 2) \\ &= e^x (x^2 - 2x + 3 + 2x - 2) \\ &= e^x (x^2 + 1),\end{aligned}$$

which is the required expression.

Question 9

Evaluate

$$\int_{-1}^1 x \arctan(x) dx.$$

Before using integration by parts, exploit the symmetry of the integrand to simplify the calculation.

Question 9 – Solution

Let

$$f(x) = x \arctan(x).$$

Both x and $\arctan(x)$ are odd functions. Therefore, their product is even:

$$\begin{aligned} f(-x) &= (-x) \arctan(-x) \\ &= (-x)(-\arctan(x)) \\ &= x \arctan(x) = f(x). \end{aligned}$$

Consequently,

$$\int_{-1}^1 x \arctan(x) dx = 2 \int_0^1 x \arctan(x) dx. \quad (1)$$

Now use integration by parts on the integral over $[0, 1]$. Choose

$$u = \arctan(x), \quad dv = x dx, \quad du = \frac{1}{1+x^2} dx, \quad v = \frac{x^2}{2}.$$

Then

$$\int_0^1 x \arctan(x) dx = \frac{x^2}{2} \arctan(x) \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx.$$

Multiplying by 2 as in equation (1) gives

$$\int_{-1}^1 x \arctan(x) dx = x^2 \arctan(x) \Big|_0^1 - \int_0^1 \frac{x^2}{1+x^2} dx. \quad (2)$$

For the remaining integral, rewrite the integrand:

$$\frac{x^2}{1+x^2} = \frac{(1+x^2) - 1}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^2} dx &= \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \\ &= [x - \arctan(x)]_0^1 \\ &= 1 - \frac{\pi}{4}. \end{aligned}$$

The boundary term in equation (2) is

$$x^2 \arctan(x) \Big|_0^1 = 1^2 \arctan(1) - 0^2 \arctan(0) = \frac{\pi}{4}.$$

Substituting both results into equation (2),

$$\begin{aligned} \int_{-1}^1 x \arctan(x) dx &= \frac{\pi}{4} - \left(1 - \frac{\pi}{4} \right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Thus,

$$\boxed{\int_{-1}^1 x \arctan(x) dx = \frac{\pi}{2} - 1}.$$

The answer is positive, as expected, because the even integrand is nonnegative on $[-1, 1]$.

Question 10

For $n > 0$, evaluate the improper integral

$$J_n = \int_0^1 x^n \ln(x) \, dx.$$

Then describe the sign and limiting behavior of J_n as $n \rightarrow \infty$.

Question 10 – Solution

Because $\ln(x) \rightarrow -\infty$ as $x \rightarrow 0^+$, write the integral as

$$J_n = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^n \ln(x) dx.$$

For $\varepsilon > 0$, use integration by parts with

$$u = \ln(x), \quad dv = x^n dx, \quad du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}.$$

This choice is valid because $n > 0$, so in particular $n+1 \neq 0$. Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 x^n \ln(x) dx &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{1}{n+1} \int_{\varepsilon}^1 x^n dx \\ &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{x^{n+1}}{(n+1)^2} \Big|_{\varepsilon}^1. \end{aligned} \tag{1}$$

We must evaluate the boundary term at $x = 0$. Since $n+1 > 0$,

$$\lim_{x \rightarrow 0^+} x^{n+1} \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}}.$$

This has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}} &= \lim_{x \rightarrow 0^+} \frac{1/x}{-(n+1)x^{-(n+2)}} \\ &= \lim_{x \rightarrow 0^+} -\frac{x^{n+1}}{n+1} = 0. \end{aligned}$$

Also, $\lim_{x \rightarrow 0^+} x^{n+1} = 0$. At the upper endpoint, $\ln(1) = 0$. Therefore, taking $\varepsilon \rightarrow 0^+$ in equation (1) gives

$$\begin{aligned} J_n &= \frac{1^{n+1} \ln(1)}{n+1} - 0 - \frac{1^{n+1} - 0}{(n+1)^2} \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\boxed{J_n = -\frac{1}{(n+1)^2}}.$$

For every $n > 0$, $J_n < 0$, which agrees with $\ln(x) < 0$ on $0 < x < 1$. Moreover, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} -\frac{1}{(n+1)^2} = 0.$$

The values increase toward zero through negative values. In short,

$$\boxed{J_n \rightarrow 0^- \quad \text{as} \quad n \rightarrow \infty}.$$

Question 1

Evaluate

$$\int \sin^5(x) \cos^4(x) dx.$$

Explain why separating one factor of $\sin(x)$ is an efficient strategy.

Question 1 – Solution

The power of sine is odd, so separate one sine factor:

$$\sin^5(x) = \sin^4(x) \sin(x).$$

The remaining even power can be converted using $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^4(x) = (1 - \cos^2(x))^2.$$

Thus,

$$\int \sin^5(x) \cos^4(x) dx = \int (1 - \cos^2(x))^2 \cos^4(x) \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

Then the integral becomes

$$\begin{aligned} - \int (1 - u^2)^2 u^4 du &= - \int (1 - 2u^2 + u^4) u^4 du \\ &= - \int (u^4 - 2u^6 + u^8) du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \sin^5(x) \cos^4(x) dx = -\frac{\cos^5(x)}{5} + \frac{2 \cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C.}$$

Separating one sine factor is efficient because it supplies the differential $-\sin(x) dx$ while every remaining factor becomes a polynomial in u .

Question 2

Use power-reduction identities to evaluate

$$\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx.$$

Question 2 – Solution

Begin with the double-angle identity

$$\sin(2x) = 2 \sin(x) \cos(x).$$

Squaring both sides gives

$$\sin^2(x) \cos^2(x) = \frac{1}{4} \sin^2(2x).$$

Now use $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ with $\theta = 2x$:

$$\sin^2(x) \cos^2(x) = \frac{1}{8} (1 - \cos(4x)).$$

Therefore,

$$\begin{aligned} \int_0^{\pi/2} \sin^2(x) \cos^2(x) dx &= \frac{1}{8} \int_0^{\pi/2} (1 - \cos(4x)) dx \\ &= \frac{1}{8} \left[x - \frac{1}{4} \sin(4x) \right]_0^{\pi/2}. \end{aligned}$$

Evaluate the endpoints:

$$\begin{aligned} \frac{1}{8} \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left(0 - \frac{1}{4} \sin(0) \right) \right] &= \frac{1}{8} \left(\frac{\pi}{2} \right) \\ &= \frac{\pi}{16}. \end{aligned}$$

Hence,

$$\boxed{\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx = \frac{\pi}{16}}.$$

Question 3

Find the average value of

$$f(x) = \sin^4(x)$$

over one full period.

Question 3 – Solution

We may use the interval $[0, 2\pi]$, whose length is 2π . The average-value formula is

$$f_{\text{avg}} = \frac{1}{2\pi} \int_0^{2\pi} \sin^4(x) \, dx.$$

Apply the power-reduction identity twice:

$$\begin{aligned} \sin^4(x) &= \left(\frac{1 - \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 - 2\cos(2x) + \cos^2(2x)) \\ &= \frac{1}{4} \left(1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3 - 4\cos(2x) + \cos(4x)}{8}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{3 - 4\cos(2x) + \cos(4x)}{8} \, dx \\ &= \frac{1}{16\pi} \left[3x - 2\sin(2x) + \frac{1}{4}\sin(4x) \right]_0^{2\pi} \\ &= \frac{1}{16\pi} (6\pi) = \frac{3}{8}. \end{aligned}$$

The sine terms vanish because their values are zero at both endpoints. Therefore,

$$\boxed{f_{\text{avg}} = \frac{3}{8}}.$$

Question 4

Evaluate

$$\int \tan^7(x) \sec^2(x) \, dx$$

without converting the integrand entirely to sines and cosines.

Question 4 – Solution

The factor $\sec^2(x) dx$ is exactly the differential of $\tan(x)$, so let

$$u = \tan(x), \quad du = \sec^2(x) dx.$$

Then

$$\begin{aligned} \int \tan^7(x) \sec^2(x) dx &= \int u^7 du \\ &= \frac{u^8}{8} + C. \end{aligned}$$

Substitute $u = \tan(x)$:

$$\boxed{\int \tan^7(x) \sec^2(x) dx = \frac{1}{8} \tan^8(x) + C}.$$

As a check,

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{8} \tan^8(x) \right] &= \frac{1}{8} \cdot 8 \tan^7(x) \sec^2(x) \\ &= \tan^7(x) \sec^2(x), \end{aligned}$$

which is the original integrand.

Question 5

Evaluate

$$\int \sec^5(x) \tan^3(x) \, dx$$

by saving a factor of $\sec(x) \tan(x)$ for substitution.

Question 5 – Solution

Separate one factor of $\sec(x) \tan(x)$:

$$\sec^5(x) \tan^3(x) = \sec^4(x) \tan^2(x) (\sec(x) \tan(x)).$$

Use $\tan^2(x) = \sec^2(x) - 1$ to rewrite the remaining tangent power:

$$\sec^4(x) \tan^2(x) = \sec^4(x) (\sec^2(x) - 1).$$

Now let

$$u = \sec(x), \quad du = \sec(x) \tan(x) dx.$$

The integral becomes

$$\begin{aligned} \int u^4(u^2 - 1) du &= \int (u^6 - u^4) du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C. \end{aligned}$$

Substituting $u = \sec(x)$ gives

$$\boxed{\int \sec^5(x) \tan^3(x) dx = \frac{1}{7} \sec^7(x) - \frac{1}{5} \sec^5(x) + C}.$$

Saving $\sec(x) \tan(x)$ is useful because it supplies du , while the identity converts all remaining factors into powers of u .

Question 6

Use a product-to-sum identity to evaluate

$$\int \sin(5x) \cos(2x) \, dx.$$

Question 6 – Solution

Use the product-to-sum identity

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

With $A = 5x$ and $B = 2x$,

$$\sin(5x) \cos(2x) = \frac{1}{2} [\sin(7x) + \sin(3x)].$$

Therefore,

$$\begin{aligned} \int \sin(5x) \cos(2x) dx &= \frac{1}{2} \int [\sin(7x) + \sin(3x)] dx \\ &= \frac{1}{2} \left(-\frac{1}{7} \cos(7x) - \frac{1}{3} \cos(3x) \right) + C \\ &= -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C. \end{aligned}$$

Hence,

$$\boxed{\int \sin(5x) \cos(2x) dx = -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C}.$$

Differentiating gives

$$\frac{1}{2} \sin(7x) + \frac{1}{2} \sin(3x) = \sin(5x) \cos(2x),$$

confirming the answer by the same product-to-sum identity.

Question 7

Determine all integers $m, n \geq 1$ for which

$$\int_0^{2\pi} \sin(mx) \cos(nx) \, dx \neq 0.$$

Justify your conclusion using a product-to-sum identity.

Question 7 – Solution

Use

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Setting $A = mx$ and $B = nx$ gives

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m + n)x) + \sin((m - n)x)]. \quad (1)$$

For every nonzero integer k ,

$$\begin{aligned} \int_0^{2\pi} \sin(kx) dx &= \left[-\frac{\cos(kx)}{k} \right]_0^{2\pi} \\ &= \frac{-\cos(2\pi k) + \cos(0)}{k} = 0, \end{aligned}$$

because $\cos(2\pi k) = 1$. Since $m + n \geq 2$, the first sine term in equation (1) always integrates to zero.

If $m \neq n$, then $m - n$ is a nonzero integer, so the second term also integrates to zero. If $m = n$, then

$$\sin((m - n)x) = \sin(0) = 0,$$

so it again contributes zero. Thus, in every case,

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0.$$

Therefore, there are no integers $m, n \geq 1$ for which the integral is nonzero:

There are no such integer pairs (m, n) .

Question 8

Evaluate

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx$$

using a substitution that produces reciprocal powers of the new variable.

Question 8 – Solution

Separate one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^3(x) = \sin(x)(1 - \cos^2(x)).$$

Then

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx = \int \frac{1 - \cos^2(x)}{\cos^4(x)} \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

The integral becomes

$$\begin{aligned} - \int \frac{1 - u^2}{u^4} du &= - \int (u^{-4} - u^{-2}) du \\ &= - \left(-\frac{1}{3}u^{-3} + u^{-1} \right) + C \\ &= \frac{1}{3}u^{-3} - u^{-1} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \frac{\sin^3(x)}{\cos^4(x)} dx = \frac{1}{3 \cos^3(x)} - \frac{1}{\cos(x)} + C}.$$

Equivalently, the answer is $\frac{1}{3} \sec^3(x) - \sec(x) + C$ on intervals where the original integrand is defined.

Question 9

Find constants A and B such that

$$\cos^4(x) = A + B \cos(2x) + \frac{1}{8} \cos(4x).$$

Then use the identity to evaluate $\int \cos^4(x) \, dx$.

Question 9 – Solution

Begin with the power-reduction identity

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

Squaring both sides,

$$\begin{aligned}\cos^4(x) &= \left(\frac{1 + \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 + 2 \cos(2x) + \cos^2(2x)) .\end{aligned}$$

Apply power reduction again:

$$\cos^2(2x) = \frac{1 + \cos(4x)}{2}.$$

Therefore,

$$\begin{aligned}\cos^4(x) &= \frac{1}{4} \left(1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x).\end{aligned}$$

Matching this with the required form gives

$$\boxed{A = \frac{3}{8}, \quad B = \frac{1}{2}}.$$

Now integrate the identity term by term:

$$\begin{aligned}\int \cos^4(x) dx &= \int \left(\frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x) \right) dx \\ &= \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C.\end{aligned}$$

Hence,

$$\boxed{\int \cos^4(x) dx = \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C}.$$

Question 10

A signal is defined by

$$s(t) = \sin(3t) + \sin(5t).$$

Compute its energy over $0 \leq t \leq 2\pi$:

$$E = \int_0^{2\pi} s^2(t) dt.$$

Question 10 – Solution

First expand the square:

$$\begin{aligned} E &= \int_0^{2\pi} (\sin(3t) + \sin(5t))^2 dt \\ &= \int_0^{2\pi} [\sin^2(3t) + 2\sin(3t)\sin(5t) + \sin^2(5t)] dt. \end{aligned} \tag{1}$$

For the cross term, use

$$2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B).$$

Thus,

$$\begin{aligned} 2 \int_0^{2\pi} \sin(3t)\sin(5t) dt &= \int_0^{2\pi} (\cos(2t) - \cos(8t)) dt \\ &= \left[\frac{1}{2}\sin(2t) - \frac{1}{8}\sin(8t) \right]_0^{2\pi} = 0. \end{aligned}$$

For each squared term, use $\sin^2(kt) = \frac{1}{2}(1 - \cos(2kt))$:

$$\begin{aligned} \int_0^{2\pi} \sin^2(kt) dt &= \frac{1}{2} \int_0^{2\pi} (1 - \cos(2kt)) dt \\ &= \frac{1}{2} \left[t - \frac{\sin(2kt)}{2k} \right]_0^{2\pi} = \pi \end{aligned}$$

for any positive integer k . In particular, the $k = 3$ and $k = 5$ terms each contribute π . Substituting into equation (1),

$$E = \pi + 0 + \pi = 2\pi.$$

Therefore,

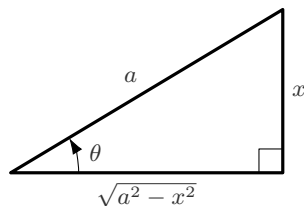
$$\boxed{E = 2\pi}.$$

Question 1

Evaluate the integral both geometrically and analytically:

$$\int_0^a \sqrt{a^2 - x^2} \, dx, \quad a > 0.$$

Question 1 – Solution



Geometrically, $y = \sqrt{a^2 - x^2}$ is the upper semicircle $x^2 + y^2 = a^2$. On $0 \leq x \leq a$, the region is one quarter of a disk of radius a . Hence its area is

$$I = \frac{1}{4}\pi a^2.$$

For an analytic verification, let

$$x = a \sin \theta, \quad dx = a \cos \theta \, d\theta.$$

Since $0 \leq x \leq a$, the new bounds are $0 \leq \theta \leq \pi/2$. Also,

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a \cos \theta, \end{aligned}$$

because $\cos \theta \geq 0$ on this interval. Therefore,

$$\begin{aligned} I &= a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta \\ &= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi a^2}{4}. \end{aligned}$$

Thus both methods give

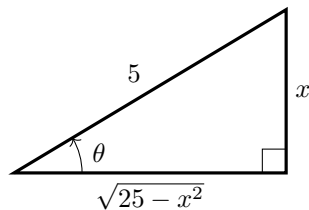
$$\boxed{\int_0^a \sqrt{a^2 - x^2} \, dx = \frac{\pi a^2}{4}}.$$

Question 2

Evaluate using a trigonometric substitution:

$$\int \frac{x^2}{\sqrt{25 - x^2}} dx.$$

Question 2 – Solution



The radical has the form $\sqrt{a^2 - x^2}$, so let

$$x = 5 \sin \theta, \quad dx = 5 \cos \theta \, d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\sqrt{25 - x^2} = \sqrt{25 - 25 \sin^2 \theta} = 5 \cos \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{25 \sin^2 \theta}{5 \cos \theta} (5 \cos \theta \, d\theta) \\ &= 25 \int \sin^2 \theta \, d\theta \\ &= \frac{25}{2} \int (1 - \cos 2\theta) \, d\theta \\ &= \frac{25}{2} \theta - \frac{25}{4} \sin 2\theta + C. \end{aligned}$$

Since $\theta = \arcsin(x/5)$ and

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{x}{5} \right) \left(\frac{\sqrt{25 - x^2}}{5} \right),$$

we obtain

$$I = \frac{25}{2} \arcsin \left(\frac{x}{5} \right) - \frac{x}{2} \sqrt{25 - x^2} + C.$$

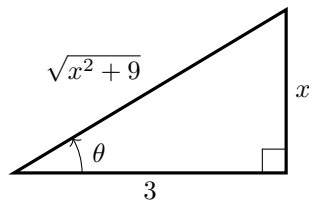
Question 3

Find the total area under

$$y = (x^2 + 9)^{-3/2}$$

on $0 \leq x < \infty$.

Question 3 – Solution



The required area is the improper integral

$$I = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x^2 + 9)^{3/2}}.$$

Let $x = 3 \tan \theta$. Then

$$dx = 3 \sec^2 \theta \, d\theta, \quad (x^2 + 9)^{3/2} = 27 \sec^3 \theta.$$

The bounds transform as $x = 0 \Rightarrow \theta = 0$ and $x \rightarrow \infty \Rightarrow \theta \rightarrow \pi/2$. Thus,

$$\begin{aligned} I &= \frac{1}{9} \int_0^{\pi/2} \cos \theta \, d\theta \\ &= \frac{1}{9} [\sin \theta]_0^{\pi/2} \\ &= \frac{1}{9}. \end{aligned}$$

The integrand is positive, so this convergent improper integral equals the geometric area. Therefore,

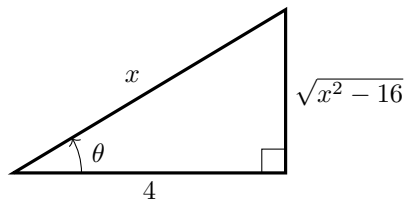
$$\boxed{I = \frac{1}{9}}.$$

Question 4

Evaluate using a trigonometric substitution:

$$\int \frac{dx}{x^2 \sqrt{x^2 - 16}}, \quad x > 4.$$

Question 4 – Solution



Because the radical has the form $\sqrt{x^2 - 16}$, let

$$x = 4 \sec \theta, \quad dx = 4 \sec \theta \tan \theta \, d\theta.$$

Since $x > 4$, take $0 < \theta < \pi/2$. Then

$$\sqrt{x^2 - 16} = \sqrt{16 \sec^2 \theta - 16} = 4 \tan \theta.$$

Therefore,

$$\begin{aligned} I &= \int \frac{4 \sec \theta \tan \theta \, d\theta}{(16 \sec^2 \theta)(4 \tan \theta)} \\ &= \frac{1}{16} \int \cos \theta \, d\theta \\ &= \frac{1}{16} \sin \theta + C. \end{aligned}$$

From the reference triangle,

$$\sec \theta = \frac{x}{4}, \quad \sin \theta = \frac{\sqrt{x^2 - 16}}{x}.$$

Hence,

$$\boxed{I = \frac{\sqrt{x^2 - 16}}{16x} + C}.$$

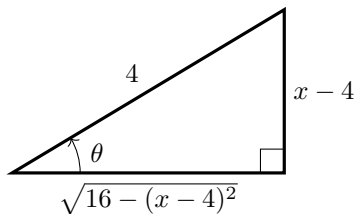
Differentiation recovers $1/(x^2\sqrt{x^2 - 16})$.

Question 5

Complete the square, choose an appropriate trigonometric substitution, and evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Question 5 – Solution



Complete the square inside the radical:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) \\ &= -((x - 4)^2 - 16) \\ &= 16 - (x - 4)^2. \end{aligned}$$

Let

$$x - 4 = 4 \sin \theta, \quad dx = 4 \cos \theta d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\begin{aligned} \sqrt{16 - (x - 4)^2} &= \sqrt{16 - 16 \sin^2 \theta} \\ &= 4 \cos \theta. \end{aligned}$$

Thus,

$$\begin{aligned} I &= \int \frac{4 \cos \theta d\theta}{4 \cos \theta} \\ &= \int d\theta = \theta + C. \end{aligned}$$

Since $\sin \theta = (x - 4)/4$,

$$I = \arcsin \left(\frac{x - 4}{4} \right) + C.$$

This formula applies on intervals inside $0 < x < 8$, where the radical is real and nonzero.

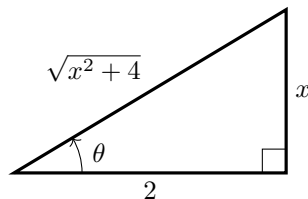
Question 6

Evaluate

$$\int \sqrt{x^2 + 4} \, dx$$

using trigonometric substitution, and verify the result by differentiation.

Question 6 – Solution



Let $x = 2 \tan \theta$, with $-\pi/2 < \theta < \pi/2$. Then

$$dx = 2 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 4} = 2 \sec \theta.$$

Therefore,

$$I = 4 \int \sec^3 \theta d\theta.$$

Using $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$,

$$I = 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C.$$

Now $\tan \theta = x/2$ and $\sec \theta = \sqrt{x^2 + 4}/2$. Hence,

$$\begin{aligned} I &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| + C \\ &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C, \end{aligned}$$

where $-2 \ln 2$ is absorbed into C . Thus,

$$\boxed{I = \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C}.$$

Differentiating and simplifying gives $I' = \sqrt{x^2 + 4}$, as required.

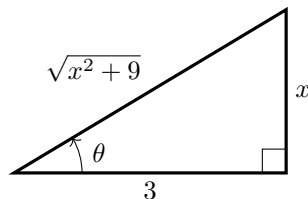
Question 7

For

$$\int \frac{dx}{\sqrt{x^2 + 9}},$$

two substitutions are proposed: $x = 3 \tan \theta$ and $x = 3 \sin \theta$. Select the efficient substitution, explain the identity it uses, and evaluate the integral.

Question 7 – Solution



Choose

$$x = 3 \tan \theta.$$

This substitution uses $1 + \tan^2 \theta = \sec^2 \theta$:

$$x^2 + 9 = 9 \tan^2 \theta + 9 = 9 \sec^2 \theta.$$

Taking $-\pi/2 < \theta < \pi/2$ gives $\sec \theta > 0$, so

$$dx = 3 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 9} = 3 \sec \theta.$$

Hence,

$$\begin{aligned} I &= \int \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Since $\tan \theta = x/3$ and $\sec \theta = \sqrt{x^2 + 9}/3$,

$$\begin{aligned} I &= \ln \left| \frac{x + \sqrt{x^2 + 9}}{3} \right| + C \\ &= \ln |x + \sqrt{x^2 + 9}| + C. \end{aligned}$$

The constant $-\ln 3$ is absorbed into C . Thus,

$$\int \frac{dx}{\sqrt{x^2 + 9}} = \ln |x + \sqrt{x^2 + 9}| + C.$$

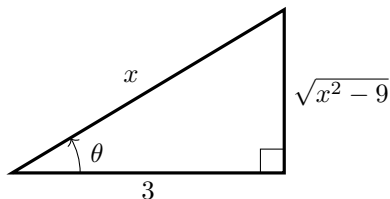
The sine substitution does not turn $x^2 + 9$ into a single squared trigonometric function, so it does not simplify the radical.

Question 8

Evaluate the definite integral exactly:

$$\int_3^5 \frac{\sqrt{x^2 - 9}}{x} dx.$$

Question 8 – Solution



Let

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta d\theta.$$

For $x \geq 3$, take $0 \leq \theta < \pi/2$. Then

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

Transform the bounds:

$$x = 3 \Rightarrow \sec \theta = 1 \Rightarrow \theta = 0, \quad x = 5 \Rightarrow \cos \theta = \frac{3}{5} \Rightarrow \theta = \arccos\left(\frac{3}{5}\right).$$

Thus,

$$\begin{aligned} I &= \int_0^{\arccos(3/5)} \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) d\theta \\ &= 3 \int_0^{\arccos(3/5)} \tan^2 \theta d\theta \\ &= 3 [\tan \theta - \theta]_0^{\arccos(3/5)}. \end{aligned}$$

At the upper bound, a 3-4-5 triangle gives $\tan \theta = 4/3$. Therefore,

$$\begin{aligned} I &= 3 \left(\frac{4}{3} - \arccos\left(\frac{3}{5}\right) \right) \\ &= 4 - 3 \arccos\left(\frac{3}{5}\right). \end{aligned}$$

Hence,

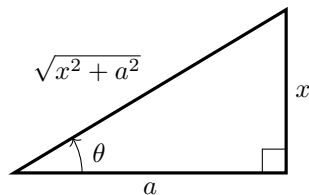
$$\boxed{I = 4 - 3 \arccos\left(\frac{3}{5}\right)}.$$

Question 9

For $a > 0$, derive the antiderivative

$$\int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

Question 9 – Solution



Let

$$x = a \tan \theta, \quad dx = a \sec^2 \theta d\theta.$$

Since $a > 0$, choose $-\pi/2 < \theta < \pi/2$. Then

$$\begin{aligned} (a^2 + x^2)^{3/2} &= (a^2 + a^2 \tan^2 \theta)^{3/2} \\ &= (a^2 \sec^2 \theta)^{3/2} = a^3 \sec^3 \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \int \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta \\ &= \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{1}{a^2} \sin \theta + C. \end{aligned}$$

From a reference triangle,

$$\tan \theta = \frac{x}{a}, \quad \sin \theta = \frac{x}{\sqrt{a^2 + x^2}}.$$

Thus,

$$I = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$$

Indeed,

$$\frac{d}{dx} \left(\frac{x}{a^2 \sqrt{a^2 + x^2}} \right) = \frac{1}{(a^2 + x^2)^{3/2}},$$

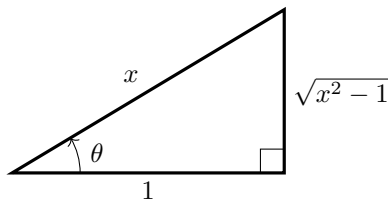
which verifies the result.

Question 10

Use a trigonometric or hyperbolic substitution to evaluate

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx, \quad x > 1.$$

Question 10 – Solution



Because $x > 1$, let

$$x = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Then

$$dx = \sec \theta \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \tan \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{\tan \theta}{\sec^2 \theta} (\sec \theta \tan \theta) \, d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} \, d\theta \\ &= \int \frac{\sec^2 \theta - 1}{\sec \theta} \, d\theta \\ &= \int (\sec \theta - \cos \theta) \, d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C. \end{aligned}$$

Back-substitute using

$$\sec \theta = x, \quad \tan \theta = \sqrt{x^2 - 1}, \quad \sin \theta = \frac{\sqrt{x^2 - 1}}{x}.$$

Since $x > 1$, the logarithm's argument is positive. Therefore,

$$I = \ln \left(x + \sqrt{x^2 - 1} \right) - \frac{\sqrt{x^2 - 1}}{x} + C.$$

Question 1

Find A and B , then evaluate the integral:

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

$$\int \frac{7x+1}{(x-1)(x+2)} dx.$$

Question 1 – Solution

Step 1: Set up the decomposition.

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Step 2: Clear the denominators. Multiply by $(x-1)(x+2)$:

$$7x+1 = A(x+2) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad 8 &= 3A \quad \implies \quad A = \frac{8}{3}, \\ x=-2: \quad -13 &= -3B \quad \implies \quad B = \frac{13}{3}. \end{aligned}$$

Step 4: Substitute and integrate each term.

$$\begin{aligned} \int \frac{7x+1}{(x-1)(x+2)} dx &= \int \left(\frac{8/3}{x-1} + \frac{13/3}{x+2} \right) dx \\ &= \frac{8}{3} \int \frac{dx}{x-1} + \frac{13}{3} \int \frac{dx}{x+2} \\ &= \boxed{\frac{8}{3} \ln|x-1| + \frac{13}{3} \ln|x+2| + C}. \end{aligned}$$

Question 2

Use polynomial division, then evaluate:

$$\int \frac{x^3 + 1}{x^2 - 1} dx.$$

State the excluded values of x .

Question 2 – Solution

Step 1: Divide the polynomials. Since $x^3/x^2 = x$,

$$\begin{aligned}x(x^2 - 1) &= x^3 - x, \\(x^3 + 1) - (x^3 - x) &= x + 1.\end{aligned}$$

Thus $x^3 + 1 = x(x^2 - 1) + (x + 1)$, so

$$\frac{x^3 + 1}{x^2 - 1} = x + \frac{x + 1}{x^2 - 1}.$$

Step 2: Factor and simplify.

$$\begin{aligned}\frac{x^3 + 1}{x^2 - 1} &= x + \frac{x + 1}{(x - 1)(x + 1)} \\&= x + \frac{1}{x - 1},\end{aligned}$$

because the original denominator is zero at both $x = -1$ and $x = 1$.

Step 3: Integrate term by term.

$$\begin{aligned}\int \frac{x^3 + 1}{x^2 - 1} dx &= \int x dx + \int \frac{dx}{x - 1} \\&= \boxed{\frac{x^2}{2} + \ln |x - 1| + C}, \quad x \neq \pm 1.\end{aligned}$$

Question 3

Decompose into partial fractions. Include every power of $(x - 1)$:

$$\frac{2x + 3}{x(x - 1)^2}.$$

Do not integrate.

Question 3 – Solution

Step 1: Include a term for every factor and repeated power.

$$\frac{2x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Step 2: Clear the denominators.

$$2x+3 = A(x-1)^2 + Bx(x-1) + Cx.$$

Step 3: Find A and C .

$$x=0: \quad 3 = A(-1)^2 \implies A=3,$$

$$x=1: \quad 5 = C(1) \implies C=5.$$

Step 4: Find B by using $x=2$.

$$\begin{aligned} 7 &= 3(1)^2 + B(2)(1) + 5(2) \\ &= 3 + 2B + 10, \\ -6 &= 2B \implies B = -3. \end{aligned}$$

Step 5: State the result.

$$\boxed{\frac{2x+3}{x(x-1)^2} = \frac{3}{x} - \frac{3}{x-1} + \frac{5}{(x-1)^2}},$$

and the requested coefficients are

$$\boxed{A=3, \quad B=-3, \quad C=5}.$$

Question 4

Use partial fractions to evaluate:

$$\int \frac{x^2 + 1}{x(x^2 + 4)} dx.$$

Question 4 – Solution

Step 1: Set up the decomposition. Use a linear numerator over the irreducible quadratic:

$$\frac{x^2 + 1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Step 2: Clear denominators and collect like terms.

$$\begin{aligned}x^2 + 1 &= A(x^2 + 4) + x(Bx + C) \\&= (A + B)x^2 + Cx + 4A.\end{aligned}$$

Step 3: Match coefficients.

$$A + B = 1, \quad C = 0, \quad 4A = 1.$$

Therefore $A = 1/4$, $B = 1 - 1/4 = 3/4$, and $C = 0$.

Step 4: Integrate. For the second term, let $u = x^2 + 4$; then $du = 2x \, dx$.

$$\begin{aligned}I &= \frac{1}{4} \int \frac{dx}{x} + \frac{3}{4} \int \frac{x}{x^2 + 4} \, dx \\&= \frac{1}{4} \ln|x| + \frac{3}{4} \left(\frac{1}{2} \int \frac{du}{u} \right) \\&= \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C.\end{aligned}$$

Thus,

$$\boxed{I = \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C}.$$

Question 5

Find the value of k that removes the $\ln|x+1|$ term. Then evaluate:

$$\int \frac{kx+1}{x^2-1} dx.$$

Question 5 – Solution**Step 1: Set up the decomposition.**

$$\frac{kx+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Step 2: Clear the denominators.

$$kx+1 = A(x+1) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad k+1 &= 2A \implies A = \frac{k+1}{2}, \\ x=-1: \quad -k+1 &= -2B \implies B = \frac{k-1}{2}. \end{aligned}$$

Step 4: Remove the $\ln|x+1|$ term. Since integrating $B/(x+1)$ gives $B \ln|x+1|$, set $B=0$:

$$B=0 \implies \frac{k-1}{2} = 0 \implies \boxed{k=1}.$$

Step 5: Substitute $k=1$, simplify, and integrate.

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}, \quad x \neq \pm 1.$$

$$\int \frac{x+1}{x^2-1} dx = \int \frac{dx}{x-1} = \boxed{\ln|x-1| + C}.$$

Question 6

A student proposes the decomposition

$$\frac{1}{(x^2 + 1)^2} = \frac{A}{x^2 + 1} + \frac{B}{(x^2 + 1)^2}.$$

1. Why is this decomposition not useful? Give the correct general form.
2. Use $x = \tan \theta$ to evaluate:

$$\int \frac{dx}{(x^2 + 1)^2}$$

Question 6 – Solution**Part 1: Check the decomposition.**

Each power of an irreducible quadratic needs a linear numerator. The general form is

$$\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

The original fraction already has this form: $A = B = C = 0$ and $D = 1$. Therefore, decomposing it only reproduces the original integrand and does not make the integral easier.

Part 2: Evaluate the integral.

Step 1: Substitute $x = \tan \theta$. Then

$$dx = \sec^2 \theta \, d\theta, \quad x^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta.$$

Step 2: Simplify and integrate.

$$\begin{aligned} I &= \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int \cos^2 \theta \, d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C. \end{aligned}$$

Step 3: Return to x .

$$\theta = \arctan x, \quad \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2x}{1 + x^2}.$$

Substitute both expressions:

$$I = \frac{x}{2(x^2 + 1)} + \frac{1}{2} \arctan x + C.$$

Question 7

Use partial fractions to evaluate exactly:

$$\int_0^{1/2} \frac{dx}{1-x^2}.$$

Why is this integral proper on $[0, 1/2]$?

Question 7 – Solution

Step 1: Factor and set up partial fractions.

$$1 - x^2 = (1 - x)(1 + x),$$

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}.$$

Step 2: Clear denominators and find A and B .

$$1 = A(1 + x) + B(1 - x).$$

$$x = 1 : \quad 1 = 2A \implies A = \frac{1}{2},$$

$$x = -1 : \quad 1 = 2B \implies B = \frac{1}{2}.$$

Step 3: Integrate and evaluate both bounds.

$$\begin{aligned} I &= \frac{1}{2} \int_0^{1/2} \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) dx \\ &= \frac{1}{2} [-\ln(1 - x) + \ln(1 + x)]_0^{1/2} \\ &= \frac{1}{2} \left[-\ln \frac{1}{2} + \ln \frac{3}{2} - (-\ln 1 + \ln 1) \right] \\ &= \frac{1}{2} \ln \left(\frac{3/2}{1/2} \right) = \frac{1}{2} \ln 3. \end{aligned}$$

Step 4: Check whether the integral is proper. The singularities $x = \pm 1$ are outside $[0, 1/2]$, so the integrand is continuous on the whole interval and no improper limit is needed.

$$\boxed{\int_0^{1/2} \frac{dx}{1 - x^2} = \frac{1}{2} \ln 3.}$$

Question 8

Find A , B , and C in the decomposition

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Use $x = 0$, 2 , and -3 . State (A, B, C) .

Question 8 – Solution

Step 1: Start with the decomposition.

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

Step 2: Multiply every term by $x(x-2)(x+3)$.

$$1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2).$$

Step 3: Use values that eliminate two terms at a time.

$$\begin{array}{llll} x = 0 : & 1 = -6A & \implies & A = -\frac{1}{6}, \\ x = 2 : & 1 = 10B & \implies & B = \frac{1}{10}, \\ x = -3 : & 1 = 15C & \implies & C = \frac{1}{15}. \end{array}$$

Step 4: State the ordered triple.

$$(A, B, C) = \left(-\frac{1}{6}, \frac{1}{10}, \frac{1}{15} \right).$$

Question 9

Use partial fractions to evaluate:

$$\int \frac{dx}{x^3 - x^2} = \int \frac{dx}{x^2(x - 1)}.$$

Question 9 – Solution

Step 1: Set up partial fractions. The repeated factor x^2 requires both $1/x$ and $1/x^2$:

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Step 2: Clear the denominators.

$$1 = Ax(x-1) + B(x-1) + Cx^2.$$

Step 3: Find B and C .

$$x = 0: \quad 1 = -B \implies B = -1,$$

$$x = 1: \quad 1 = C \implies C = 1.$$

Step 4: Find A by setting $x = 2$.

$$\begin{aligned} 1 &= A(2)(1) + (-1)(1) + (1)(4) \\ &= 2A + 3, \\ -2 &= 2A \implies A = -1. \end{aligned}$$

Step 5: Substitute and integrate each term.

$$\begin{aligned} I &= \int \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx \\ &= -\int \frac{dx}{x} - \int x^{-2} dx + \int \frac{dx}{x-1} \\ &= -\ln|x| - \frac{x^{-1}}{-1} + \ln|x-1| + C \\ &= -\ln|x| + \frac{1}{x} + \ln|x-1| + C. \end{aligned}$$

Therefore,

$$\boxed{I = -\ln|x| + \frac{1}{x} + \ln|x-1| + C}.$$

Question 10

1. Define and evaluate the Cauchy principal value:

$$\text{PV} \int_{-1}^1 \frac{dx}{x}$$

2. Does the ordinary improper integral converge? Check both sides of $x = 0$.

Question 10 – Solution

Part 1: Cauchy principal value.

Step 1: Approach zero by the same distance ε from both sides.

$$\begin{aligned} \text{PV} \int_{-1}^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^1 \frac{dx}{x} \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} ([\ln |x|]_{-1}^{-\varepsilon} + [\ln |x|]_{\varepsilon}^1) \\ &= \lim_{\varepsilon \rightarrow 0^+} ((\ln \varepsilon - \ln 1) + (\ln 1 - \ln \varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0^+} 0 = 0. \end{aligned}$$

$$\boxed{\text{PV} \int_{-1}^1 \frac{dx}{x} = 0}.$$

Part 2: Ordinary improper integral.

The two one-sided integrals must converge separately. On the right,

$$\begin{aligned} \int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (0 - \ln \varepsilon) = +\infty. \end{aligned}$$

On the left,

$$\begin{aligned} \int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln |x|]_{-1}^{-\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - 0) = -\infty. \end{aligned}$$

Therefore, the ordinary improper integral diverges. The principal value is finite only because these opposite divergences cancel in a symmetric limit.

Question 1

Use $u = \sqrt{x}$ to evaluate

$$\int \frac{dx}{1 + \sqrt{x}}.$$

Question 1 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\int \frac{dx}{1 + \sqrt{x}} = \int \frac{2u}{1 + u} \, du.$$

Step 3: Divide the numerator.

$$\frac{u}{u+1} = \frac{(u+1)-1}{u+1} = 1 - \frac{1}{u+1}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} 2 \int \left(1 - \frac{1}{u+1} \right) du &= 2u - 2 \ln |u+1| + C \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C. \end{aligned}$$

$$\boxed{2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C}$$

Question 2

Use $u = x + 2$ to evaluate:

$$\int x\sqrt{x+2} \, dx.$$

Question 2 – Solution

Step 1: Substitute $u = x + 2$. Then

$$x = u - 2, \quad du = dx.$$

Step 2: Rewrite and expand the integrand.

$$\begin{aligned} \int x\sqrt{x+2} \, dx &= \int (u-2)u^{1/2} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du. \end{aligned}$$

Step 3: Integrate each power.

$$\begin{aligned} I &= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C. \end{aligned}$$

Step 4: Substitute $u = x + 2$.

$$\boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$$

Question 3

Find the geometric area between $y = \sqrt[3]{x}$ and the x -axis on $[-1, 8]$.

Question 3 – Solution

Step 1: Split at the x -intercept. The function is negative on $[-1, 0]$ and positive on $[0, 8]$, so

$$A = - \int_{-1}^0 x^{1/3} dx + \int_0^8 x^{1/3} dx.$$

Step 2: Find an antiderivative.

$$\int x^{1/3} dx = \frac{x^{4/3}}{4/3} = \frac{3}{4}x^{4/3}.$$

Step 3: Evaluate both pieces.

$$\begin{aligned} A &= - \left[\frac{3}{4}x^{4/3} \right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} \right]_0^8 \\ &= - \left(0 - \frac{3}{4} \right) + \left(\frac{3}{4} \cdot 16 - 0 \right) \\ &= \frac{3}{4} + 12 = \frac{51}{4}. \end{aligned}$$

$$\boxed{A = \frac{51}{4}}$$

Question 4

Rationalize the integrand and evaluate

$$\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}.$$

Question 4 – Solution

Step 1: Multiply by the conjugate.

$$\begin{aligned}\frac{1}{\sqrt{x+1}-\sqrt{x}} &= \frac{1}{\sqrt{x+1}-\sqrt{x}} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\ &= \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x} \\ &= \sqrt{x+1}+\sqrt{x}.\end{aligned}$$

Step 2: Integrate both terms.

$$\begin{aligned}I &= \int (x+1)^{1/2} dx + \int x^{1/2} dx \\ &= \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + C \\ &= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C.\end{aligned}$$

$$\boxed{\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C}$$

Question 5

Evaluate

$$\int \frac{\sqrt{x}}{1+x} dx.$$

Question 5 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{u}{1+u^2} (2u \, du) \\ &= 2 \int \frac{u^2}{1+u^2} du. \end{aligned}$$

Step 3: Divide the numerator.

$$\frac{u^2}{1+u^2} = \frac{(1+u^2) - 1}{1+u^2} = 1 - \frac{1}{1+u^2}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} I &= 2 \int \left(1 - \frac{1}{1+u^2} \right) du \\ &= 2u - 2 \arctan u + C \\ &= 2\sqrt{x} - 2 \arctan \sqrt{x} + C. \end{aligned}$$

$2\sqrt{x} - 2 \arctan \sqrt{x} + C$

Question 6

Find $a > 0$ so

$$\int_0^a \sqrt{x} \, dx = 18.$$

Question 6 – Solution

Step 1: Evaluate the definite integral.

$$\begin{aligned}\int_0^a \sqrt{x} \, dx &= \int_0^a x^{1/2} \, dx \\ &= \left[\frac{x^{3/2}}{3/2} \right]_0^a \\ &= \left[\frac{2}{3} x^{3/2} \right]_0^a = \frac{2}{3} a^{3/2}.\end{aligned}$$

Step 2: Set the result equal to 18 and solve.

$$\begin{aligned}\frac{2}{3} a^{3/2} &= 18, \\ a^{3/2} &= 27, \\ \left(a^{3/2} \right)^{2/3} &= 27^{2/3}, \\ a &= (\sqrt[3]{27})^2 = 3^2 = 9.\end{aligned}$$

$a = 9$

Question 7

Evaluate

$$\int \frac{dx}{x\sqrt[3]{x+1}}$$

using $u = \sqrt[3]{x+1}$.

Question 7 – Solution

Step 1: Substitute $u = \sqrt[3]{x+1}$. Then

$$u^3 = x + 1, \quad x = u^3 - 1, \quad dx = 3u^2 du.$$

Step 2: Rewrite and factor.

$$I = \int \frac{3u^2 du}{(u^3 - 1)u} = 3 \int \frac{u}{u^3 - 1} du,$$
$$u^3 - 1 = (u - 1)(u^2 + u + 1).$$

Step 3: Use partial fractions.

$$\frac{3u}{u^3 - 1} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + u + 1}.$$

After clearing denominators,

$$3u = A(u^2 + u + 1) + (Bu + C)(u - 1).$$

Matching coefficients gives

$$A + B = 0, \quad A - B + C = 3, \quad A - C = 0,$$

so $A = 1$, $B = -1$, and $C = 1$. Therefore,

$$\frac{3u}{u^3 - 1} = \frac{1}{u - 1} + \frac{-u + 1}{u^2 + u + 1}.$$

Step 4: Prepare the quadratic term.

$$-u + 1 = -\frac{1}{2}(2u + 1) + \frac{3}{2}.$$

Thus

$$I = \int \frac{du}{u - 1} - \frac{1}{2} \int \frac{2u + 1}{u^2 + u + 1} du + \frac{3}{2} \int \frac{du}{u^2 + u + 1}.$$

Complete the square:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Step 5: Integrate and return to x .

$$I = \ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C,$$

where $u = \sqrt[3]{x+1}$.

$$\ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$

Question 8

Evaluate

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

using a substitution that clears both radicals.

Question 8 – Solution

Step 1: Choose a power that clears both radicals. Since $\text{lcm}(2, 3) = 6$, let $x = u^6$. Then

$$\sqrt{x} = u^3, \quad \sqrt[3]{x} = u^2, \quad dx = 6u^5 du.$$

The bounds remain 0 and 1 because $u = x^{1/6}$. **Step 2: Rewrite and simplify.**

$$\begin{aligned} I &= 6 \int_0^1 \frac{u^5}{u^3 + u^2} du \\ &= 6 \int_0^1 \frac{u^5}{u^2(u+1)} du = 6 \int_0^1 \frac{u^3}{u+1} du. \end{aligned}$$

Step 3: Divide the polynomials.

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Step 4: Integrate and evaluate the bounds.

$$\begin{aligned} I &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1) \right]_0^1 \\ &= 6 \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2 \right) \\ &= 6 \left(\frac{5}{6} - \ln 2 \right) = 5 - 6 \ln 2. \end{aligned}$$

$$\boxed{I = 5 - 6 \ln 2}$$

Question 9

Differentiate $F(x)$, then identify the substitution used to integrate $F'(x)$:

$$F(x) = \frac{2}{3}(1 + x^2)^{3/2}.$$

Question 9 – Solution

Step 1: Differentiate with the chain rule.

$$\begin{aligned} F'(x) &= \frac{2}{3} \cdot \frac{3}{2} (1+x^2)^{1/2} \cdot (2x) \\ &= 2x\sqrt{1+x^2}. \end{aligned}$$

Step 2: Reverse the process by integration.

$$\int 2x\sqrt{1+x^2} \, dx.$$

Let

$$u = 1 + x^2, \quad du = 2x \, dx.$$

Then

$$\begin{aligned} \int 2x\sqrt{1+x^2} \, dx &= \int u^{1/2} \, du \\ &= \frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{3}(1+x^2)^{3/2} + C. \end{aligned}$$

$$\boxed{\int 2x\sqrt{1+x^2} \, dx = F(x) + C}$$

Question 10

Use one substitution to evaluate:

$$\int \frac{x+1}{\sqrt{x^2+2x+5}} dx.$$

Question 10 – Solution

Step 1: Substitute the expression inside the square root.

$$u = x^2 + 2x + 5.$$

Differentiate:

$$du = (2x + 2) dx = 2(x + 1) dx, \quad (x + 1) dx = \frac{1}{2} du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} I &= \int \frac{x + 1}{\sqrt{x^2 + 2x + 5}} dx \\ &= \frac{1}{2} \int u^{-1/2} du. \end{aligned}$$

Step 3: Integrate and return to x .

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{1/2}}{1/2} \right) + C \\ &= u^{1/2} + C \\ &= \sqrt{x^2 + 2x + 5} + C. \end{aligned}$$

$\sqrt{x^2 + 2x + 5} + C$

Question 1

Complete the square and evaluate

$$\int \frac{dx}{x^2 + 6x + 13}.$$

Question 1 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 + 6x + 13 &= (x^2 + 6x + 9) + 4 \\&= (x + 3)^2 + 4.\end{aligned}$$

Step 2: Rewrite the integral.

$$I = \int \frac{dx}{(x + 3)^2 + 2^2}.$$

Step 3: Use the arctangent formula. Let $u = x + 3$, so $du = dx$:

$$\begin{aligned}I &= \int \frac{du}{u^2 + 2^2} \\&= \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\&= \frac{1}{2} \arctan\left(\frac{x + 3}{2}\right) + C.\end{aligned}$$

$$\frac{1}{2} \arctan \frac{x + 3}{2} + C$$

Question 2

Split the numerator, then evaluate:

$$\int \frac{3x + 1}{x^2 + 2x + 5} dx.$$

Question 2 – Solution

Step 1: Split the numerator using the derivative of the denominator. Since

$$\frac{d}{dx}(x^2 + 2x + 5) = 2x + 2,$$

write

$$3x + 1 = \frac{3}{2}(2x + 2) - 2.$$

Step 2: Split the integral.

$$I = \frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx - 2 \int \frac{dx}{x^2 + 2x + 5}.$$

Step 3: Integrate the derivative term.

$$\frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx = \frac{3}{2} \ln(x^2 + 2x + 5).$$

Step 4: Complete the square in the remaining term.

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Thus

$$\begin{aligned} -2 \int \frac{dx}{(x + 1)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 1}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 1}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\boxed{\frac{3}{2} \ln(x^2 + 2x + 5) - \arctan \frac{x + 1}{2} + C}$$

Question 3

Evaluate

$$\int \frac{x}{x^2 + 4x + 8} dx.$$

Question 3 – Solution

Step 1: Rewrite the numerator using the derivative of the denominator.

$$x = \frac{1}{2}(2x + 4) - 2.$$

Step 2: Split the integral.

$$I = \frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx - 2 \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx = \frac{1}{2} \ln(x^2 + 4x + 8).$$

Step 4: Complete the square.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\begin{aligned} -2 \int \frac{dx}{(x + 2)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 2}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 2}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\boxed{\frac{1}{2} \ln(x^2 + 4x + 8) - \arctan \frac{x + 2}{2} + C}$$

Question 4

Find the maximum of $f(x) = 1/(x^2 - 4x + 8)$, then evaluate

$$\int_{-\infty}^{\infty} f(x) \, dx.$$

Question 4 – Solution**Step 1: Complete the square.**

$$x^2 - 4x + 8 = (x - 2)^2 + 4.$$

Step 2: Find the maximum. The denominator is smallest when $(x - 2)^2 = 0$, or $x = 2$. Its minimum is 4, so

$$f(2) = \frac{1}{4}.$$

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 3: Write the improper integral as a limit.**

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(x - 2)^2 + 2^2} \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \left[\arctan \left(\frac{x - 2}{2} \right) \right]_{-R}^R. \end{aligned}$$

Step 4: Evaluate the limits.

$$I = \frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

$\max = 1/4, \quad I = \pi/2$

Question 5

Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}}.$$

Question 5 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 - 6x + 10 &= (x^2 - 6x + 9) + 1 \\&= (x - 3)^2 + 1.\end{aligned}$$

Step 2: Substitute $u = x - 3$. Then $du = dx$ and

$$I = \int \frac{du}{\sqrt{u^2 + 1}}.$$

Step 3: Use the standard formula.

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left| u + \sqrt{u^2 + a^2} \right| + C.$$

With $a = 1$,

$$\begin{aligned}I &= \ln \left| u + \sqrt{u^2 + 1} \right| + C \\&= \ln \left| x - 3 + \sqrt{x^2 - 6x + 10} \right| + C.\end{aligned}$$

$$\boxed{\ln |x - 3 + \sqrt{x^2 - 6x + 10}| + C}$$

Question 6

Find every k that makes $x^2 + kx + 9$ a perfect square. Then integrate

$$\frac{1}{x^2 + kx + 9}.$$

Question 6 – Solution

Step 1: Write the possible perfect squares. Since the constant term is 9, the square must be

$$(x + 3)^2 = x^2 + 6x + 9,$$

$$(x - 3)^2 = x^2 - 6x + 9.$$

Thus

$$k = 6 \quad \text{or} \quad k = -6.$$

Step 2: Integrate when $k = 6$.

$$\begin{aligned} \int \frac{dx}{x^2 + 6x + 9} &= \int \frac{dx}{(x + 3)^2} \\ &= \int (x + 3)^{-2} dx \\ &= -(x + 3)^{-1} + C = -\frac{1}{x + 3} + C. \end{aligned}$$

Step 3: Integrate when $k = -6$.

$$\begin{aligned} \int \frac{dx}{x^2 - 6x + 9} &= \int (x - 3)^{-2} dx \\ &= -(x - 3)^{-1} + C = -\frac{1}{x - 3} + C. \end{aligned}$$

$k = 6 : -\frac{1}{x + 3} + C; \quad k = -6 : -\frac{1}{x - 3} + C$

Question 7

Evaluate

$$\int \frac{x^2}{x^2 + 4} dx$$

without partial fractions.

Question 7 – Solution**Step 1: Rewrite the numerator.**

$$x^2 = (x^2 + 4) - 4.$$

Therefore,

$$\begin{aligned}\frac{x^2}{x^2 + 4} &= \frac{(x^2 + 4) - 4}{x^2 + 4} \\ &= 1 - \frac{4}{x^2 + 4}.\end{aligned}$$

Step 2: Split the integral.

$$I = \int 1 \, dx - 4 \int \frac{dx}{x^2 + 2^2}.$$

Step 3: Integrate both terms.

$$\begin{aligned}I &= x - 4 \left[\frac{1}{2} \arctan \left(\frac{x}{2} \right) \right] + C \\ &= x - 2 \arctan \left(\frac{x}{2} \right) + C.\end{aligned}$$

$$x - 2 \arctan \frac{x}{2} + C$$

Question 8

Determine the real domain and evaluate

$$\int \frac{dx}{\sqrt{6x - x^2}}.$$

Question 8 – Solution

Step 1: Complete the square.

$$\begin{aligned}6x - x^2 &= -(x^2 - 6x) \\&= -((x - 3)^2 - 9) \\&= 9 - (x - 3)^2.\end{aligned}$$

Step 2: Determine the domain. The radicand must be nonnegative:

$$\begin{aligned}9 - (x - 3)^2 &\geq 0, \\(x - 3)^2 &\leq 9, \\-3 &\leq x - 3 \leq 3, \\0 &\leq x \leq 6.\end{aligned}$$

The denominator cannot equal zero, so the integrand's domain is

$$\boxed{0 < x < 6}.$$

Step 3: Substitute $u = x - 3$, so $du = dx$:

$$I = \int \frac{du}{\sqrt{3^2 - u^2}}.$$

Step 4: Use the inverse-sine formula.

$$\begin{aligned}I &= \arcsin\left(\frac{u}{3}\right) + C \\&= \arcsin\left(\frac{x - 3}{3}\right) + C.\end{aligned}$$

$$\boxed{\arcsin \frac{x - 3}{3} + C}$$

Question 9

Split the numerator, then evaluate:

$$\int \frac{2x + 5}{x^2 + 4x + 8} dx.$$

Question 9 – Solution**Step 1: Split the numerator.**

$$2x + 5 = (2x + 4) + 1.$$

Step 2: Split the integral.

$$I = \int \frac{2x + 4}{x^2 + 4x + 8} dx + \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\int \frac{2x + 4}{x^2 + 4x + 8} dx = \ln(x^2 + 4x + 8).$$

Step 4: Complete the square for the remaining term.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\int \frac{dx}{(x + 2)^2 + 2^2} = \frac{1}{2} \arctan \left(\frac{x + 2}{2} \right).$$

Step 5: Combine the results.

$$\boxed{\ln(x^2 + 4x + 8) + \frac{1}{2} \arctan \frac{x + 2}{2} + C}$$

Question 10

For $a > 0$, evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2}.$$

Question 10 – Solution

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 1: Write the improper integral as a limit.**

$$I = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + a^2}.$$

Step 2: Find an antiderivative. Since $a > 0$,

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

Therefore,

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \frac{1}{a} \left[\arctan\left(\frac{x}{a}\right) \right]_{-R}^R \\ &= \frac{1}{a} \lim_{R \rightarrow \infty} \left[\arctan\left(\frac{R}{a}\right) - \arctan\left(\frac{-R}{a}\right) \right]. \end{aligned}$$

Step 3: Evaluate the limits.

$$\begin{aligned} I &= \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\ &= \frac{\pi}{a}. \end{aligned}$$

$$\boxed{I = \frac{\pi}{a}}$$

Question 1

Choose the shortest method and evaluate:

$$\int x e^{x^2} dx.$$

Question 1 – Solution

Step 1: Identify the inner function. The exponent is x^2 , and its derivative $2x$ is present except for a constant factor.

Step 2: Substitute $u = x^2$. Then

$$du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int x e^{x^2} \, dx &= \frac{1}{2} \int e^u \, du \\ &= \frac{1}{2} e^u + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{2} e^{x^2} + C}$$

Question 2

Choose substitution or integration by parts, then evaluate:

$$\int x \ln(1 + x^2) dx.$$

Question 2 – Solution

Step 1: Choose substitution. The expression $1 + x^2$ is inside the logarithm, and its derivative $2x$ matches the remaining factor.

Let

$$u = 1 + x^2, \quad du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Then

$$I = \frac{1}{2} \int \ln u \, du.$$

Step 2: Integrate $\ln u$ by parts. Choose

$$v = \ln u, \quad dw = du.$$

Then $dv = du/u$ and $w = u$, so

$$\begin{aligned} \int \ln u \, du &= u \ln u - \int u \left(\frac{1}{u} \right) du \\ &= u \ln u - u + C. \end{aligned}$$

Step 3: Restore the factor $1/2$ and return to x .

$$\begin{aligned} I &= \frac{1}{2} (u \ln u - u) + C \\ &= \frac{1}{2} [(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C. \end{aligned}$$

$$\frac{1}{2} [(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C$$

Question 3

Use $u = x^2 + 1$ to evaluate:

$$\int \frac{x^3}{\sqrt{x^2 + 1}} dx.$$

Question 3 – Solution**Step 1: Separate one factor of x .**

$$x^3 dx = x^2(x dx).$$

Step 2: Substitute $u = x^2 + 1$. Then

$$du = 2x dx, \quad x dx = \frac{1}{2} du, \quad x^2 = u - 1.$$

Step 3: Rewrite and simplify.

$$\begin{aligned} I &= \frac{1}{2} \int \frac{u-1}{\sqrt{u}} du \\ &= \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du. \end{aligned}$$

Step 4: Integrate each power.

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} \right) + C \\ &= \frac{1}{3} u^{3/2} - u^{1/2} + C. \end{aligned}$$

Step 5: Return to x .

$$\frac{1}{3}(x^2 + 1)^{3/2} - \sqrt{x^2 + 1} + C$$

Question 4

Identify the hidden derivative and evaluate:

$$\int \frac{\ln x}{x} dx.$$

Question 4 – Solution

Step 1: Identify the inner function. Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

the factor dx/x is the differential of $\ln x$.

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

Step 3: Rewrite and integrate.

$$\begin{aligned}\int \frac{\ln x}{x} dx &= \int u du \\ &= \frac{u^2}{2} + C.\end{aligned}$$

Step 4: Return to x .

$\frac{1}{2}(\ln x)^2 + C$

Question 5

Recognize a product derivative and evaluate:

$$\int e^x(x+1) dx.$$

Question 5 – Solution

Step 1: Look for a product whose derivative matches the integrand. Try xe^x .

Step 2: Differentiate with the product rule.

$$\begin{aligned}\frac{d}{dx}(xe^x) &= \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) \\ &= e^x + xe^x \\ &= e^x(x+1).\end{aligned}$$

Step 3: Reverse the derivative.

$$\int e^x(x+1) dx = xe^x + C.$$

$xe^x + C$

Question 6

Explain why $u = x^2 + 1$ does not simplify the integral directly, then evaluate:

$$\int \frac{dx}{x^2 + 1}.$$

Question 6 – Solution

Step 1: Test the proposed substitution. If

$$u = x^2 + 1,$$

then

$$du = 2x \, dx.$$

The integral contains dx , but it does not contain the required factor $x \, dx$. Thus the substitution does not produce a direct du factor. One can express dx in terms of u on a branch, but this introduces a square root and is less convenient than the standard antiderivative.

Step 2: Recognize a standard form.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.$$

Therefore,

$$\int \frac{dx}{x^2 + 1} = \arctan x + C.$$

$$\int \frac{dx}{x^2 + 1} = \arctan x + C$$

Question 7

Choose the best method and evaluate:

$$\int \frac{x^2}{x^3 + 8} dx.$$

Question 7 – Solution

Step 1: Identify the structure. The denominator is $x^3 + 8$, and its derivative is $3x^2$. The numerator contains x^2 , so substitution is the shortest method.

Step 2: Substitute $u = x^3 + 8$. Then

$$du = 3x^2 dx, \quad x^2 dx = \frac{1}{3} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \frac{x^2}{x^3 + 8} dx &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln |u| + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{3} \ln |x^3 + 8| + C}$$

Question 8

State the structural cue, then evaluate:

$$\int \sin x \cos^4 x \, dx.$$

Question 8 – Solution

Step 1: Identify the structural cue. The sine power is odd, and $\sin x \, dx$ is the differential of $-\cos x$. Keep the sine factor and use the cosine as the substitution.

Step 2: Substitute $u = \cos x$. Then

$$du = -\sin x \, dx, \quad \sin x \, dx = -du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \sin x \cos^4 x \, dx &= - \int u^4 \, du \\ &= -\frac{u^5}{5} + C. \end{aligned}$$

Step 4: Return to x .

$$-\frac{1}{5} \cos^5 x + C$$

Question 9

Choose the best method for each integral. Do not evaluate.

$$\int x^2 e^x dx, \quad \int \frac{dx}{x^2 - 4}, \quad \int \sqrt{9 - x^2} dx, \quad \int \sin^2 x dx.$$

Question 9 – Solution

1. For $\int x^2 e^x dx$, differentiating x^2 repeatedly makes it simpler, while integrating e^x changes nothing. Use **integration by parts** twice.

2. For $\int dx/(x^2 - 4)$, factor

$$x^2 - 4 = (x - 2)(x + 2).$$

The denominator has distinct linear factors, so use **partial fractions**.

3. For $\int \sqrt{9 - x^2} dx$, the form $\sqrt{a^2 - x^2}$ suggests $x = a \sin \theta$. Use **trigonometric substitution** with $x = 3 \sin \theta$.

4. For $\int \sin^2 x dx$, both sine and cosine powers are even. Use the **power-reduction identity**

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

$x^2 e^x$	integration by parts
$1/(x^2 - 4)$	partial fractions
$\sqrt{9 - x^2}$	trigonometric substitution
$\sin^2 x$	power reduction

Question 10

Find the integrand whose antiderivative is

$$F(x) = \ln(x^2 + 1) + \arctan x.$$

Then verify your result.

Question 10 – Solution

Step 1: Differentiate the logarithm using the chain rule.

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}.$$

Step 2: Differentiate the inverse tangent.

$$\frac{d}{dx} \arctan x = \frac{1}{x^2 + 1}.$$

Step 3: Add and simplify.

$$\begin{aligned} F'(x) &= \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} \\ &= \frac{2x + 1}{x^2 + 1}. \end{aligned}$$

Thus the required integrand is

$$\boxed{f(x) = \frac{2x + 1}{x^2 + 1}}$$

Step 4: Verify by integration.

$$\begin{aligned} \int \frac{2x + 1}{x^2 + 1} dx &= \int \frac{2x}{x^2 + 1} dx + \int \frac{dx}{x^2 + 1} \\ &= \ln(x^2 + 1) + \arctan x + C = F(x) + C. \end{aligned}$$

Question 1

For which real p does the integral converge? Evaluate it when possible.

$$\int_1^{\infty} x^{-p} dx$$

Question 1 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx.$$

Step 2: First suppose $p \neq 1$.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \frac{b^{1-p} - 1}{1-p}. \end{aligned}$$

If $p > 1$, then $1 - p < 0$ and $b^{1-p} \rightarrow 0$, so

$$I = \frac{-1}{1-p} = \frac{1}{p-1}.$$

If $p < 1$, then $b^{1-p} \rightarrow \infty$, so the integral diverges.

Step 3: Check $p = 1$ separately.

$$\int_1^b \frac{dx}{x} = \ln b \rightarrow \infty.$$

$p > 1 : I = \frac{1}{p-1}; \quad p \leq 1 : \text{diverges}$

Question 2

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^1 x^{-2/3} dx$$

Question 2 – Solution

Step 1: Identify the improper endpoint. The integrand is undefined at $x = 0$, so replace the lower bound with $\varepsilon > 0$:

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-2/3} dx.$$

Step 2: Integrate.

$$\int x^{-2/3} dx = \frac{x^{1/3}}{1/3} = 3x^{1/3}.$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [3x^{1/3}]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (3 - 3\varepsilon^{1/3}) = 3. \end{aligned}$$

3 (convergent)

Question 3

Determine whether the integral converges. If it does, evaluate it.

$$\int_{-1}^1 \frac{dx}{\sqrt{|x|}}$$

Question 3 – Solution

Step 1: Split at the interior singularity.

$$I = \int_{-1}^0 \frac{dx}{\sqrt{|x|}} + \int_0^1 \frac{dx}{\sqrt{|x|}}.$$

The integrand is even, so the two pieces are equal:

$$I = 2 \int_0^1 x^{-1/2} dx.$$

Step 2: Write the improper limit.

$$I = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-1/2} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{\varepsilon \rightarrow 0^+} [2x^{1/2}]_{\varepsilon}^1 \\ &= 4 \lim_{\varepsilon \rightarrow 0^+} (1 - \sqrt{\varepsilon}) = 4. \end{aligned}$$

$I = 4 \text{ (convergent)}$

Question 4

For which real a does the integral converge? Evaluate it when possible.

$$\int_0^{\infty} e^{-ax} dx$$

Question 4 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b e^{-ax} dx.$$

Step 2: Suppose $a \neq 0$ and integrate.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{a} e^{-ax} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \frac{1 - e^{-ab}}{a}. \end{aligned}$$

If $a > 0$, then $e^{-ab} \rightarrow 0$, so $I = 1/a$. If $a < 0$, then $e^{-ab} \rightarrow \infty$, so the integral diverges.

Step 3: Check $a = 0$. Then the integrand is 1, and

$$\int_0^\infty 1 dx$$

diverges.

$a > 0 : 1/a; \quad a \leq 0 : \text{diverges}$

Question 5

Determine whether the integral converges:

$$\int_2^{\infty} \frac{dx}{x \ln x}.$$

Question 5 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x}.$$

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

The bounds become $u = \ln 2$ and $u = \ln b$, so

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} \\ &= \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)). \end{aligned}$$

Step 3: Evaluate the limit. Since $\ln(\ln b) \rightarrow \infty$, the integral diverges.

diverges

Question 6

Evaluate the two-tailed integral:

$$\int_{-\infty}^{\infty} e^{-|x|} dx.$$

Question 6 – Solution

Step 1: Use symmetry. Since $e^{-|x|}$ is even,

$$I = 2 \int_0^{\infty} e^{-x} dx.$$

Step 2: Replace infinity with a limit.

$$I = 2 \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\ &= 2 \lim_{b \rightarrow \infty} (1 - e^{-b}) \\ &= 2(1 - 0) = 2. \end{aligned}$$

$$\boxed{I = 2}$$

Question 7

Determine whether the ordinary improper integral converges:

$$\int_{-1}^1 \frac{dx}{x}.$$

Question 7 – Solution

Step 1: Split at the singularity. Ordinary convergence requires both one-sided integrals to converge.

Step 2: Evaluate the left side.

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - \ln 1) = -\infty.\end{aligned}$$

Step 3: Evaluate the right side.

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln 1 - \ln \varepsilon) = +\infty.\end{aligned}$$

Since the one-sided integrals diverge, the ordinary integral diverges.

diverges

The symmetric principal value is 0, but it is not an ordinary integral value.

Question 8

Evaluate the improper integral:

$$\int_0^1 \ln x \, dx.$$

Question 8 – Solution

Step 1: Replace the improper endpoint with a limit.

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln x \, dx.$$

Step 2: Integrate by parts. Let $u = \ln x$ and $dv = dx$. Then $du = dx/x$ and $v = x$:

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - x. \end{aligned}$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [x \ln x - x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln \varepsilon + \varepsilon). \end{aligned}$$

Since $\varepsilon \ln \varepsilon \rightarrow 0$ and $\varepsilon \rightarrow 0$,

$$I = -1.$$

$$\boxed{-1}$$

Question 9

Find c so that

$$f(x) = \frac{c}{1+x^2}$$

is a probability density on $(-\infty, \infty)$.

Question 9 – Solution

Step 1: Use the normalization requirement. A probability density must satisfy

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Thus

$$1 = c \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Step 2: Evaluate the improper integral. The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{R \rightarrow \infty} [\arctan x]_{-R}^R \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi. \end{aligned}$$

Step 3: Solve for c .

$$1 = c\pi \quad \implies \quad c = \frac{1}{\pi}.$$

Because $c > 0$, $f(x) \geq 0$ for every real x .

$c = 1/\pi$

Question 10

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^{\infty} \frac{dx}{(1+x)^2}$$

Question 10 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b (1+x)^{-2} dx.$$

Step 2: Integrate.

$$\int (1+x)^{-2} dx = -(1+x)^{-1} = -\frac{1}{1+x}.$$

Step 3: Evaluate the bounds and limit.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{1+x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{1+b} + 1 \right) \\ &= 0 + 1 = 1. \end{aligned}$$

1 (convergent)

Question 1

Use direct comparison to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + \sin^2 x}.$$

Question 1 – Solution

Step 1: Find an upper comparison. Since $\sin^2 x \geq 0$,

$$x^2 + \sin^2 x \geq x^2.$$

Both sides are positive, so taking reciprocals reverses the inequality:

$$0 \leq \frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}.$$

Step 2: Test the comparison integral.

$$\begin{aligned} \int_1^\infty \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1. \end{aligned}$$

Step 3: Apply direct comparison. The larger integral converges, so the original integral also converges.

converges

Question 2

Use limit comparison to determine convergence:

$$\int_1^{\infty} \frac{3x^2 + 1}{x^4 - 2x + 5} dx.$$

Question 2 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{3x^2}{x^4} = \frac{3}{x^2},$$

so use $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(3x^2 + 1)/(x^4 - 2x + 5)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{3x^4 + x^2}{x^4 - 2x + 5} \\ &= \lim_{x \rightarrow \infty} \frac{3 + 1/x^2}{1 - 2/x^3 + 5/x^4} = 3. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and $\int_1^\infty x^{-2} dx$ converges, the original integral converges.

converges

Question 3

Use a lower comparison to determine convergence:

$$\int_1^{\infty} \frac{2 + \sin x}{x} dx.$$

Question 3 – Solution

Step 1: Bound the numerator. Since $\sin x \geq -1$,

$$2 + \sin x \geq 1.$$

For $x \geq 1$, divide by the positive number x :

$$\frac{2 + \sin x}{x} \geq \frac{1}{x} \geq 0.$$

Step 2: Test the lower comparison.

$$\int_1^\infty \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

Step 3: Apply direct comparison. The smaller integral diverges, so the original integral diverges.

diverges

Question 4

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sqrt{x} + x}.$$

Question 4 – Solution

Step 1: Choose a comparison. Near $x = 0$, $\sqrt{x}$ is the dominant denominator term, so use

$$g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}.$$

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/(\sqrt{x} + x)}{1/\sqrt{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x} + x} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{1 + \sqrt{x}} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 x^{-1/2} dx = 2 < \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral converges.

converges

Question 5

Use comparison, not integration by parts, to determine convergence:

$$\int_2^{\infty} \frac{\ln x}{x^2} dx.$$

Question 5 – Solution

Step 1: Bound the logarithm. The logarithm grows more slowly than every positive power of x . In particular,

$$\ln x \leq \sqrt{x}$$

for all sufficiently large x .

Step 2: Divide by $x^2 > 0$.

$$0 \leq \frac{\ln x}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}.$$

Step 3: Test the upper comparison.

$$\int_2^\infty x^{-3/2} dx$$

converges because it is a p -integral with $p = 3/2 > 1$.

Step 4: Apply direct comparison. Therefore, the original integral converges.

converges

Question 6

A student uses

$$\frac{1}{\sqrt{x^2 + 1}} < \frac{1}{x}$$

to claim convergence on $[1, \infty)$. Explain the error and classify the integral.

Question 6 – Solution

Step 1: Identify the error. The comparison integral

$$\int_1^{\infty} \frac{dx}{x}$$

diverges. Being smaller than a divergent function does not prove convergence; a smaller positive function may converge or diverge.

Step 2: Use limit comparison with $g(x) = 1/x$.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/\sqrt{x^2 + 1}}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and the harmonic integral diverges, the original integral also diverges.

diverges by limit comparison

Question 7

Use dominant powers to determine convergence:

$$\int_1^{\infty} \frac{x^3 + 4}{x^5 + x} dx.$$

Question 7 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{x^3}{x^5} = \frac{1}{x^2},$$

so let $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(x^3 + 4)/(x^5 + x)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{x^5 + 4x^2}{x^5 + x} \\ &= \lim_{x \rightarrow \infty} \frac{1 + 4/x^3}{1 + 1/x^4} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $\int_1^\infty x^{-2} dx$ converges and $0 < L < \infty$, the original integral converges.

converges

Question 8

Find all real p for which the integral converges:

$$\int_1^{\infty} \frac{dx}{x^p + 1}.$$

Question 8 – Solution

Step 1: Consider $p > 0$. Compare with $g(x) = 1/x^p$:

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/(x^p + 1)}{1/x^p} \\ &= \lim_{x \rightarrow \infty} \frac{x^p}{x^p + 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + 1/x^p} = 1. \end{aligned}$$

Thus the original integral has the same behavior as the p -integral $\int_1^\infty x^{-p} dx$, which converges exactly when $p > 1$.

Step 2: Consider $p = 0$. The integrand is $1/2$, so the integral diverges.

Step 3: Consider $p < 0$. Then $x^p \rightarrow 0$, so

$$\frac{1}{x^p + 1} \rightarrow 1 \neq 0.$$

For $x \geq 1$ and $p < 0$, $x^p \leq 1$, so the integrand is at least $1/2$. Comparison with $\int_1^\infty (1/2) dx$ proves divergence.

$$\boxed{p > 1}$$

Question 9

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sin x}.$$

Question 9 – Solution

Step 1: Choose a comparison. Since $\sin x \sim x$ near zero, compare with $g(x) = 1/x$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/\sin x}{1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \rightarrow 0^+} (-\ln \varepsilon) = \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral diverges.

diverges

Question 10

Use a two-sided estimate to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + 3x}.$$

Question 10 – Solution

Step 1: Bound the denominator. For $x \geq 1$, we have $x \leq x^2$, so

$$3x \leq 3x^2.$$

Adding x^2 gives

$$x^2 \leq x^2 + 3x \leq 4x^2.$$

Step 2: Take reciprocals. All terms are positive, so

$$\frac{1}{4x^2} \leq \frac{1}{x^2 + 3x} \leq \frac{1}{x^2}.$$

Step 3: Test the upper comparison.

$$\int_1^{\infty} \frac{dx}{x^2} = 1 < \infty.$$

Step 4: Apply direct comparison. Since the original integrand is nonnegative and bounded above by an integrable function, the integral converges.

converges

Question 1

Given

$$f(0) = 1, \ f(1) = 2, \ f(2) = 5, \ f(3) = 10, \ f(4) = 17,$$

use T_4 to approximate $\int_0^4 f(x) \, dx$.

Question 1 – Solution

Step 1: Find the subinterval width.

$$h = \frac{b - a}{n} = \frac{4 - 0}{4} = 1.$$

Step 2: Write the trapezoidal formula.

$$T_4 = \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)].$$

Step 3: Substitute the table values.

$$\begin{aligned} T_4 &= \frac{1}{2} [1 + 2(2) + 2(5) + 2(10) + 17] \\ &= \frac{1}{2} [1 + 4 + 10 + 20 + 17] \\ &= \frac{1}{2} (52) = 26. \end{aligned}$$

$$\boxed{T_4 = 26}$$

Question 2

Use M_4 to approximate

$$\int_0^2 e^{-x^2} dx.$$

Give the exponential form and a decimal.

Question 2 – Solution

Step 1: Find the width.

$$h = \frac{2 - 0}{4} = 0.5.$$

Step 2: Find the four midpoints.

$$0.25, \quad 0.75, \quad 1.25, \quad 1.75.$$

Step 3: Apply the midpoint formula.

$$\begin{aligned} M_4 &= h \sum_{i=1}^4 f(m_i) \\ &= 0.5 \left(e^{-(0.25)^2} + e^{-(0.75)^2} + e^{-(1.25)^2} + e^{-(1.75)^2} \right). \end{aligned}$$

Step 4: Evaluate.

$$M_4 \approx 0.88278895.$$

$M_4 \approx 0.8828$

Question 3

Use Simpson's Rule with $n = 4$ to approximate

$$\int_0^2 \sqrt{1+x^3} \, dx.$$

Question 3 – Solution

Step 1: Find the width and nodes.

$$h = \frac{2-0}{4} = 0.5, \quad x_i = 0, 0.5, 1, 1.5, 2.$$

Step 2: Evaluate $f(x) = \sqrt{1+x^3}$.

$$\begin{array}{lll} f(0) = 1, & f(0.5) = \sqrt{1.125}, & f(1) = \sqrt{2}, \\ f(1.5) = \sqrt{4.375}, & f(2) = 3. & \end{array}$$

Step 3: Apply Simpson's Rule.

$$\begin{aligned} S_4 &= \frac{h}{3} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \\ &= \frac{0.5}{3} [1 + 4\sqrt{1.125} + 2\sqrt{2} + 4\sqrt{4.375} + 3] \\ &\approx 3.23961135. \end{aligned}$$

$$\boxed{S_4 \approx 3.2396}$$

Question 4

For increasing f , explain why

$$L_n \leq \int_a^b f(x) \, dx \leq R_n.$$

Then bound $\int_0^1 e^x \, dx$ using $n = 2$.

Question 4 – Solution

Step 1: Justify the inequality. On each subinterval, an increasing function satisfies

$$f(x_{i-1}) \leq f(x) \leq f(x_i).$$

Multiplying by $\Delta x > 0$ and adding over all subintervals gives

$$L_n \leq \int_a^b f(x) dx \leq R_n.$$

Step 2: Find the width and nodes.

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}, \quad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1.$$

Step 3: Compute both sums.

$$\begin{aligned} L_2 &= \frac{1}{2}[f(0) + f(1/2)] = \frac{1}{2}(1 + e^{1/2}) \approx 1.3244, \\ R_2 &= \frac{1}{2}[f(1/2) + f(1)] = \frac{1}{2}(e^{1/2} + e) \approx 2.1835. \end{aligned}$$

Step 4: State the bound.

$$1.3244 \leq \int_0^1 e^x dx \leq 2.1835$$

Question 5

For $\int_0^2 x^4 dx$, determine whether T_n and M_n are overestimates or underestimates. Use concavity.

Question 5 – Solution

Step 1: Determine concavity. For $f(x) = x^4$,

$$f'(x) = 4x^3, \quad f''(x) = 12x^2.$$

On $[0, 2]$, $f''(x) \geq 0$, so the graph is concave up.

Step 2: Interpret the trapezoidal approximation. For a concave-up function, each secant line lies above the graph. Therefore,

$$T_n > \int_0^2 x^4 dx.$$

Step 3: Interpret the midpoint approximation. For a concave-up function, midpoint rectangles lie below the graph on average. Therefore,

$$M_n < \int_0^2 x^4 dx.$$

$T_n > I, \quad M_n < I$

Question 6

How large must n be for T_n to approximate

$$\int_0^1 e^x dx$$

within 10^{-4} ? Use $|f''(x)| \leq e$.

Question 6 – Solution

Step 1: Write the trapezoidal error bound.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Here $K = e$ and $b - a = 1$, so

$$|E_T| \leq \frac{e}{12n^2}.$$

Step 2: Require the bound to be at most 10^{-4} .

$$\frac{e}{12n^2} \leq 10^{-4},$$

$$e \leq 0.0012n^2,$$

$$n^2 \geq \frac{e}{0.0012} \approx 2265.23,$$

$$n \geq \sqrt{2265.23} \approx 47.594.$$

Step 3: Choose the smallest integer.

$$\boxed{n = 48}$$

Question 7

Show that Simpson's Rule with $n = 2$ is exact for

$$\int_0^2 x^3 dx.$$

Question 7 – Solution

Step 1: Find the width and nodes.

$$h = \frac{2-0}{2} = 1, \quad x_0 = 0, \ x_1 = 1, \ x_2 = 2.$$

Step 2: Apply Simpson's Rule.

$$\begin{aligned} S_2 &= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \\ &= \frac{1}{3}[0^3 + 4(1^3) + 2^3] \\ &= \frac{1}{3}(0 + 4 + 8) = 4. \end{aligned}$$

Step 3: Evaluate the exact integral.

$$\begin{aligned} I &= \int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 \\ &= \frac{16}{4} - 0 = 4. \end{aligned}$$

Since $S_2 = I$, Simpson's Rule is exact here.

$$\boxed{S_2 = I = 4}$$

Question 8

For an increasing function on $[0, 3]$, suppose

$$L_6 = 7.2, \quad R_6 = 8.7.$$

Find T_6 and bound the integral.

Question 8 – Solution

Step 1: Use the trapezoidal-average identity. With equal-width subintervals,

$$T_n = \frac{L_n + R_n}{2}.$$

Thus

$$\begin{aligned} T_6 &= \frac{7.2 + 8.7}{2} \\ &= \frac{15.9}{2} = 7.95. \end{aligned}$$

Step 2: Use monotonicity to bound the integral. Because f is increasing, left endpoints underestimate and right endpoints overestimate:

$$L_6 \leq I \leq R_6.$$

Therefore,

$$7.2 \leq I \leq 8.7.$$

$7.2 \leq I \leq 8.7, \quad T_6 = 7.95$

Question 9

A velocity is measured every second:

t (s)	0	1	2	3	4
v (m/s)	0	3	5	6	4

Use T_4 to estimate displacement.

Question 9 – Solution

Step 1: Find the time width.

$$h = \frac{4 - 0}{4} = 1 \text{ second.}$$

Step 2: Apply the trapezoidal formula.

$$\begin{aligned} T_4 &= \frac{h}{2}[v_0 + 2v_1 + 2v_2 + 2v_3 + v_4] \\ &= \frac{1}{2}[0 + 2(3) + 2(5) + 2(6) + 4] \\ &= \frac{1}{2}[0 + 6 + 10 + 12 + 4] \\ &= \frac{1}{2}(32) = 16. \end{aligned}$$

Step 3: Include units. Velocity times time gives displacement in meters.

$$\boxed{16 \text{ m}}$$

Question 10

For Simpson's Rule on $[0, \pi]$, suppose

$$|f^{(4)}(x)| \leq 1.$$

Find the smallest even n that guarantees error at most 10^{-6} .

Question 10 – Solution

Step 1: Write Simpson's error bound.

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Here $K = 1$ and $b - a = \pi$, so

$$|E_S| \leq \frac{\pi^5}{180n^4}.$$

Step 2: Require the bound to be at most 10^{-6} .

$$\begin{aligned}\frac{\pi^5}{180n^4} &\leq 10^{-6}, \\ n^4 &\geq \frac{\pi^5}{180 \cdot 10^{-6}}, \\ n &\geq \left(\frac{\pi^5}{180 \cdot 10^{-6}} \right)^{1/4} \\ &\approx 36.109.\end{aligned}$$

Step 3: Enforce Simpson's even- n requirement. The first even integer at least 36.109 is 38.

$$\boxed{n = 38}$$