

Calculus II

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Question 1

Evaluate by tabular integration:

$$\int x^3 e^{2x} dx.$$

Question 1 – Solution

We use tabular integration because repeatedly differentiating the polynomial x^3 eventually gives zero. Differentiate in the D column and integrate in the I column. The signs alternate, beginning with a plus sign.

Sign	D	I
+	x^3	$\frac{e^{2x}}{2}$
–	$3x^2$	$\frac{e^{2x}}{4}$
+	$6x$	$\frac{e^{2x}}{8}$
–	6	$\frac{e^{2x}}{16}$
	0	

For example, each entry in the integration column is obtained using

$$\int e^{2x} dx = \frac{1}{2}e^{2x},$$

so every step down that column introduces another factor of $\frac{1}{2}$.

Now multiply across each row, pairing each entry in the derivative column with the entry in the integration column on the same row, and use the indicated signs:

$$\begin{aligned} \int x^3 e^{2x} dx &= x^3 \left(\frac{e^{2x}}{2} \right) - 3x^2 \left(\frac{e^{2x}}{4} \right) + 6x \left(\frac{e^{2x}}{8} \right) - 6 \left(\frac{e^{2x}}{16} \right) + C \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{6x e^{2x}}{8} - \frac{6e^{2x}}{16} + C \\ &= e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C. \end{aligned}$$

Therefore,

$$\boxed{\int x^3 e^{2x} dx = e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C}$$

As a check, differentiating the answer gives

$$\frac{d}{dx} \left[e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) \right] = x^3 e^{2x},$$

which is the original integrand.

Question 2

Find the unique constant a such that

$$\int_0^1 x e^{ax} dx = 1.$$

Give the defining equation and a numerical value.

Question 2 – Solution

First consider the special case $a = 0$. In that case,

$$\int_0^1 xe^{0x} dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \neq 1.$$

Therefore, $a = 0$ is not a solution, so we may assume that $a \neq 0$.

To evaluate the integral, use integration by parts with

$$u = x, \quad dv = e^{ax} dx.$$

Then

$$du = dx, \quad v = \frac{e^{ax}}{a}.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_0^1 xe^{ax} dx &= \frac{xe^{ax}}{a} \Big|_0^1 - \frac{1}{a} \int_0^1 e^{ax} dx \\ &= \frac{xe^{ax}}{a} \Big|_0^1 - \frac{e^{ax}}{a^2} \Big|_0^1 \\ &= \frac{e^a}{a} - \frac{e^a - 1}{a^2} \\ &= \frac{ae^a - e^a + 1}{a^2} \\ &= \frac{e^a(a - 1) + 1}{a^2}. \end{aligned}$$

We want this integral to equal 1, so

$$\begin{aligned} \frac{e^a(a - 1) + 1}{a^2} &= 1, \\ e^a(a - 1) + 1 &= a^2. \end{aligned}$$

Thus, the defining equation is

$$\boxed{e^a(a - 1) + 1 = a^2}.$$

Substituting $a = 1$ verifies that it satisfies the equation:

$$e^1(1 - 1) + 1 = 1 = 1^2.$$

Hence, $a = 1$ is a solution.

To see that it is the only solution, define

$$F(a) = \int_0^1 xe^{ax} dx.$$

Differentiating with respect to a gives

$$F'(a) = \int_0^1 x^2 e^{ax} dx > 0$$

for every real a . Therefore, F is strictly increasing and can equal 1 for at most one value of a . Since $F(1) = 1$, the unique constant is

$$\boxed{a = 1.0000}.$$

Question 3

Consider the integral

$$\int x^2 e^x dx.$$

A student makes the integration-by-parts choice

$$u = e^x, \quad dv = x^2 dx.$$

Explain why this choice does not simplify the integral. Then evaluate the integral efficiently using a better choice of u and dv .

Question 3 – Solution

Recall the integration-by-parts formula

$$\int u \, dv = uv - \int v \, du.$$

With the student's choice $u = e^x$ and $dv = x^2 \, dx$, we have

$$du = e^x \, dx, \quad v = \frac{x^3}{3}.$$

Therefore,

$$\begin{aligned} \int x^2 e^x \, dx &= e^x \left(\frac{x^3}{3} \right) - \int \frac{x^3}{3} e^x \, dx \\ &= \frac{x^3 e^x}{3} - \frac{1}{3} \int x^3 e^x \, dx. \end{aligned}$$

The polynomial degree has increased from 2 to 3, so the remaining integral is more complicated. Repeating this choice would keep increasing the degree.

Instead, choose

$$u = x^2, \quad dv = e^x \, dx, \quad du = 2x \, dx, \quad v = e^x.$$

Applying integration by parts gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

For the remaining integral, use integration by parts again:

$$u = x, \quad dv = e^x \, dx, \quad du = dx, \quad v = e^x.$$

Thus,

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x. \end{aligned}$$

Substituting this result into the original calculation gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C. \end{aligned}$$

$$\boxed{\int x^2 e^x \, dx = e^x (x^2 - 2x + 2) + C}$$

Check by differentiating:

$$\begin{aligned} \frac{d}{dx} [e^x (x^2 - 2x + 2)] &= e^x (x^2 - 2x + 2) + e^x (2x - 2) \\ &= x^2 e^x, \end{aligned}$$

which is the original integrand.

Question 4

Define

$$I_n = \int x^n e^x dx, \quad n \geq 1.$$

Use integration by parts to derive a recurrence relation connecting I_n and I_{n-1} . Then use the recurrence to evaluate I_4 .

Question 4 – Solution

Start with

$$I_n = \int x^n e^x dx.$$

To reduce the power of x , choose

$$u = x^n, \quad dv = e^x dx, \quad du = nx^{n-1} dx, \quad v = e^x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} I_n &= x^n e^x - \int e^x (nx^{n-1}) dx \\ &= x^n e^x - n \int x^{n-1} e^x dx. \end{aligned}$$

Since $I_{n-1} = \int x^{n-1} e^x dx$, the recurrence relation is

$$\boxed{I_n = x^n e^x - n I_{n-1}}.$$

Here, as usual for indefinite integrals, the constant of integration is added after the recurrence has been applied. The base case is

$$I_0 = \int e^x dx = e^x.$$

Now apply the recurrence repeatedly:

$$\begin{aligned} I_1 &= x e^x - I_0 = x e^x - e^x = e^x (x - 1), \\ I_2 &= x^2 e^x - 2 I_1 \\ &= x^2 e^x - 2 e^x (x - 1) = e^x (x^2 - 2x + 2), \\ I_3 &= x^3 e^x - 3 I_2 \\ &= x^3 e^x - 3 e^x (x^2 - 2x + 2) \\ &= e^x (x^3 - 3x^2 + 6x - 6), \\ I_4 &= x^4 e^x - 4 I_3 \\ &= x^4 e^x - 4 e^x (x^3 - 3x^2 + 6x - 6) \\ &= e^x (x^4 - 4x^3 + 12x^2 - 24x + 24). \end{aligned}$$

Therefore,

$$\boxed{I_4 = \int x^4 e^x dx = e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) + C}$$

Indeed, differentiating the result gives

$$\begin{aligned} \frac{d}{dx} [e^x (x^4 - 4x^3 + 12x^2 - 24x + 24)] &= e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) \\ &\quad + e^x (4x^3 - 12x^2 + 24x - 24) \\ &= x^4 e^x, \end{aligned}$$

which confirms the answer.

Question 5

Evaluate the cyclic integral

$$\int e^{2x} \cos(3x) dx$$

using integration by parts. Do not guess the form of the antiderivative.

Question 5 – Solution

Let

$$I = \int e^{2x} \cos(3x) dx.$$

For the first integration by parts, choose

$$u = \cos(3x), \quad dv = e^{2x} dx, \quad du = -3 \sin(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Using $\int u dv = uv - \int v du$, we get

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) - \int \frac{1}{2} e^{2x} (-3 \sin(3x)) dx \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) dx. \end{aligned} \tag{1}$$

Now evaluate the remaining integral. Let

$$J = \int e^{2x} \sin(3x) dx.$$

Use integration by parts again with

$$u = \sin(3x), \quad dv = e^{2x} dx, \quad du = 3 \cos(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Then

$$\begin{aligned} J &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I. \end{aligned} \tag{2}$$

Substitute equation (2) into equation (1):

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I \right) \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) - \frac{9}{4} I. \end{aligned}$$

Move the I -term to the left and solve:

$$\begin{aligned} I + \frac{9}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ \frac{13}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ I &= \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)). \end{aligned}$$

Therefore,

$$\boxed{\int e^{2x} \cos(3x) dx = \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) + C}$$

To check, differentiate the antiderivative:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) \right] &= \frac{e^{2x}}{13} [2(2 \cos(3x) + 3 \sin(3x)) \\ &\quad - 6 \sin(3x) + 9 \cos(3x)] \\ &= e^{2x} \cos(3x), \end{aligned}$$

which is the original integrand.

Question 6

Use integration by parts to prove that

$$\int_0^1 \ln(x) \, dx = -1.$$

Because $\ln(x)$ is unbounded as $x \rightarrow 0^+$, be sure to treat the lower endpoint correctly as an improper integral.

Question 6 – Solution

The function $\ln(x)$ is not defined at $x = 0$ and satisfies

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty.$$

Therefore, we must interpret the integral as

$$\int_0^1 \ln(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln(x) dx.$$

For $\varepsilon > 0$, evaluate the proper integral by integration by parts. Choose

$$u = \ln(x), \quad dv = dx, \quad du = \frac{1}{x} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 x \left(\frac{1}{x} \right) dx \\ &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 1 dx \\ &= [x \ln(x) - x]_{\varepsilon}^1. \end{aligned}$$

Now substitute the bounds:

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= (1 \ln(1) - 1) - (\varepsilon \ln(\varepsilon) - \varepsilon) \\ &= -1 - \varepsilon \ln(\varepsilon) + \varepsilon. \end{aligned}$$

It remains to evaluate the indeterminate endpoint term. Rewrite it as

$$\varepsilon \ln(\varepsilon) = \frac{\ln(\varepsilon)}{1/\varepsilon}.$$

As $\varepsilon \rightarrow 0^+$, this has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon \ln(\varepsilon) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} \\ &= \lim_{\varepsilon \rightarrow 0^+} (-\varepsilon) = 0. \end{aligned}$$

Also, $\lim_{\varepsilon \rightarrow 0^+} \varepsilon = 0$. Hence,

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln(\varepsilon) + \varepsilon) \\ &= -1. \end{aligned}$$

Therefore, the improper integral converges and

$$\boxed{\int_0^1 \ln(x) dx = -1}.$$

This sign is reasonable because $\ln(x) < 0$ for $0 < x < 1$.

Question 7

Find an antiderivative F of

$$f(x) = \arcsin(x), \quad -1 < x < 1,$$

that satisfies the initial condition

$$F(0) = 2.$$

Question 7 – Solution

We begin by evaluating

$$\int \arcsin(x) dx.$$

Since the integrand can be viewed as $\arcsin(x) \cdot 1$, use integration by parts with

$$u = \arcsin(x), \quad dv = dx, \quad du = \frac{1}{\sqrt{1-x^2}} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\int \arcsin(x) dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx. \quad (1)$$

To evaluate the remaining integral, let

$$w = 1 - x^2.$$

Then

$$dw = -2x dx, \quad x dx = -\frac{1}{2} dw.$$

Therefore,

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int w^{-1/2} dw \\ &= -\frac{1}{2} (2w^{1/2}) \\ &= -\sqrt{w} \\ &= -\sqrt{1-x^2}. \end{aligned}$$

Substitute this result into equation (1):

$$\begin{aligned} \int \arcsin(x) dx &= x \arcsin(x) - (-\sqrt{1-x^2}) + C \\ &= x \arcsin(x) + \sqrt{1-x^2} + C. \end{aligned}$$

Thus, the general antiderivative is

$$F(x) = x \arcsin(x) + \sqrt{1-x^2} + C.$$

Now apply the condition $F(0) = 2$:

$$\begin{aligned} F(0) &= 0 \cdot \arcsin(0) + \sqrt{1-0^2} + C \\ &= 1 + C. \end{aligned}$$

Hence,

$$1 + C = 2 \quad \implies \quad C = 1.$$

Therefore, the required antiderivative is

$$\boxed{F(x) = x \arcsin(x) + \sqrt{1-x^2} + 1}.$$

To verify, differentiate:

$$\begin{aligned} F'(x) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x). \end{aligned}$$

Also, $F(0) = 0 + 1 + 1 = 2$, so both requirements are satisfied.

Question 8

Reverse engineer an antiderivative by finding a quadratic polynomial $P(x)$ such that

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1).$$

Determine $P(x)$ and verify that it satisfies the equation.

Question 8 – Solution

We want to choose $P(x)$ so that differentiating $e^x P(x)$ produces the given expression. By the product rule,

$$\begin{aligned}\frac{d}{dx} [e^x P(x)] &= e^x P(x) + e^x P'(x) \\ &= e^x (P(x) + P'(x)).\end{aligned}$$

Thus, the required identity

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1)$$

will hold if

$$P(x) + P'(x) = x^2 + 1. \tag{1}$$

Because P is quadratic, write

$$P(x) = ax^2 + bx + c,$$

where a , b , and c are constants. Then

$$P'(x) = 2ax + b.$$

Adding $P(x)$ and $P'(x)$ gives

$$\begin{aligned}P(x) + P'(x) &= (ax^2 + bx + c) + (2ax + b) \\ &= ax^2 + (b + 2a)x + (c + b).\end{aligned}$$

Equation (1) requires this polynomial to equal

$$x^2 + 1 = 1x^2 + 0x + 1.$$

Matching coefficients of like powers of x yields

$$\begin{array}{rcl}a & = & 1, \quad \text{coefficient of } x^2, \\ b + 2a & = & 0, \quad \text{coefficient of } x, \\ c + b & = & 1, \quad \text{constant term.}\end{array}$$

Solve the equations in order. Since $a = 1$,

$$b + 2(1) = 0 \quad \implies \quad b = -2.$$

Then

$$c + (-2) = 1 \quad \implies \quad c = 3.$$

Therefore,

$$\boxed{P(x) = x^2 - 2x + 3}.$$

Finally, verify the result. Since $P'(x) = 2x - 2$,

$$\begin{aligned}\frac{d}{dx} [e^x (x^2 - 2x + 3)] &= e^x (x^2 - 2x + 3) + e^x (2x - 2) \\ &= e^x (x^2 - 2x + 3 + 2x - 2) \\ &= e^x (x^2 + 1),\end{aligned}$$

which is the required expression.

Question 9

Evaluate

$$\int_{-1}^1 x \arctan(x) dx.$$

Before using integration by parts, exploit the symmetry of the integrand to simplify the calculation.

Question 9 – Solution

Let

$$f(x) = x \arctan(x).$$

Both x and $\arctan(x)$ are odd functions. Therefore, their product is even:

$$\begin{aligned} f(-x) &= (-x) \arctan(-x) \\ &= (-x)(-\arctan(x)) \\ &= x \arctan(x) = f(x). \end{aligned}$$

Consequently,

$$\int_{-1}^1 x \arctan(x) dx = 2 \int_0^1 x \arctan(x) dx. \quad (1)$$

Now use integration by parts on the integral over $[0, 1]$. Choose

$$u = \arctan(x), \quad dv = x dx, \quad du = \frac{1}{1+x^2} dx, \quad v = \frac{x^2}{2}.$$

Then

$$\int_0^1 x \arctan(x) dx = \frac{x^2}{2} \arctan(x) \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx.$$

Multiplying by 2 as in equation (1) gives

$$\int_{-1}^1 x \arctan(x) dx = x^2 \arctan(x) \Big|_0^1 - \int_0^1 \frac{x^2}{1+x^2} dx. \quad (2)$$

For the remaining integral, rewrite the integrand:

$$\frac{x^2}{1+x^2} = \frac{(1+x^2) - 1}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^2} dx &= \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \\ &= [x - \arctan(x)]_0^1 \\ &= 1 - \frac{\pi}{4}. \end{aligned}$$

The boundary term in equation (2) is

$$x^2 \arctan(x) \Big|_0^1 = 1^2 \arctan(1) - 0^2 \arctan(0) = \frac{\pi}{4}.$$

Substituting both results into equation (2),

$$\begin{aligned} \int_{-1}^1 x \arctan(x) dx &= \frac{\pi}{4} - \left(1 - \frac{\pi}{4} \right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Thus,

$$\boxed{\int_{-1}^1 x \arctan(x) dx = \frac{\pi}{2} - 1}.$$

The answer is positive, as expected, because the even integrand is nonnegative on $[-1, 1]$.

Question 10

For $n > 0$, evaluate the improper integral

$$J_n = \int_0^1 x^n \ln(x) \, dx.$$

Then describe the sign and limiting behavior of J_n as $n \rightarrow \infty$.

Question 10 – Solution

Because $\ln(x) \rightarrow -\infty$ as $x \rightarrow 0^+$, write the integral as

$$J_n = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^n \ln(x) dx.$$

For $\varepsilon > 0$, use integration by parts with

$$u = \ln(x), \quad dv = x^n dx, \quad du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}.$$

This choice is valid because $n > 0$, so in particular $n+1 \neq 0$. Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 x^n \ln(x) dx &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{1}{n+1} \int_{\varepsilon}^1 x^n dx \\ &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{x^{n+1}}{(n+1)^2} \Big|_{\varepsilon}^1. \end{aligned} \tag{1}$$

We must evaluate the boundary term at $x = 0$. Since $n+1 > 0$,

$$\lim_{x \rightarrow 0^+} x^{n+1} \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}}.$$

This has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}} &= \lim_{x \rightarrow 0^+} \frac{1/x}{-(n+1)x^{-(n+2)}} \\ &= \lim_{x \rightarrow 0^+} -\frac{x^{n+1}}{n+1} = 0. \end{aligned}$$

Also, $\lim_{x \rightarrow 0^+} x^{n+1} = 0$. At the upper endpoint, $\ln(1) = 0$. Therefore, taking $\varepsilon \rightarrow 0^+$ in equation (1) gives

$$\begin{aligned} J_n &= \frac{1^{n+1} \ln(1)}{n+1} - 0 - \frac{1^{n+1} - 0}{(n+1)^2} \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\boxed{J_n = -\frac{1}{(n+1)^2}}.$$

For every $n > 0$, $J_n < 0$, which agrees with $\ln(x) < 0$ on $0 < x < 1$. Moreover, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} -\frac{1}{(n+1)^2} = 0.$$

The values increase toward zero through negative values. In short,

$$\boxed{J_n \rightarrow 0^- \quad \text{as} \quad n \rightarrow \infty}.$$

Question 1

Evaluate

$$\int \sin^5(x) \cos^4(x) dx.$$

Explain why separating one factor of $\sin(x)$ is an efficient strategy.

Question 1 – Solution

The power of sine is odd, so separate one sine factor:

$$\sin^5(x) = \sin^4(x) \sin(x).$$

The remaining even power can be converted using $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^4(x) = (1 - \cos^2(x))^2.$$

Thus,

$$\int \sin^5(x) \cos^4(x) dx = \int (1 - \cos^2(x))^2 \cos^4(x) \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

Then the integral becomes

$$\begin{aligned} - \int (1 - u^2)^2 u^4 du &= - \int (1 - 2u^2 + u^4) u^4 du \\ &= - \int (u^4 - 2u^6 + u^8) du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \sin^5(x) \cos^4(x) dx = -\frac{\cos^5(x)}{5} + \frac{2 \cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C.}$$

Separating one sine factor is efficient because it supplies the differential $-\sin(x) dx$ while every remaining factor becomes a polynomial in u .

Question 2

Use power-reduction identities to evaluate

$$\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx.$$

Question 2 – Solution

Begin with the double-angle identity

$$\sin(2x) = 2 \sin(x) \cos(x).$$

Squaring both sides gives

$$\sin^2(x) \cos^2(x) = \frac{1}{4} \sin^2(2x).$$

Now use $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ with $\theta = 2x$:

$$\sin^2(x) \cos^2(x) = \frac{1}{8} (1 - \cos(4x)).$$

Therefore,

$$\begin{aligned} \int_0^{\pi/2} \sin^2(x) \cos^2(x) dx &= \frac{1}{8} \int_0^{\pi/2} (1 - \cos(4x)) dx \\ &= \frac{1}{8} \left[x - \frac{1}{4} \sin(4x) \right]_0^{\pi/2}. \end{aligned}$$

Evaluate the endpoints:

$$\begin{aligned} \frac{1}{8} \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left(0 - \frac{1}{4} \sin(0) \right) \right] &= \frac{1}{8} \left(\frac{\pi}{2} \right) \\ &= \frac{\pi}{16}. \end{aligned}$$

Hence,

$$\boxed{\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx = \frac{\pi}{16}}.$$

Question 3

Find the average value of

$$f(x) = \sin^4(x)$$

over one full period.

Question 3 – Solution

We may use the interval $[0, 2\pi]$, whose length is 2π . The average-value formula is

$$f_{\text{avg}} = \frac{1}{2\pi} \int_0^{2\pi} \sin^4(x) dx.$$

Apply the power-reduction identity twice:

$$\begin{aligned} \sin^4(x) &= \left(\frac{1 - \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 - 2\cos(2x) + \cos^2(2x)) \\ &= \frac{1}{4} \left(1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3 - 4\cos(2x) + \cos(4x)}{8}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{3 - 4\cos(2x) + \cos(4x)}{8} dx \\ &= \frac{1}{16\pi} \left[3x - 2\sin(2x) + \frac{1}{4}\sin(4x) \right]_0^{2\pi} \\ &= \frac{1}{16\pi} (6\pi) = \frac{3}{8}. \end{aligned}$$

The sine terms vanish because their values are zero at both endpoints. Therefore,

$$\boxed{f_{\text{avg}} = \frac{3}{8}}.$$

Question 4

Evaluate

$$\int \tan^7(x) \sec^2(x) \, dx$$

without converting the integrand entirely to sines and cosines.

Question 4 – Solution

The factor $\sec^2(x) dx$ is exactly the differential of $\tan(x)$, so let

$$u = \tan(x), \quad du = \sec^2(x) dx.$$

Then

$$\begin{aligned} \int \tan^7(x) \sec^2(x) dx &= \int u^7 du \\ &= \frac{u^8}{8} + C. \end{aligned}$$

Substitute $u = \tan(x)$:

$$\boxed{\int \tan^7(x) \sec^2(x) dx = \frac{1}{8} \tan^8(x) + C}.$$

As a check,

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{8} \tan^8(x) \right] &= \frac{1}{8} \cdot 8 \tan^7(x) \sec^2(x) \\ &= \tan^7(x) \sec^2(x), \end{aligned}$$

which is the original integrand.

Question 5

Evaluate

$$\int \sec^5(x) \tan^3(x) \, dx$$

by saving a factor of $\sec(x) \tan(x)$ for substitution.

Question 5 – Solution

Separate one factor of $\sec(x) \tan(x)$:

$$\sec^5(x) \tan^3(x) = \sec^4(x) \tan^2(x) (\sec(x) \tan(x)).$$

Use $\tan^2(x) = \sec^2(x) - 1$ to rewrite the remaining tangent power:

$$\sec^4(x) \tan^2(x) = \sec^4(x) (\sec^2(x) - 1).$$

Now let

$$u = \sec(x), \quad du = \sec(x) \tan(x) dx.$$

The integral becomes

$$\begin{aligned} \int u^4(u^2 - 1) du &= \int (u^6 - u^4) du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C. \end{aligned}$$

Substituting $u = \sec(x)$ gives

$$\boxed{\int \sec^5(x) \tan^3(x) dx = \frac{1}{7} \sec^7(x) - \frac{1}{5} \sec^5(x) + C}.$$

Saving $\sec(x) \tan(x)$ is useful because it supplies du , while the identity converts all remaining factors into powers of u .

Question 6

Use a product-to-sum identity to evaluate

$$\int \sin(5x) \cos(2x) \, dx.$$

Question 6 – Solution

Use the product-to-sum identity

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

With $A = 5x$ and $B = 2x$,

$$\sin(5x) \cos(2x) = \frac{1}{2} [\sin(7x) + \sin(3x)].$$

Therefore,

$$\begin{aligned} \int \sin(5x) \cos(2x) dx &= \frac{1}{2} \int [\sin(7x) + \sin(3x)] dx \\ &= \frac{1}{2} \left(-\frac{1}{7} \cos(7x) - \frac{1}{3} \cos(3x) \right) + C \\ &= -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C. \end{aligned}$$

Hence,

$$\boxed{\int \sin(5x) \cos(2x) dx = -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C.}$$

Differentiating gives

$$\frac{1}{2} \sin(7x) + \frac{1}{2} \sin(3x) = \sin(5x) \cos(2x),$$

confirming the answer by the same product-to-sum identity.

Question 7

Determine all integers $m, n \geq 1$ for which

$$\int_0^{2\pi} \sin(mx) \cos(nx) \, dx \neq 0.$$

Justify your conclusion using a product-to-sum identity.

Question 7 – Solution

Use

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Setting $A = mx$ and $B = nx$ gives

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m + n)x) + \sin((m - n)x)]. \quad (1)$$

For every nonzero integer k ,

$$\begin{aligned} \int_0^{2\pi} \sin(kx) dx &= \left[-\frac{\cos(kx)}{k} \right]_0^{2\pi} \\ &= \frac{-\cos(2\pi k) + \cos(0)}{k} = 0, \end{aligned}$$

because $\cos(2\pi k) = 1$. Since $m + n \geq 2$, the first sine term in equation (1) always integrates to zero.

If $m \neq n$, then $m - n$ is a nonzero integer, so the second term also integrates to zero. If $m = n$, then

$$\sin((m - n)x) = \sin(0) = 0,$$

so it again contributes zero. Thus, in every case,

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0.$$

Therefore, there are no integers $m, n \geq 1$ for which the integral is nonzero:

There are no such integer pairs (m, n) .

Question 8

Evaluate

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx$$

using a substitution that produces reciprocal powers of the new variable.

Question 8 – Solution

Separate one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^3(x) = \sin(x)(1 - \cos^2(x)).$$

Then

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx = \int \frac{1 - \cos^2(x)}{\cos^4(x)} \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

The integral becomes

$$\begin{aligned} - \int \frac{1 - u^2}{u^4} du &= - \int (u^{-4} - u^{-2}) du \\ &= - \left(-\frac{1}{3}u^{-3} + u^{-1} \right) + C \\ &= \frac{1}{3}u^{-3} - u^{-1} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \frac{\sin^3(x)}{\cos^4(x)} dx = \frac{1}{3 \cos^3(x)} - \frac{1}{\cos(x)} + C}.$$

Equivalently, the answer is $\frac{1}{3} \sec^3(x) - \sec(x) + C$ on intervals where the original integrand is defined.

Question 9

Find constants A and B such that

$$\cos^4(x) = A + B \cos(2x) + \frac{1}{8} \cos(4x).$$

Then use the identity to evaluate $\int \cos^4(x) dx$.

Question 9 – Solution

Begin with the power-reduction identity

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

Squaring both sides,

$$\begin{aligned}\cos^4(x) &= \left(\frac{1 + \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 + 2 \cos(2x) + \cos^2(2x)).\end{aligned}$$

Apply power reduction again:

$$\cos^2(2x) = \frac{1 + \cos(4x)}{2}.$$

Therefore,

$$\begin{aligned}\cos^4(x) &= \frac{1}{4} \left(1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x).\end{aligned}$$

Matching this with the required form gives

$$\boxed{A = \frac{3}{8}, \quad B = \frac{1}{2}}.$$

Now integrate the identity term by term:

$$\begin{aligned}\int \cos^4(x) dx &= \int \left(\frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x) \right) dx \\ &= \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C.\end{aligned}$$

Hence,

$$\boxed{\int \cos^4(x) dx = \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C}.$$

Question 10

A signal is defined by

$$s(t) = \sin(3t) + \sin(5t).$$

Compute its energy over $0 \leq t \leq 2\pi$:

$$E = \int_0^{2\pi} s^2(t) dt.$$

Question 10 – Solution

First expand the square:

$$\begin{aligned} E &= \int_0^{2\pi} (\sin(3t) + \sin(5t))^2 dt \\ &= \int_0^{2\pi} [\sin^2(3t) + 2\sin(3t)\sin(5t) + \sin^2(5t)] dt. \end{aligned} \tag{1}$$

For the cross term, use

$$2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B).$$

Thus,

$$\begin{aligned} 2 \int_0^{2\pi} \sin(3t)\sin(5t) dt &= \int_0^{2\pi} (\cos(2t) - \cos(8t)) dt \\ &= \left[\frac{1}{2}\sin(2t) - \frac{1}{8}\sin(8t) \right]_0^{2\pi} = 0. \end{aligned}$$

For each squared term, use $\sin^2(kt) = \frac{1}{2}(1 - \cos(2kt))$:

$$\begin{aligned} \int_0^{2\pi} \sin^2(kt) dt &= \frac{1}{2} \int_0^{2\pi} (1 - \cos(2kt)) dt \\ &= \frac{1}{2} \left[t - \frac{\sin(2kt)}{2k} \right]_0^{2\pi} = \pi \end{aligned}$$

for any positive integer k . In particular, the $k = 3$ and $k = 5$ terms each contribute π . Substituting into equation (1),

$$E = \pi + 0 + \pi = 2\pi.$$

Therefore,

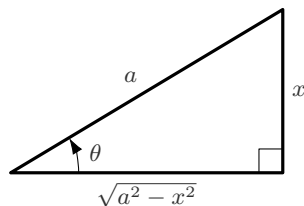
$$\boxed{E = 2\pi}.$$

Question 1

Evaluate the integral both geometrically and analytically:

$$\int_0^a \sqrt{a^2 - x^2} \, dx, \quad a > 0.$$

Question 1 – Solution



Geometrically, $y = \sqrt{a^2 - x^2}$ is the upper semicircle $x^2 + y^2 = a^2$. On $0 \leq x \leq a$, the region is one quarter of a disk of radius a . Hence its area is

$$I = \frac{1}{4}\pi a^2.$$

For an analytic verification, let

$$x = a \sin \theta, \quad dx = a \cos \theta \, d\theta.$$

Since $0 \leq x \leq a$, the new bounds are $0 \leq \theta \leq \pi/2$. Also,

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a \cos \theta, \end{aligned}$$

because $\cos \theta \geq 0$ on this interval. Therefore,

$$\begin{aligned} I &= a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta \\ &= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi a^2}{4}. \end{aligned}$$

Thus both methods give

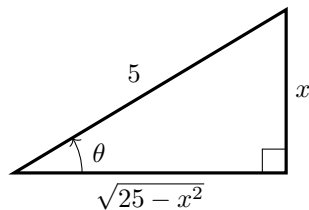
$$\boxed{\int_0^a \sqrt{a^2 - x^2} \, dx = \frac{\pi a^2}{4}}.$$

Question 2

Evaluate using a trigonometric substitution:

$$\int \frac{x^2}{\sqrt{25-x^2}} dx.$$

Question 2 – Solution



The radical has the form $\sqrt{a^2 - x^2}$, so let

$$x = 5 \sin \theta, \quad dx = 5 \cos \theta \, d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\sqrt{25 - x^2} = \sqrt{25 - 25 \sin^2 \theta} = 5 \cos \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{25 \sin^2 \theta}{5 \cos \theta} (5 \cos \theta \, d\theta) \\ &= 25 \int \sin^2 \theta \, d\theta \\ &= \frac{25}{2} \int (1 - \cos 2\theta) \, d\theta \\ &= \frac{25}{2} \theta - \frac{25}{4} \sin 2\theta + C. \end{aligned}$$

Since $\theta = \arcsin(x/5)$ and

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{x}{5} \right) \left(\frac{\sqrt{25 - x^2}}{5} \right),$$

we obtain

$$\boxed{I = \frac{25}{2} \arcsin \left(\frac{x}{5} \right) - \frac{x}{2} \sqrt{25 - x^2} + C}.$$

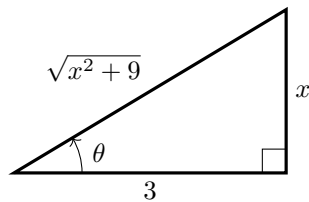
Question 3

Find the total area under

$$y = (x^2 + 9)^{-3/2}$$

on $0 \leq x < \infty$.

Question 3 – Solution



The required area is the improper integral

$$I = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x^2 + 9)^{3/2}}.$$

Let $x = 3 \tan \theta$. Then

$$dx = 3 \sec^2 \theta \, d\theta, \quad (x^2 + 9)^{3/2} = 27 \sec^3 \theta.$$

The bounds transform as $x = 0 \Rightarrow \theta = 0$ and $x \rightarrow \infty \Rightarrow \theta \rightarrow \pi/2$. Thus,

$$\begin{aligned} I &= \frac{1}{9} \int_0^{\pi/2} \cos \theta \, d\theta \\ &= \frac{1}{9} [\sin \theta]_0^{\pi/2} \\ &= \frac{1}{9}. \end{aligned}$$

The integrand is positive, so this convergent improper integral equals the geometric area. Therefore,

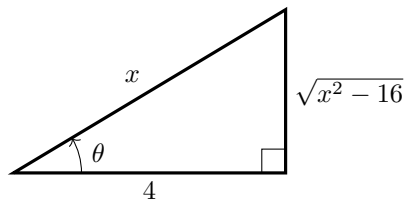
$$\boxed{I = \frac{1}{9}}.$$

Question 4

Evaluate using a trigonometric substitution:

$$\int \frac{dx}{x^2 \sqrt{x^2 - 16}}, \quad x > 4.$$

Question 4 – Solution



Because the radical has the form $\sqrt{x^2 - 16}$, let

$$x = 4 \sec \theta, \quad dx = 4 \sec \theta \tan \theta d\theta.$$

Since $x > 4$, take $0 < \theta < \pi/2$. Then

$$\sqrt{x^2 - 16} = \sqrt{16 \sec^2 \theta - 16} = 4 \tan \theta.$$

Therefore,

$$\begin{aligned} I &= \int \frac{4 \sec \theta \tan \theta d\theta}{(16 \sec^2 \theta)(4 \tan \theta)} \\ &= \frac{1}{16} \int \cos \theta d\theta \\ &= \frac{1}{16} \sin \theta + C. \end{aligned}$$

From the reference triangle,

$$\sec \theta = \frac{x}{4}, \quad \sin \theta = \frac{\sqrt{x^2 - 16}}{x}.$$

Hence,

$$\boxed{I = \frac{\sqrt{x^2 - 16}}{16x} + C}.$$

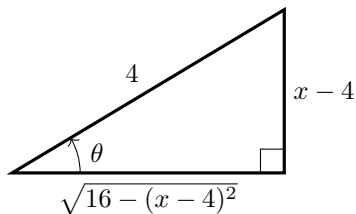
Differentiation recovers $1/(x^2\sqrt{x^2 - 16})$.

Question 5

Complete the square, choose an appropriate trigonometric substitution, and evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Question 5 – Solution



Complete the square inside the radical:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) \\ &= -((x - 4)^2 - 16) \\ &= 16 - (x - 4)^2. \end{aligned}$$

Let

$$x - 4 = 4 \sin \theta, \quad dx = 4 \cos \theta \, d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\begin{aligned} \sqrt{16 - (x - 4)^2} &= \sqrt{16 - 16 \sin^2 \theta} \\ &= 4 \cos \theta. \end{aligned}$$

Thus,

$$\begin{aligned} I &= \int \frac{4 \cos \theta \, d\theta}{4 \cos \theta} \\ &= \int d\theta = \theta + C. \end{aligned}$$

Since $\sin \theta = (x - 4)/4$,

$$I = \arcsin \left(\frac{x - 4}{4} \right) + C.$$

This formula applies on intervals inside $0 < x < 8$, where the radical is real and nonzero.

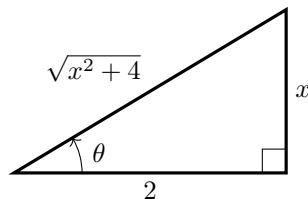
Question 6

Evaluate

$$\int \sqrt{x^2 + 4} \, dx$$

using trigonometric substitution, and verify the result by differentiation.

Question 6 – Solution



Let $x = 2 \tan \theta$, with $-\pi/2 < \theta < \pi/2$. Then

$$dx = 2 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 4} = 2 \sec \theta.$$

Therefore,

$$I = 4 \int \sec^3 \theta d\theta.$$

Using $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$,

$$I = 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C.$$

Now $\tan \theta = x/2$ and $\sec \theta = \sqrt{x^2 + 4}/2$. Hence,

$$\begin{aligned} I &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| + C \\ &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C, \end{aligned}$$

where $-2 \ln 2$ is absorbed into C . Thus,

$$\boxed{I = \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C}.$$

Differentiating and simplifying gives $I' = \sqrt{x^2 + 4}$, as required.

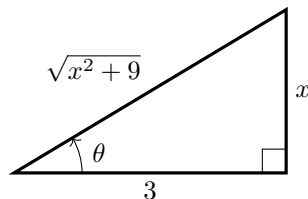
Question 7

For

$$\int \frac{dx}{\sqrt{x^2 + 9}},$$

two substitutions are proposed: $x = 3 \tan \theta$ and $x = 3 \sin \theta$. Select the efficient substitution, explain the identity it uses, and evaluate the integral.

Question 7 – Solution



Choose

$$x = 3 \tan \theta.$$

This substitution uses $1 + \tan^2 \theta = \sec^2 \theta$:

$$x^2 + 9 = 9 \tan^2 \theta + 9 = 9 \sec^2 \theta.$$

Taking $-\pi/2 < \theta < \pi/2$ gives $\sec \theta > 0$, so

$$dx = 3 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 9} = 3 \sec \theta.$$

Hence,

$$\begin{aligned} I &= \int \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Since $\tan \theta = x/3$ and $\sec \theta = \sqrt{x^2 + 9}/3$,

$$\begin{aligned} I &= \ln \left| \frac{x + \sqrt{x^2 + 9}}{3} \right| + C \\ &= \ln |x + \sqrt{x^2 + 9}| + C. \end{aligned}$$

The constant $-\ln 3$ is absorbed into C . Thus,

$$\int \frac{dx}{\sqrt{x^2 + 9}} = \ln |x + \sqrt{x^2 + 9}| + C.$$

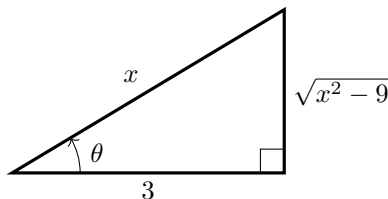
The sine substitution does not turn $x^2 + 9$ into a single squared trigonometric function, so it does not simplify the radical.

Question 8

Evaluate the definite integral exactly:

$$\int_3^5 \frac{\sqrt{x^2 - 9}}{x} dx.$$

Question 8 – Solution



Let

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta \, d\theta.$$

For $x \geq 3$, take $0 \leq \theta < \pi/2$. Then

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

Transform the bounds:

$$x = 3 \Rightarrow \sec \theta = 1 \Rightarrow \theta = 0, \quad x = 5 \Rightarrow \cos \theta = \frac{3}{5} \Rightarrow \theta = \arccos\left(\frac{3}{5}\right).$$

Thus,

$$\begin{aligned} I &= \int_0^{\arccos(3/5)} \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) \, d\theta \\ &= 3 \int_0^{\arccos(3/5)} \tan^2 \theta \, d\theta \\ &= 3 [\tan \theta - \theta]_0^{\arccos(3/5)}. \end{aligned}$$

At the upper bound, a 3-4-5 triangle gives $\tan \theta = 4/3$. Therefore,

$$\begin{aligned} I &= 3 \left(\frac{4}{3} - \arccos\left(\frac{3}{5}\right) \right) \\ &= 4 - 3 \arccos\left(\frac{3}{5}\right). \end{aligned}$$

Hence,

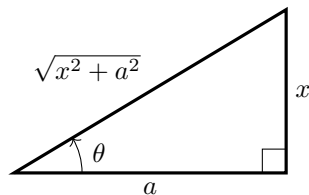
$$\boxed{I = 4 - 3 \arccos\left(\frac{3}{5}\right)}.$$

Question 9

For $a > 0$, derive the antiderivative

$$\int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

Question 9 – Solution



Let

$$x = a \tan \theta, \quad dx = a \sec^2 \theta d\theta.$$

Since $a > 0$, choose $-\pi/2 < \theta < \pi/2$. Then

$$\begin{aligned} (a^2 + x^2)^{3/2} &= (a^2 + a^2 \tan^2 \theta)^{3/2} \\ &= (a^2 \sec^2 \theta)^{3/2} = a^3 \sec^3 \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \int \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta \\ &= \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{1}{a^2} \sin \theta + C. \end{aligned}$$

From a reference triangle,

$$\tan \theta = \frac{x}{a}, \quad \sin \theta = \frac{x}{\sqrt{a^2 + x^2}}.$$

Thus,

$$I = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$$

Indeed,

$$\frac{d}{dx} \left(\frac{x}{a^2 \sqrt{a^2 + x^2}} \right) = \frac{1}{(a^2 + x^2)^{3/2}},$$

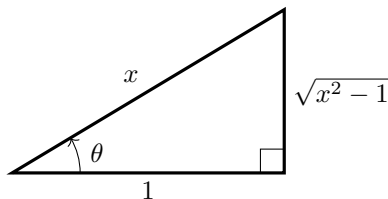
which verifies the result.

Question 10

Use a trigonometric or hyperbolic substitution to evaluate

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx, \quad x > 1.$$

Question 10 – Solution



Because $x > 1$, let

$$x = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Then

$$dx = \sec \theta \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \tan \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{\tan \theta}{\sec^2 \theta} (\sec \theta \tan \theta) \, d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} \, d\theta \\ &= \int \frac{\sec^2 \theta - 1}{\sec \theta} \, d\theta \\ &= \int (\sec \theta - \cos \theta) \, d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C. \end{aligned}$$

Back-substitute using

$$\sec \theta = x, \quad \tan \theta = \sqrt{x^2 - 1}, \quad \sin \theta = \frac{\sqrt{x^2 - 1}}{x}.$$

Since $x > 1$, the logarithm's argument is positive. Therefore,

$$I = \ln \left(x + \sqrt{x^2 - 1} \right) - \frac{\sqrt{x^2 - 1}}{x} + C.$$

Question 1

Find A and B , then evaluate the integral:

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

$$\int \frac{7x+1}{(x-1)(x+2)} dx.$$

Question 1 – Solution

Step 1: Set up the decomposition.

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Step 2: Clear the denominators. Multiply by $(x-1)(x+2)$:

$$7x+1 = A(x+2) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad 8 &= 3A \quad \implies \quad A = \frac{8}{3}, \\ x=-2: \quad -13 &= -3B \quad \implies \quad B = \frac{13}{3}. \end{aligned}$$

Step 4: Substitute and integrate each term.

$$\begin{aligned} \int \frac{7x+1}{(x-1)(x+2)} dx &= \int \left(\frac{8/3}{x-1} + \frac{13/3}{x+2} \right) dx \\ &= \frac{8}{3} \int \frac{dx}{x-1} + \frac{13}{3} \int \frac{dx}{x+2} \\ &= \boxed{\frac{8}{3} \ln|x-1| + \frac{13}{3} \ln|x+2| + C}. \end{aligned}$$

Question 2

Use polynomial division, then evaluate:

$$\int \frac{x^3 + 1}{x^2 - 1} dx.$$

State the excluded values of x .

Question 2 – Solution

Step 1: Divide the polynomials. Since $x^3/x^2 = x$,

$$\begin{aligned}x(x^2 - 1) &= x^3 - x, \\(x^3 + 1) - (x^3 - x) &= x + 1.\end{aligned}$$

Thus $x^3 + 1 = x(x^2 - 1) + (x + 1)$, so

$$\frac{x^3 + 1}{x^2 - 1} = x + \frac{x + 1}{x^2 - 1}.$$

Step 2: Factor and simplify.

$$\begin{aligned}\frac{x^3 + 1}{x^2 - 1} &= x + \frac{x + 1}{(x - 1)(x + 1)} \\&= x + \frac{1}{x - 1},\end{aligned}$$

because the original denominator is zero at both $x = -1$ and $x = 1$.

Step 3: Integrate term by term.

$$\begin{aligned}\int \frac{x^3 + 1}{x^2 - 1} dx &= \int x dx + \int \frac{dx}{x - 1} \\&= \boxed{\frac{x^2}{2} + \ln |x - 1| + C}, \quad x \neq \pm 1.\end{aligned}$$

Question 3

Decompose into partial fractions. Include every power of $(x - 1)$:

$$\frac{2x + 3}{x(x - 1)^2}.$$

Do not integrate.

Question 3 – Solution

Step 1: Include a term for every factor and repeated power.

$$\frac{2x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Step 2: Clear the denominators.

$$2x+3 = A(x-1)^2 + Bx(x-1) + Cx.$$

Step 3: Find A and C .

$$x = 0 : \quad 3 = A(-1)^2 \implies A = 3,$$

$$x = 1 : \quad 5 = C(1) \implies C = 5.$$

Step 4: Find B by using $x = 2$.

$$\begin{aligned} 7 &= 3(1)^2 + B(2)(1) + 5(2) \\ &= 3 + 2B + 10, \\ -6 &= 2B \implies B = -3. \end{aligned}$$

Step 5: State the result.

$$\boxed{\frac{2x+3}{x(x-1)^2} = \frac{3}{x} - \frac{3}{x-1} + \frac{5}{(x-1)^2}},$$

and the requested coefficients are

$$\boxed{A = 3, \quad B = -3, \quad C = 5}.$$

Question 4

Use partial fractions to evaluate:

$$\int \frac{x^2 + 1}{x(x^2 + 4)} dx.$$

Question 4 – Solution

Step 1: Set up the decomposition. Use a linear numerator over the irreducible quadratic:

$$\frac{x^2 + 1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Step 2: Clear denominators and collect like terms.

$$\begin{aligned}x^2 + 1 &= A(x^2 + 4) + x(Bx + C) \\&= (A + B)x^2 + Cx + 4A.\end{aligned}$$

Step 3: Match coefficients.

$$A + B = 1, \quad C = 0, \quad 4A = 1.$$

Therefore $A = 1/4$, $B = 1 - 1/4 = 3/4$, and $C = 0$.

Step 4: Integrate. For the second term, let $u = x^2 + 4$; then $du = 2x \, dx$.

$$\begin{aligned}I &= \frac{1}{4} \int \frac{dx}{x} + \frac{3}{4} \int \frac{x}{x^2 + 4} \, dx \\&= \frac{1}{4} \ln|x| + \frac{3}{4} \left(\frac{1}{2} \int \frac{du}{u} \right) \\&= \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C.\end{aligned}$$

Thus,

$$\boxed{I = \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C}.$$

Question 5

Find the value of k that removes the $\ln|x+1|$ term. Then evaluate:

$$\int \frac{kx+1}{x^2-1} dx.$$

Question 5 – Solution

Step 1: Set up the decomposition.

$$\frac{kx+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Step 2: Clear the denominators.

$$kx+1 = A(x+1) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad k+1 &= 2A \implies A = \frac{k+1}{2}, \\ x=-1: \quad -k+1 &= -2B \implies B = \frac{k-1}{2}. \end{aligned}$$

Step 4: Remove the $\ln|x+1|$ term. Since integrating $B/(x+1)$ gives $B \ln|x+1|$, set $B=0$:

$$B=0 \implies \frac{k-1}{2} = 0 \implies \boxed{k=1}.$$

Step 5: Substitute $k=1$, simplify, and integrate.

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}, \quad x \neq \pm 1.$$

$$\int \frac{x+1}{x^2-1} dx = \int \frac{dx}{x-1} = \boxed{\ln|x-1| + C}.$$

Question 6

A student proposes the decomposition

$$\frac{1}{(x^2 + 1)^2} = \frac{A}{x^2 + 1} + \frac{B}{(x^2 + 1)^2}.$$

1. Why is this decomposition not useful? Give the correct general form.
2. Use $x = \tan \theta$ to evaluate:

$$\int \frac{dx}{(x^2 + 1)^2}$$

Question 6 – Solution**Part 1: Check the decomposition.**

Each power of an irreducible quadratic needs a linear numerator. The general form is

$$\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

The original fraction already has this form: $A = B = C = 0$ and $D = 1$. Therefore, decomposing it only reproduces the original integrand and does not make the integral easier.

Part 2: Evaluate the integral.

Step 1: Substitute $x = \tan \theta$. Then

$$dx = \sec^2 \theta \, d\theta, \quad x^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta.$$

Step 2: Simplify and integrate.

$$\begin{aligned} I &= \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int \cos^2 \theta \, d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C. \end{aligned}$$

Step 3: Return to x .

$$\theta = \arctan x, \quad \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2x}{1 + x^2}.$$

Substitute both expressions:

$$I = \frac{x}{2(x^2 + 1)} + \frac{1}{2} \arctan x + C.$$

Question 7

Use partial fractions to evaluate exactly:

$$\int_0^{1/2} \frac{dx}{1-x^2}.$$

Why is this integral proper on $[0, 1/2]$?

Question 7 – Solution

Step 1: Factor and set up partial fractions.

$$1 - x^2 = (1 - x)(1 + x),$$

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}.$$

Step 2: Clear denominators and find A and B .

$$1 = A(1 + x) + B(1 - x).$$

$$x = 1 : \quad 1 = 2A \implies A = \frac{1}{2},$$

$$x = -1 : \quad 1 = 2B \implies B = \frac{1}{2}.$$

Step 3: Integrate and evaluate both bounds.

$$\begin{aligned} I &= \frac{1}{2} \int_0^{1/2} \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) dx \\ &= \frac{1}{2} [-\ln(1 - x) + \ln(1 + x)]_0^{1/2} \\ &= \frac{1}{2} \left[-\ln \frac{1}{2} + \ln \frac{3}{2} - (-\ln 1 + \ln 1) \right] \\ &= \frac{1}{2} \ln \left(\frac{3/2}{1/2} \right) = \frac{1}{2} \ln 3. \end{aligned}$$

Step 4: Check whether the integral is proper. The singularities $x = \pm 1$ are outside $[0, 1/2]$, so the integrand is continuous on the whole interval and no improper limit is needed.

$$\boxed{\int_0^{1/2} \frac{dx}{1 - x^2} = \frac{1}{2} \ln 3.}$$

Question 8

Find A , B , and C in the decomposition

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Use $x = 0$, 2 , and -3 . State (A, B, C) .

Question 8 – Solution

Step 1: Start with the decomposition.

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

Step 2: Multiply every term by $x(x-2)(x+3)$.

$$1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2).$$

Step 3: Use values that eliminate two terms at a time.

$$\begin{array}{llll} x = 0 : & 1 = -6A & \implies & A = -\frac{1}{6}, \\ x = 2 : & 1 = 10B & \implies & B = \frac{1}{10}, \\ x = -3 : & 1 = 15C & \implies & C = \frac{1}{15}. \end{array}$$

Step 4: State the ordered triple.

$$(A, B, C) = \left(-\frac{1}{6}, \frac{1}{10}, \frac{1}{15} \right).$$

Question 9

Use partial fractions to evaluate:

$$\int \frac{dx}{x^3 - x^2} = \int \frac{dx}{x^2(x - 1)}.$$

Question 9 – Solution

Step 1: Set up partial fractions. The repeated factor x^2 requires both $1/x$ and $1/x^2$:

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Step 2: Clear the denominators.

$$1 = Ax(x-1) + B(x-1) + Cx^2.$$

Step 3: Find B and C .

$$x = 0: \quad 1 = -B \implies B = -1,$$

$$x = 1: \quad 1 = C \implies C = 1.$$

Step 4: Find A by setting $x = 2$.

$$\begin{aligned} 1 &= A(2)(1) + (-1)(1) + (1)(4) \\ &= 2A + 3, \\ -2 &= 2A \implies A = -1. \end{aligned}$$

Step 5: Substitute and integrate each term.

$$\begin{aligned} I &= \int \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx \\ &= -\int \frac{dx}{x} - \int x^{-2} dx + \int \frac{dx}{x-1} \\ &= -\ln|x| - \frac{x^{-1}}{-1} + \ln|x-1| + C \\ &= -\ln|x| + \frac{1}{x} + \ln|x-1| + C. \end{aligned}$$

Therefore,

$$\boxed{I = -\ln|x| + \frac{1}{x} + \ln|x-1| + C}.$$

Question 10

1. Define and evaluate the Cauchy principal value:

$$\text{PV} \int_{-1}^1 \frac{dx}{x}$$

2. Does the ordinary improper integral converge? Check both sides of $x = 0$.

Question 10 – Solution

Part 1: Cauchy principal value.

Step 1: Approach zero by the same distance ε from both sides.

$$\begin{aligned}\text{PV} \int_{-1}^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^1 \frac{dx}{x} \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} ([\ln |x|]_{-1}^{-\varepsilon} + [\ln |x|]_{\varepsilon}^1) \\ &= \lim_{\varepsilon \rightarrow 0^+} ((\ln \varepsilon - \ln 1) + (\ln 1 - \ln \varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0^+} 0 = 0.\end{aligned}$$

$$\boxed{\text{PV} \int_{-1}^1 \frac{dx}{x} = 0}.$$

Part 2: Ordinary improper integral.

The two one-sided integrals must converge separately. On the right,

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (0 - \ln \varepsilon) = +\infty.\end{aligned}$$

On the left,

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln |x|]_{-1}^{-\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - 0) = -\infty.\end{aligned}$$

Therefore, the ordinary improper integral diverges. The principal value is finite only because these opposite divergences cancel in a symmetric limit.

Question 1

Use $u = \sqrt{x}$ to evaluate

$$\int \frac{dx}{1 + \sqrt{x}}.$$

Question 1 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\int \frac{dx}{1 + \sqrt{x}} = \int \frac{2u}{1 + u} \, du.$$

Step 3: Divide the numerator.

$$\frac{u}{u + 1} = \frac{(u + 1) - 1}{u + 1} = 1 - \frac{1}{u + 1}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} 2 \int \left(1 - \frac{1}{u + 1} \right) du &= 2u - 2 \ln |u + 1| + C \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C. \end{aligned}$$

$$\boxed{2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C}$$

Question 2

Use $u = x + 2$ to evaluate:

$$\int x\sqrt{x+2} \, dx.$$

Question 2 – Solution

Step 1: Substitute $u = x + 2$. Then

$$x = u - 2, \quad du = dx.$$

Step 2: Rewrite and expand the integrand.

$$\begin{aligned} \int x\sqrt{x+2} \, dx &= \int (u-2)u^{1/2} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du. \end{aligned}$$

Step 3: Integrate each power.

$$\begin{aligned} I &= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C. \end{aligned}$$

Step 4: Substitute $u = x + 2$.

$$\boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$$

Question 3

Find the geometric area between $y = \sqrt[3]{x}$ and the x -axis on $[-1, 8]$.

Question 3 – Solution

Step 1: Split at the x -intercept. The function is negative on $[-1, 0]$ and positive on $[0, 8]$, so

$$A = - \int_{-1}^0 x^{1/3} dx + \int_0^8 x^{1/3} dx.$$

Step 2: Find an antiderivative.

$$\int x^{1/3} dx = \frac{x^{4/3}}{4/3} = \frac{3}{4}x^{4/3}.$$

Step 3: Evaluate both pieces.

$$\begin{aligned} A &= - \left[\frac{3}{4}x^{4/3} \right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} \right]_0^8 \\ &= - \left(0 - \frac{3}{4} \right) + \left(\frac{3}{4} \cdot 16 - 0 \right) \\ &= \frac{3}{4} + 12 = \frac{51}{4}. \end{aligned}$$

$$\boxed{A = \frac{51}{4}}$$

Question 4

Rationalize the integrand and evaluate

$$\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}.$$

Question 4 – Solution

Step 1: Multiply by the conjugate.

$$\begin{aligned}\frac{1}{\sqrt{x+1}-\sqrt{x}} &= \frac{1}{\sqrt{x+1}-\sqrt{x}} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\ &= \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x} \\ &= \sqrt{x+1}+\sqrt{x}.\end{aligned}$$

Step 2: Integrate both terms.

$$\begin{aligned}I &= \int (x+1)^{1/2} dx + \int x^{1/2} dx \\ &= \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + C \\ &= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C.\end{aligned}$$

$$\boxed{\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C}$$

Question 5

Evaluate

$$\int \frac{\sqrt{x}}{1+x} dx.$$

Question 5 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{u}{1+u^2} (2u \, du) \\ &= 2 \int \frac{u^2}{1+u^2} du. \end{aligned}$$

Step 3: Divide the numerator.

$$\frac{u^2}{1+u^2} = \frac{(1+u^2) - 1}{1+u^2} = 1 - \frac{1}{1+u^2}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} I &= 2 \int \left(1 - \frac{1}{1+u^2} \right) du \\ &= 2u - 2 \arctan u + C \\ &= 2\sqrt{x} - 2 \arctan \sqrt{x} + C. \end{aligned}$$

$2\sqrt{x} - 2 \arctan \sqrt{x} + C$

Question 6

Find $a > 0$ so

$$\int_0^a \sqrt{x} \, dx = 18.$$

Question 6 – Solution

Step 1: Evaluate the definite integral.

$$\begin{aligned}\int_0^a \sqrt{x} \, dx &= \int_0^a x^{1/2} \, dx \\&= \left[\frac{x^{3/2}}{3/2} \right]_0^a \\&= \left[\frac{2}{3} x^{3/2} \right]_0^a = \frac{2}{3} a^{3/2}.\end{aligned}$$

Step 2: Set the result equal to 18 and solve.

$$\begin{aligned}\frac{2}{3} a^{3/2} &= 18, \\a^{3/2} &= 27, \\(a^{3/2})^{2/3} &= 27^{2/3}, \\a &= (\sqrt[3]{27})^2 = 3^2 = 9.\end{aligned}$$

$a = 9$

Question 7

Evaluate

$$\int \frac{dx}{x\sqrt[3]{x+1}}$$

using $u = \sqrt[3]{x+1}$.

Question 7 – Solution

Step 1: Substitute $u = \sqrt[3]{x+1}$. Then

$$u^3 = x + 1, \quad x = u^3 - 1, \quad dx = 3u^2 du.$$

Step 2: Rewrite and factor.

$$I = \int \frac{3u^2 du}{(u^3 - 1)u} = 3 \int \frac{u}{u^3 - 1} du,$$
$$u^3 - 1 = (u - 1)(u^2 + u + 1).$$

Step 3: Use partial fractions.

$$\frac{3u}{u^3 - 1} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + u + 1}.$$

After clearing denominators,

$$3u = A(u^2 + u + 1) + (Bu + C)(u - 1).$$

Matching coefficients gives

$$A + B = 0, \quad A - B + C = 3, \quad A - C = 0,$$

so $A = 1$, $B = -1$, and $C = 1$. Therefore,

$$\frac{3u}{u^3 - 1} = \frac{1}{u - 1} + \frac{-u + 1}{u^2 + u + 1}.$$

Step 4: Prepare the quadratic term.

$$-u + 1 = -\frac{1}{2}(2u + 1) + \frac{3}{2}.$$

Thus

$$I = \int \frac{du}{u - 1} - \frac{1}{2} \int \frac{2u + 1}{u^2 + u + 1} du + \frac{3}{2} \int \frac{du}{u^2 + u + 1}.$$

Complete the square:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Step 5: Integrate and return to x .

$$I = \ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C,$$

where $u = \sqrt[3]{x+1}$.

$$\ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$

Question 8

Evaluate

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

using a substitution that clears both radicals.

Question 8 – Solution

Step 1: Choose a power that clears both radicals. Since $\text{lcm}(2, 3) = 6$, let $x = u^6$. Then

$$\sqrt{x} = u^3, \quad \sqrt[3]{x} = u^2, \quad dx = 6u^5 du.$$

The bounds remain 0 and 1 because $u = x^{1/6}$. **Step 2: Rewrite and simplify.**

$$\begin{aligned} I &= 6 \int_0^1 \frac{u^5}{u^3 + u^2} du \\ &= 6 \int_0^1 \frac{u^5}{u^2(u+1)} du = 6 \int_0^1 \frac{u^3}{u+1} du. \end{aligned}$$

Step 3: Divide the polynomials.

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Step 4: Integrate and evaluate the bounds.

$$\begin{aligned} I &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1) \right]_0^1 \\ &= 6 \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2 \right) \\ &= 6 \left(\frac{5}{6} - \ln 2 \right) = 5 - 6 \ln 2. \end{aligned}$$

$$\boxed{I = 5 - 6 \ln 2}$$

Question 9

Differentiate $F(x)$, then identify the substitution used to integrate $F'(x)$:

$$F(x) = \frac{2}{3}(1 + x^2)^{3/2}.$$

Question 9 – Solution

Step 1: Differentiate with the chain rule.

$$\begin{aligned}F'(x) &= \frac{2}{3} \cdot \frac{3}{2}(1+x^2)^{1/2} \cdot (2x) \\&= 2x\sqrt{1+x^2}.\end{aligned}$$

Step 2: Reverse the process by integration.

$$\int 2x\sqrt{1+x^2} \, dx.$$

Let

$$u = 1 + x^2, \quad du = 2x \, dx.$$

Then

$$\begin{aligned}\int 2x\sqrt{1+x^2} \, dx &= \int u^{1/2} \, du \\&= \frac{u^{3/2}}{3/2} + C \\&= \frac{2}{3}(1+x^2)^{3/2} + C.\end{aligned}$$

$$\boxed{\int 2x\sqrt{1+x^2} \, dx = F(x) + C}$$

Question 10

Use one substitution to evaluate:

$$\int \frac{x+1}{\sqrt{x^2+2x+5}} dx.$$

Question 10 – Solution

Step 1: Substitute the expression inside the square root.

$$u = x^2 + 2x + 5.$$

Differentiate:

$$du = (2x + 2) dx = 2(x + 1) dx, \quad (x + 1) dx = \frac{1}{2} du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} I &= \int \frac{x + 1}{\sqrt{x^2 + 2x + 5}} dx \\ &= \frac{1}{2} \int u^{-1/2} du. \end{aligned}$$

Step 3: Integrate and return to x .

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{1/2}}{1/2} \right) + C \\ &= u^{1/2} + C \\ &= \sqrt{x^2 + 2x + 5} + C. \end{aligned}$$

$\sqrt{x^2 + 2x + 5} + C$

Question 1

Complete the square and evaluate

$$\int \frac{dx}{x^2 + 6x + 13}.$$

Question 1 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 + 6x + 13 &= (x^2 + 6x + 9) + 4 \\&= (x + 3)^2 + 4.\end{aligned}$$

Step 2: Rewrite the integral.

$$I = \int \frac{dx}{(x + 3)^2 + 2^2}.$$

Step 3: Use the arctangent formula. Let $u = x + 3$, so $du = dx$:

$$\begin{aligned}I &= \int \frac{du}{u^2 + 2^2} \\&= \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\&= \frac{1}{2} \arctan\left(\frac{x + 3}{2}\right) + C.\end{aligned}$$

$$\frac{1}{2} \arctan \frac{x + 3}{2} + C$$

Question 2

Split the numerator, then evaluate:

$$\int \frac{3x + 1}{x^2 + 2x + 5} dx.$$

Question 2 – Solution

Step 1: Split the numerator using the derivative of the denominator. Since

$$\frac{d}{dx}(x^2 + 2x + 5) = 2x + 2,$$

write

$$3x + 1 = \frac{3}{2}(2x + 2) - 2.$$

Step 2: Split the integral.

$$I = \frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx - 2 \int \frac{dx}{x^2 + 2x + 5}.$$

Step 3: Integrate the derivative term.

$$\frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx = \frac{3}{2} \ln(x^2 + 2x + 5).$$

Step 4: Complete the square in the remaining term.

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Thus

$$\begin{aligned} -2 \int \frac{dx}{(x + 1)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 1}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 1}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\frac{3}{2} \ln(x^2 + 2x + 5) - \arctan \frac{x + 1}{2} + C$$

Question 3

Evaluate

$$\int \frac{x}{x^2 + 4x + 8} dx.$$

Question 3 – Solution

Step 1: Rewrite the numerator using the derivative of the denominator.

$$x = \frac{1}{2}(2x + 4) - 2.$$

Step 2: Split the integral.

$$I = \frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx - 2 \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx = \frac{1}{2} \ln(x^2 + 4x + 8).$$

Step 4: Complete the square.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\begin{aligned} -2 \int \frac{dx}{(x + 2)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 2}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 2}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\boxed{\frac{1}{2} \ln(x^2 + 4x + 8) - \arctan \frac{x + 2}{2} + C}$$

Question 4

Find the maximum of $f(x) = 1/(x^2 - 4x + 8)$, then evaluate

$$\int_{-\infty}^{\infty} f(x) \, dx.$$

Question 4 – Solution**Step 1: Complete the square.**

$$x^2 - 4x + 8 = (x - 2)^2 + 4.$$

Step 2: Find the maximum. The denominator is smallest when $(x - 2)^2 = 0$, or $x = 2$. Its minimum is 4, so

$$f(2) = \frac{1}{4}.$$

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 3: Write the improper integral as a limit.**

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(x - 2)^2 + 2^2} \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \left[\arctan \left(\frac{x - 2}{2} \right) \right]_{-R}^R. \end{aligned}$$

Step 4: Evaluate the limits.

$$I = \frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

$\max = 1/4, \quad I = \pi/2$

Question 5

Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}}.$$

Question 5 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 - 6x + 10 &= (x^2 - 6x + 9) + 1 \\&= (x - 3)^2 + 1.\end{aligned}$$

Step 2: Substitute $u = x - 3$. Then $du = dx$ and

$$I = \int \frac{du}{\sqrt{u^2 + 1}}.$$

Step 3: Use the standard formula.

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left| u + \sqrt{u^2 + a^2} \right| + C.$$

With $a = 1$,

$$\begin{aligned}I &= \ln \left| u + \sqrt{u^2 + 1} \right| + C \\&= \ln \left| x - 3 + \sqrt{x^2 - 6x + 10} \right| + C.\end{aligned}$$

$$\boxed{\ln |x - 3 + \sqrt{x^2 - 6x + 10}| + C}$$

Question 6

Find every k that makes $x^2 + kx + 9$ a perfect square. Then integrate

$$\frac{1}{x^2 + kx + 9}.$$

Question 6 – Solution

Step 1: Write the possible perfect squares. Since the constant term is 9, the square must be

$$\begin{aligned}(x+3)^2 &= x^2 + 6x + 9, \\ (x-3)^2 &= x^2 - 6x + 9.\end{aligned}$$

Thus

$$k = 6 \quad \text{or} \quad k = -6.$$

Step 2: Integrate when $k = 6$.

$$\begin{aligned}\int \frac{dx}{x^2 + 6x + 9} &= \int \frac{dx}{(x+3)^2} \\ &= \int (x+3)^{-2} dx \\ &= -(x+3)^{-1} + C = -\frac{1}{x+3} + C.\end{aligned}$$

Step 3: Integrate when $k = -6$.

$$\begin{aligned}\int \frac{dx}{x^2 - 6x + 9} &= \int (x-3)^{-2} dx \\ &= -(x-3)^{-1} + C = -\frac{1}{x-3} + C.\end{aligned}$$

$k = 6 : -\frac{1}{x+3} + C; \quad k = -6 : -\frac{1}{x-3} + C$

Question 7

Evaluate

$$\int \frac{x^2}{x^2 + 4} dx$$

without partial fractions.

Question 7 – Solution**Step 1: Rewrite the numerator.**

$$x^2 = (x^2 + 4) - 4.$$

Therefore,

$$\begin{aligned}\frac{x^2}{x^2 + 4} &= \frac{(x^2 + 4) - 4}{x^2 + 4} \\ &= 1 - \frac{4}{x^2 + 4}.\end{aligned}$$

Step 2: Split the integral.

$$I = \int 1 \, dx - 4 \int \frac{dx}{x^2 + 2^2}.$$

Step 3: Integrate both terms.

$$\begin{aligned}I &= x - 4 \left[\frac{1}{2} \arctan \left(\frac{x}{2} \right) \right] + C \\ &= x - 2 \arctan \left(\frac{x}{2} \right) + C.\end{aligned}$$

$$x - 2 \arctan \frac{x}{2} + C$$

Question 8

Determine the real domain and evaluate

$$\int \frac{dx}{\sqrt{6x - x^2}}.$$

Question 8 – Solution

Step 1: Complete the square.

$$\begin{aligned}6x - x^2 &= -(x^2 - 6x) \\&= -((x - 3)^2 - 9) \\&= 9 - (x - 3)^2.\end{aligned}$$

Step 2: Determine the domain. The radicand must be nonnegative:

$$\begin{aligned}9 - (x - 3)^2 &\geq 0, \\(x - 3)^2 &\leq 9, \\-3 &\leq x - 3 \leq 3, \\0 &\leq x \leq 6.\end{aligned}$$

The denominator cannot equal zero, so the integrand's domain is

$$\boxed{0 < x < 6}.$$

Step 3: Substitute $u = x - 3$, so $du = dx$:

$$I = \int \frac{du}{\sqrt{3^2 - u^2}}.$$

Step 4: Use the inverse-sine formula.

$$\begin{aligned}I &= \arcsin\left(\frac{u}{3}\right) + C \\&= \arcsin\left(\frac{x - 3}{3}\right) + C.\end{aligned}$$

$$\boxed{\arcsin \frac{x - 3}{3} + C}$$

Question 9

Split the numerator, then evaluate:

$$\int \frac{2x + 5}{x^2 + 4x + 8} dx.$$

Question 9 – Solution**Step 1: Split the numerator.**

$$2x + 5 = (2x + 4) + 1.$$

Step 2: Split the integral.

$$I = \int \frac{2x + 4}{x^2 + 4x + 8} dx + \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\int \frac{2x + 4}{x^2 + 4x + 8} dx = \ln(x^2 + 4x + 8).$$

Step 4: Complete the square for the remaining term.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\int \frac{dx}{(x + 2)^2 + 2^2} = \frac{1}{2} \arctan \left(\frac{x + 2}{2} \right).$$

Step 5: Combine the results.

$$\boxed{\ln(x^2 + 4x + 8) + \frac{1}{2} \arctan \frac{x + 2}{2} + C}$$

Question 10

For $a > 0$, evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2}.$$

Question 10 – Solution

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 1: Write the improper integral as a limit.**

$$I = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + a^2}.$$

Step 2: Find an antiderivative. Since $a > 0$,

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

Therefore,

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \frac{1}{a} \left[\arctan\left(\frac{x}{a}\right) \right]_{-R}^R \\ &= \frac{1}{a} \lim_{R \rightarrow \infty} \left[\arctan\left(\frac{R}{a}\right) - \arctan\left(\frac{-R}{a}\right) \right]. \end{aligned}$$

Step 3: Evaluate the limits.

$$\begin{aligned} I &= \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\ &= \frac{\pi}{a}. \end{aligned}$$

$$\boxed{I = \frac{\pi}{a}}$$

Question 1

Choose the shortest method and evaluate:

$$\int x e^{x^2} dx.$$

Question 1 – Solution

Step 1: Identify the inner function. The exponent is x^2 , and its derivative $2x$ is present except for a constant factor.

Step 2: Substitute $u = x^2$. Then

$$du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int x e^{x^2} \, dx &= \frac{1}{2} \int e^u \, du \\ &= \frac{1}{2} e^u + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{2} e^{x^2} + C}$$

Question 2

Choose substitution or integration by parts, then evaluate:

$$\int x \ln(1 + x^2) dx.$$

Question 2 – Solution

Step 1: Choose substitution. The expression $1 + x^2$ is inside the logarithm, and its derivative $2x$ matches the remaining factor.

Let

$$u = 1 + x^2, \quad du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Then

$$I = \frac{1}{2} \int \ln u \, du.$$

Step 2: Integrate $\ln u$ by parts. Choose

$$v = \ln u, \quad dw = du.$$

Then $dv = du/u$ and $w = u$, so

$$\begin{aligned} \int \ln u \, du &= u \ln u - \int u \left(\frac{1}{u} \right) du \\ &= u \ln u - u + C. \end{aligned}$$

Step 3: Restore the factor $1/2$ and return to x .

$$\begin{aligned} I &= \frac{1}{2}(u \ln u - u) + C \\ &= \frac{1}{2}[(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C. \end{aligned}$$

$$\frac{1}{2}[(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C$$

Question 3

Use $u = x^2 + 1$ to evaluate:

$$\int \frac{x^3}{\sqrt{x^2 + 1}} dx.$$

Question 3 – Solution**Step 1: Separate one factor of x .**

$$x^3 dx = x^2(x dx).$$

Step 2: Substitute $u = x^2 + 1$. Then

$$du = 2x dx, \quad x dx = \frac{1}{2} du, \quad x^2 = u - 1.$$

Step 3: Rewrite and simplify.

$$\begin{aligned} I &= \frac{1}{2} \int \frac{u-1}{\sqrt{u}} du \\ &= \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du. \end{aligned}$$

Step 4: Integrate each power.

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} \right) + C \\ &= \frac{1}{3} u^{3/2} - u^{1/2} + C. \end{aligned}$$

Step 5: Return to x .

$$\frac{1}{3}(x^2 + 1)^{3/2} - \sqrt{x^2 + 1} + C$$

Question 4

Identify the hidden derivative and evaluate:

$$\int \frac{\ln x}{x} dx.$$

Question 4 – Solution

Step 1: Identify the inner function. Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

the factor dx/x is the differential of $\ln x$.

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

Step 3: Rewrite and integrate.

$$\begin{aligned}\int \frac{\ln x}{x} dx &= \int u du \\ &= \frac{u^2}{2} + C.\end{aligned}$$

Step 4: Return to x .

$\frac{1}{2}(\ln x)^2 + C$

Question 5

Recognize a product derivative and evaluate:

$$\int e^x(x+1) dx.$$

Question 5 – Solution

Step 1: Look for a product whose derivative matches the integrand. Try xe^x .

Step 2: Differentiate with the product rule.

$$\begin{aligned}\frac{d}{dx}(xe^x) &= \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) \\ &= e^x + xe^x \\ &= e^x(x+1).\end{aligned}$$

Step 3: Reverse the derivative.

$$\int e^x(x+1) dx = xe^x + C.$$

$xe^x + C$

Question 6

Explain why $u = x^2 + 1$ does not simplify the integral directly, then evaluate:

$$\int \frac{dx}{x^2 + 1}.$$

Question 6 – Solution

Step 1: Test the proposed substitution. If

$$u = x^2 + 1,$$

then

$$du = 2x \, dx.$$

The integral contains dx , but it does not contain the required factor $x \, dx$. Thus the substitution does not produce a direct du factor. One can express dx in terms of u on a branch, but this introduces a square root and is less convenient than the standard antiderivative.

Step 2: Recognize a standard form.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Therefore,

$$\int \frac{dx}{x^2+1} = \arctan x + C.$$

$\int \frac{dx}{x^2+1} = \arctan x + C$

Question 7

Choose the best method and evaluate:

$$\int \frac{x^2}{x^3 + 8} dx.$$

Question 7 – Solution

Step 1: Identify the structure. The denominator is $x^3 + 8$, and its derivative is $3x^2$. The numerator contains x^2 , so substitution is the shortest method.

Step 2: Substitute $u = x^3 + 8$. Then

$$du = 3x^2 dx, \quad x^2 dx = \frac{1}{3} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \frac{x^2}{x^3 + 8} dx &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln |u| + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{3} \ln |x^3 + 8| + C}$$

Question 8

State the structural cue, then evaluate:

$$\int \sin x \cos^4 x \, dx.$$

Question 8 – Solution

Step 1: Identify the structural cue. The sine power is odd, and $\sin x \, dx$ is the differential of $-\cos x$. Keep the sine factor and use the cosine as the substitution.

Step 2: Substitute $u = \cos x$. Then

$$du = -\sin x \, dx, \quad \sin x \, dx = -du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \sin x \cos^4 x \, dx &= - \int u^4 \, du \\ &= -\frac{u^5}{5} + C. \end{aligned}$$

Step 4: Return to x .

$$-\frac{1}{5} \cos^5 x + C$$

Question 9

Choose the best method for each integral. Do not evaluate.

$$\int x^2 e^x dx, \quad \int \frac{dx}{x^2 - 4}, \quad \int \sqrt{9 - x^2} dx, \quad \int \sin^2 x dx.$$

Question 9 – Solution

1. For $\int x^2 e^x dx$, differentiating x^2 repeatedly makes it simpler, while integrating e^x changes nothing. Use **integration by parts** twice.

2. For $\int dx/(x^2 - 4)$, factor

$$x^2 - 4 = (x - 2)(x + 2).$$

The denominator has distinct linear factors, so use **partial fractions**.

3. For $\int \sqrt{9 - x^2} dx$, the form $\sqrt{a^2 - x^2}$ suggests $x = a \sin \theta$. Use **trigonometric substitution** with $x = 3 \sin \theta$.

4. For $\int \sin^2 x dx$, both sine and cosine powers are even. Use the **power-reduction identity**

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

$x^2 e^x$	integration by parts
$1/(x^2 - 4)$	partial fractions
$\sqrt{9 - x^2}$	trigonometric substitution
$\sin^2 x$	power reduction

Question 10

Find the integrand whose antiderivative is

$$F(x) = \ln(x^2 + 1) + \arctan x.$$

Then verify your result.

Question 10 – Solution

Step 1: Differentiate the logarithm using the chain rule.

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}.$$

Step 2: Differentiate the inverse tangent.

$$\frac{d}{dx} \arctan x = \frac{1}{x^2 + 1}.$$

Step 3: Add and simplify.

$$\begin{aligned} F'(x) &= \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} \\ &= \frac{2x + 1}{x^2 + 1}. \end{aligned}$$

Thus the required integrand is

$$\boxed{f(x) = \frac{2x + 1}{x^2 + 1}}$$

Step 4: Verify by integration.

$$\begin{aligned} \int \frac{2x + 1}{x^2 + 1} dx &= \int \frac{2x}{x^2 + 1} dx + \int \frac{dx}{x^2 + 1} \\ &= \ln(x^2 + 1) + \arctan x + C = F(x) + C. \end{aligned}$$

Question 1

For which real p does the integral converge? Evaluate it when possible.

$$\int_1^{\infty} x^{-p} dx$$

Question 1 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx.$$

Step 2: First suppose $p \neq 1$.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \frac{b^{1-p} - 1}{1-p}. \end{aligned}$$

If $p > 1$, then $1 - p < 0$ and $b^{1-p} \rightarrow 0$, so

$$I = \frac{-1}{1-p} = \frac{1}{p-1}.$$

If $p < 1$, then $b^{1-p} \rightarrow \infty$, so the integral diverges.

Step 3: Check $p = 1$ separately.

$$\int_1^b \frac{dx}{x} = \ln b \rightarrow \infty.$$

$p > 1 : I = \frac{1}{p-1}; \quad p \leq 1 : \text{diverges}$

Question 2

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^1 x^{-2/3} dx$$

Question 2 – Solution

Step 1: Identify the improper endpoint. The integrand is undefined at $x = 0$, so replace the lower bound with $\varepsilon > 0$:

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-2/3} dx.$$

Step 2: Integrate.

$$\int x^{-2/3} dx = \frac{x^{1/3}}{1/3} = 3x^{1/3}.$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [3x^{1/3}]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (3 - 3\varepsilon^{1/3}) = 3. \end{aligned}$$

3 (convergent)

Question 3

Determine whether the integral converges. If it does, evaluate it.

$$\int_{-1}^1 \frac{dx}{\sqrt{|x|}}$$

Question 3 – Solution

Step 1: Split at the interior singularity.

$$I = \int_{-1}^0 \frac{dx}{\sqrt{|x|}} + \int_0^1 \frac{dx}{\sqrt{|x|}}.$$

The integrand is even, so the two pieces are equal:

$$I = 2 \int_0^1 x^{-1/2} dx.$$

Step 2: Write the improper limit.

$$I = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-1/2} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{\varepsilon \rightarrow 0^+} [2x^{1/2}]_{\varepsilon}^1 \\ &= 4 \lim_{\varepsilon \rightarrow 0^+} (1 - \sqrt{\varepsilon}) = 4. \end{aligned}$$

$I = 4 \text{ (convergent)}$

Question 4

For which real a does the integral converge? Evaluate it when possible.

$$\int_0^{\infty} e^{-ax} dx$$

Question 4 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b e^{-ax} dx.$$

Step 2: Suppose $a \neq 0$ and integrate.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{a} e^{-ax} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \frac{1 - e^{-ab}}{a}. \end{aligned}$$

If $a > 0$, then $e^{-ab} \rightarrow 0$, so $I = 1/a$. If $a < 0$, then $e^{-ab} \rightarrow \infty$, so the integral diverges.

Step 3: Check $a = 0$. Then the integrand is 1, and

$$\int_0^\infty 1 dx$$

diverges.

$a > 0 : 1/a; \quad a \leq 0 : \text{diverges}$

Question 5

Determine whether the integral converges:

$$\int_2^{\infty} \frac{dx}{x \ln x}.$$

Question 5 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x}.$$

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

The bounds become $u = \ln 2$ and $u = \ln b$, so

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} \\ &= \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)). \end{aligned}$$

Step 3: Evaluate the limit. Since $\ln(\ln b) \rightarrow \infty$, the integral diverges.

diverges

Question 6

Evaluate the two-tailed integral:

$$\int_{-\infty}^{\infty} e^{-|x|} dx.$$

Question 6 – Solution

Step 1: Use symmetry. Since $e^{-|x|}$ is even,

$$I = 2 \int_0^{\infty} e^{-x} dx.$$

Step 2: Replace infinity with a limit.

$$I = 2 \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\ &= 2 \lim_{b \rightarrow \infty} (1 - e^{-b}) \\ &= 2(1 - 0) = 2. \end{aligned}$$

$$\boxed{I = 2}$$

Question 7

Determine whether the ordinary improper integral converges:

$$\int_{-1}^1 \frac{dx}{x}.$$

Question 7 – Solution

Step 1: Split at the singularity. Ordinary convergence requires both one-sided integrals to converge.

Step 2: Evaluate the left side.

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - \ln 1) = -\infty.\end{aligned}$$

Step 3: Evaluate the right side.

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln 1 - \ln \varepsilon) = +\infty.\end{aligned}$$

Since the one-sided integrals diverge, the ordinary integral diverges.

diverges

The symmetric principal value is 0, but it is not an ordinary integral value.

Question 8

Evaluate the improper integral:

$$\int_0^1 \ln x \, dx.$$

Question 8 – Solution

Step 1: Replace the improper endpoint with a limit.

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln x \, dx.$$

Step 2: Integrate by parts. Let $u = \ln x$ and $dv = dx$. Then $du = dx/x$ and $v = x$:

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - x. \end{aligned}$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [x \ln x - x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln \varepsilon + \varepsilon). \end{aligned}$$

Since $\varepsilon \ln \varepsilon \rightarrow 0$ and $\varepsilon \rightarrow 0$,

$$I = -1.$$

$$\boxed{-1}$$

Question 9

Find c so that

$$f(x) = \frac{c}{1+x^2}$$

is a probability density on $(-\infty, \infty)$.

Question 9 – Solution

Step 1: Use the normalization requirement. A probability density must satisfy

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Thus

$$1 = c \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Step 2: Evaluate the improper integral. The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{R \rightarrow \infty} [\arctan x]_{-R}^R \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi. \end{aligned}$$

Step 3: Solve for c .

$$1 = c\pi \quad \implies \quad c = \frac{1}{\pi}.$$

Because $c > 0$, $f(x) \geq 0$ for every real x .

$c = 1/\pi$

Question 10

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^{\infty} \frac{dx}{(1+x)^2}$$

Question 10 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b (1+x)^{-2} dx.$$

Step 2: Integrate.

$$\int (1+x)^{-2} dx = -(1+x)^{-1} = -\frac{1}{1+x}.$$

Step 3: Evaluate the bounds and limit.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{1+x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{1+b} + 1 \right) \\ &= 0 + 1 = 1. \end{aligned}$$

1 (convergent)

Question 1

Use direct comparison to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + \sin^2 x}.$$

Question 1 – Solution

Step 1: Find an upper comparison. Since $\sin^2 x \geq 0$,

$$x^2 + \sin^2 x \geq x^2.$$

Both sides are positive, so taking reciprocals reverses the inequality:

$$0 \leq \frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}.$$

Step 2: Test the comparison integral.

$$\begin{aligned} \int_1^\infty \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1. \end{aligned}$$

Step 3: Apply direct comparison. The larger integral converges, so the original integral also converges.

converges

Question 2

Use limit comparison to determine convergence:

$$\int_1^{\infty} \frac{3x^2 + 1}{x^4 - 2x + 5} dx.$$

Question 2 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{3x^2}{x^4} = \frac{3}{x^2},$$

so use $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(3x^2 + 1)/(x^4 - 2x + 5)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{3x^4 + x^2}{x^4 - 2x + 5} \\ &= \lim_{x \rightarrow \infty} \frac{3 + 1/x^2}{1 - 2/x^3 + 5/x^4} = 3. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and $\int_1^\infty x^{-2} dx$ converges, the original integral converges.

converges

Question 3

Use a lower comparison to determine convergence:

$$\int_1^{\infty} \frac{2 + \sin x}{x} dx.$$

Question 3 – Solution

Step 1: Bound the numerator. Since $\sin x \geq -1$,

$$2 + \sin x \geq 1.$$

For $x \geq 1$, divide by the positive number x :

$$\frac{2 + \sin x}{x} \geq \frac{1}{x} \geq 0.$$

Step 2: Test the lower comparison.

$$\int_1^\infty \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

Step 3: Apply direct comparison. The smaller integral diverges, so the original integral diverges.

diverges

Question 4

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sqrt{x} + x}.$$

Question 4 – Solution

Step 1: Choose a comparison. Near $x = 0$, $\sqrt{x}$ is the dominant denominator term, so use

$$g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}.$$

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/(\sqrt{x} + x)}{1/\sqrt{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x} + x} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{1 + \sqrt{x}} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 x^{-1/2} dx = 2 < \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral converges.

converges

Question 5

Use comparison, not integration by parts, to determine convergence:

$$\int_2^{\infty} \frac{\ln x}{x^2} dx.$$

Question 5 – Solution

Step 1: Bound the logarithm. The logarithm grows more slowly than every positive power of x . In particular,

$$\ln x \leq \sqrt{x}$$

for all sufficiently large x .

Step 2: Divide by $x^2 > 0$.

$$0 \leq \frac{\ln x}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}.$$

Step 3: Test the upper comparison.

$$\int_2^\infty x^{-3/2} dx$$

converges because it is a p -integral with $p = 3/2 > 1$.

Step 4: Apply direct comparison. Therefore, the original integral converges.

converges

Question 6

A student uses

$$\frac{1}{\sqrt{x^2 + 1}} < \frac{1}{x}$$

to claim convergence on $[1, \infty)$. Explain the error and classify the integral.

Question 6 – Solution

Step 1: Identify the error. The comparison integral

$$\int_1^{\infty} \frac{dx}{x}$$

diverges. Being smaller than a divergent function does not prove convergence; a smaller positive function may converge or diverge.

Step 2: Use limit comparison with $g(x) = 1/x$.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/\sqrt{x^2 + 1}}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and the harmonic integral diverges, the original integral also diverges.

diverges by limit comparison

Question 7

Use dominant powers to determine convergence:

$$\int_1^{\infty} \frac{x^3 + 4}{x^5 + x} dx.$$

Question 7 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{x^3}{x^5} = \frac{1}{x^2},$$

so let $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(x^3 + 4)/(x^5 + x)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{x^5 + 4x^2}{x^5 + x} \\ &= \lim_{x \rightarrow \infty} \frac{1 + 4/x^3}{1 + 1/x^4} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $\int_1^\infty x^{-2} dx$ converges and $0 < L < \infty$, the original integral converges.

converges

Question 8

Find all real p for which the integral converges:

$$\int_1^{\infty} \frac{dx}{x^p + 1}.$$

Question 8 – Solution

Step 1: Consider $p > 0$. Compare with $g(x) = 1/x^p$:

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/(x^p + 1)}{1/x^p} \\ &= \lim_{x \rightarrow \infty} \frac{x^p}{x^p + 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + 1/x^p} = 1. \end{aligned}$$

Thus the original integral has the same behavior as the p -integral $\int_1^\infty x^{-p} dx$, which converges exactly when $p > 1$.

Step 2: Consider $p = 0$. The integrand is $1/2$, so the integral diverges.

Step 3: Consider $p < 0$. Then $x^p \rightarrow 0$, so

$$\frac{1}{x^p + 1} \rightarrow 1 \neq 0.$$

For $x \geq 1$ and $p < 0$, $x^p \leq 1$, so the integrand is at least $1/2$. Comparison with $\int_1^\infty (1/2) dx$ proves divergence.

$$\boxed{p > 1}$$

Question 9

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sin x}.$$

Question 9 – Solution

Step 1: Choose a comparison. Since $\sin x \sim x$ near zero, compare with $g(x) = 1/x$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/\sin x}{1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \rightarrow 0^+} (-\ln \varepsilon) = \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral diverges.

diverges

Question 10

Use a two-sided estimate to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + 3x}.$$

Question 10 – Solution

Step 1: Bound the denominator. For $x \geq 1$, we have $x \leq x^2$, so

$$3x \leq 3x^2.$$

Adding x^2 gives

$$x^2 \leq x^2 + 3x \leq 4x^2.$$

Step 2: Take reciprocals. All terms are positive, so

$$\frac{1}{4x^2} \leq \frac{1}{x^2 + 3x} \leq \frac{1}{x^2}.$$

Step 3: Test the upper comparison.

$$\int_1^\infty \frac{dx}{x^2} = 1 < \infty.$$

Step 4: Apply direct comparison. Since the original integrand is nonnegative and bounded above by an integrable function, the integral converges.

converges

Question 1

Given

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 5, \quad f(3) = 10, \quad f(4) = 17,$$

use T_4 to approximate $\int_0^4 f(x) \, dx$.

Question 1 – Solution

Step 1: Find the subinterval width.

$$h = \frac{b-a}{n} = \frac{4-0}{4} = 1.$$

Step 2: Write the trapezoidal formula.

$$T_4 = \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)].$$

Step 3: Substitute the table values.

$$\begin{aligned} T_4 &= \frac{1}{2} [1 + 2(2) + 2(5) + 2(10) + 17] \\ &= \frac{1}{2} [1 + 4 + 10 + 20 + 17] \\ &= \frac{1}{2} (52) = 26. \end{aligned}$$

$T_4 = 26$

Question 2

Use M_4 to approximate

$$\int_0^2 e^{-x^2} dx.$$

Give the exponential form and a decimal.

Question 2 – Solution

Step 1: Find the width.

$$h = \frac{2 - 0}{4} = 0.5.$$

Step 2: Find the four midpoints.

$$0.25, \quad 0.75, \quad 1.25, \quad 1.75.$$

Step 3: Apply the midpoint formula.

$$\begin{aligned} M_4 &= h \sum_{i=1}^4 f(m_i) \\ &= 0.5 \left(e^{-(0.25)^2} + e^{-(0.75)^2} + e^{-(1.25)^2} + e^{-(1.75)^2} \right). \end{aligned}$$

Step 4: Evaluate.

$$M_4 \approx 0.88278895.$$

$M_4 \approx 0.8828$

Question 3

Use Simpson's Rule with $n = 4$ to approximate

$$\int_0^2 \sqrt{1+x^3} \, dx.$$

Question 3 – Solution**Step 1: Find the width and nodes.**

$$h = \frac{2-0}{4} = 0.5, \quad x_i = 0, 0.5, 1, 1.5, 2.$$

Step 2: Evaluate $f(x) = \sqrt{1+x^3}$.

$$\begin{aligned} f(0) &= 1, & f(0.5) &= \sqrt{1.125}, & f(1) &= \sqrt{2}, \\ f(1.5) &= \sqrt{4.375}, & f(2) &= 3. \end{aligned}$$

Step 3: Apply Simpson's Rule.

$$\begin{aligned} S_4 &= \frac{h}{3} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \\ &= \frac{0.5}{3} [1 + 4\sqrt{1.125} + 2\sqrt{2} + 4\sqrt{4.375} + 3] \\ &\approx 3.23961135. \end{aligned}$$

$$\boxed{S_4 \approx 3.2396}$$

Question 4

For increasing f , explain why

$$L_n \leq \int_a^b f(x) \, dx \leq R_n.$$

Then bound $\int_0^1 e^x \, dx$ using $n = 2$.

Question 4 – Solution

Step 1: Justify the inequality. On each subinterval, an increasing function satisfies

$$f(x_{i-1}) \leq f(x) \leq f(x_i).$$

Multiplying by $\Delta x > 0$ and adding over all subintervals gives

$$L_n \leq \int_a^b f(x) dx \leq R_n.$$

Step 2: Find the width and nodes.

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}, \quad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1.$$

Step 3: Compute both sums.

$$\begin{aligned} L_2 &= \frac{1}{2}[f(0) + f(1/2)] = \frac{1}{2}(1 + e^{1/2}) \approx 1.3244, \\ R_2 &= \frac{1}{2}[f(1/2) + f(1)] = \frac{1}{2}(e^{1/2} + e) \approx 2.1835. \end{aligned}$$

Step 4: State the bound.

$$1.3244 \leq \int_0^1 e^x dx \leq 2.1835$$

Question 5

For $\int_0^2 x^4 dx$, determine whether T_n and M_n are overestimates or underestimates. Use concavity.

Question 5 – Solution

Step 1: Determine concavity. For $f(x) = x^4$,

$$f'(x) = 4x^3, \quad f''(x) = 12x^2.$$

On $[0, 2]$, $f''(x) \geq 0$, so the graph is concave up.

Step 2: Interpret the trapezoidal approximation. For a concave-up function, each secant line lies above the graph. Therefore,

$$T_n > \int_0^2 x^4 dx.$$

Step 3: Interpret the midpoint approximation. For a concave-up function, midpoint rectangles lie below the graph on average. Therefore,

$$M_n < \int_0^2 x^4 dx.$$

$T_n > I, \quad M_n < I$

Question 6

How large must n be for T_n to approximate

$$\int_0^1 e^x dx$$

within 10^{-4} ? Use $|f''(x)| \leq e$.

Question 6 – Solution

Step 1: Write the trapezoidal error bound.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Here $K = e$ and $b - a = 1$, so

$$|E_T| \leq \frac{e}{12n^2}.$$

Step 2: Require the bound to be at most 10^{-4} .

$$\frac{e}{12n^2} \leq 10^{-4},$$

$$e \leq 0.0012n^2,$$

$$n^2 \geq \frac{e}{0.0012} \approx 2265.23,$$

$$n \geq \sqrt{2265.23} \approx 47.594.$$

Step 3: Choose the smallest integer.

$$\boxed{n = 48}$$

Question 7

Show that Simpson's Rule with $n = 2$ is exact for

$$\int_0^2 x^3 dx.$$

Question 7 – Solution

Step 1: Find the width and nodes.

$$h = \frac{2-0}{2} = 1, \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 2.$$

Step 2: Apply Simpson's Rule.

$$\begin{aligned} S_2 &= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \\ &= \frac{1}{3}[0^3 + 4(1^3) + 2^3] \\ &= \frac{1}{3}(0 + 4 + 8) = 4. \end{aligned}$$

Step 3: Evaluate the exact integral.

$$\begin{aligned} I &= \int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 \\ &= \frac{16}{4} - 0 = 4. \end{aligned}$$

Since $S_2 = I$, Simpson's Rule is exact here.

$$\boxed{S_2 = I = 4}$$

Question 8

For an increasing function on $[0, 3]$, suppose

$$L_6 = 7.2, \quad R_6 = 8.7.$$

Find T_6 and bound the integral.

Question 8 – Solution

Step 1: Use the trapezoidal-average identity. With equal-width subintervals,

$$T_n = \frac{L_n + R_n}{2}.$$

Thus

$$\begin{aligned} T_6 &= \frac{7.2 + 8.7}{2} \\ &= \frac{15.9}{2} = 7.95. \end{aligned}$$

Step 2: Use monotonicity to bound the integral. Because f is increasing, left endpoints underestimate and right endpoints overestimate:

$$L_6 \leq I \leq R_6.$$

Therefore,

$$7.2 \leq I \leq 8.7.$$

$7.2 \leq I \leq 8.7, \quad T_6 = 7.95$

Question 9

A velocity is measured every second:

t (s)	0	1	2	3	4
v (m/s)	0	3	5	6	4

Use T_4 to estimate displacement.

Question 9 – Solution

Step 1: Find the time width.

$$h = \frac{4 - 0}{4} = 1 \text{ second.}$$

Step 2: Apply the trapezoidal formula.

$$\begin{aligned} T_4 &= \frac{h}{2} [v_0 + 2v_1 + 2v_2 + 2v_3 + v_4] \\ &= \frac{1}{2} [0 + 2(3) + 2(5) + 2(6) + 4] \\ &= \frac{1}{2} [0 + 6 + 10 + 12 + 4] \\ &= \frac{1}{2} (32) = 16. \end{aligned}$$

Step 3: Include units. Velocity times time gives displacement in meters.

$$\boxed{16 \text{ m}}$$

Question 10

For Simpson's Rule on $[0, \pi]$, suppose

$$|f^{(4)}(x)| \leq 1.$$

Find the smallest even n that guarantees error at most 10^{-6} .

Question 10 – Solution

Step 1: Write Simpson's error bound.

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Here $K = 1$ and $b - a = \pi$, so

$$|E_S| \leq \frac{\pi^5}{180n^4}.$$

Step 2: Require the bound to be at most 10^{-6} .

$$\begin{aligned}\frac{\pi^5}{180n^4} &\leq 10^{-6}, \\ n^4 &\geq \frac{\pi^5}{180 \cdot 10^{-6}}, \\ n &\geq \left(\frac{\pi^5}{180 \cdot 10^{-6}} \right)^{1/4} \\ &\approx 36.109.\end{aligned}$$

Step 3: Enforce Simpson's even- n requirement. The first even integer at least 36.109 is 38.

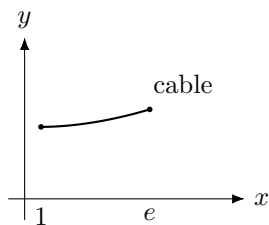
$$\boxed{n = 38}$$

Question 1

A cable follows

$$y = \frac{x^2}{4} - \frac{1}{2} \ln x, \quad 1 \leq x \leq e.$$

Find its exact length.



Question 1 – Solution**Step 1: Differentiate.**

$$y' = \frac{x}{2} - \frac{1}{2x}.$$

Step 2: Simplify the arc-length integrand.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2 \\ &= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \\ &= \left(\frac{x}{2} + \frac{1}{2x} \right)^2. \end{aligned}$$

Since $x > 0$,

$$\sqrt{1 + (y')^2} = \frac{x}{2} + \frac{1}{2x}.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} L &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x} \right) dx \\ &= \left[\frac{x^2}{4} + \frac{1}{2} \ln x \right]_1^e \\ &= \left(\frac{e^2}{4} + \frac{1}{2} \right) - \frac{1}{4} = \frac{e^2 + 1}{4}. \end{aligned}$$

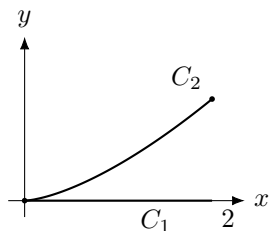
$$\boxed{L = \frac{e^2 + 1}{4}}$$

Question 2

For $0 \leq x \leq 2$, compare

$$C_1 : y = 0, \quad C_2 : y = \frac{2}{3}x^{3/2}.$$

Find $L_2 - L_1$.



Question 2 – Solution

Step 1: Find the length of C_1 . Since $y' = 0$,

$$L_1 = \int_0^2 \sqrt{1 + 0^2} \, dx = \int_0^2 1 \, dx = 2.$$

Step 2: Differentiate C_2 .

$$y' = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = \sqrt{x}.$$

Step 3: Find the length of C_2 .

$$\begin{aligned} L_2 &= \int_0^2 \sqrt{1 + (\sqrt{x})^2} \, dx \\ &= \int_0^2 \sqrt{1 + x} \, dx \\ &= \left[\frac{2}{3} (1 + x)^{3/2} \right]_0^2 \\ &= \frac{2}{3} (3\sqrt{3} - 1) = 2\sqrt{3} - \frac{2}{3}. \end{aligned}$$

Step 4: Subtract.

$$L_2 - L_1 = 2\sqrt{3} - \frac{2}{3} - 2 = 2\sqrt{3} - \frac{8}{3}.$$

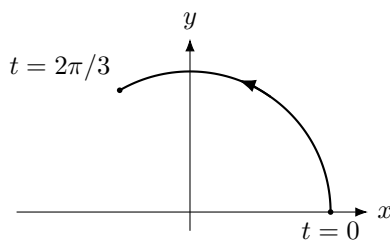
$$L_2 - L_1 = 2\sqrt{3} - \frac{8}{3}$$

Question 3

A bead moves along

$$x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \frac{2\pi}{3}.$$

Find the distance traveled.



Question 3 – Solution

Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = -3 \sin t, \quad \frac{dy}{dt} = 3 \cos t.$$

Step 2: Compute the speed.

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{9 \sin^2 t + 9 \cos^2 t} \\ &= \sqrt{9} = 3.\end{aligned}$$

Step 3: Integrate the speed.

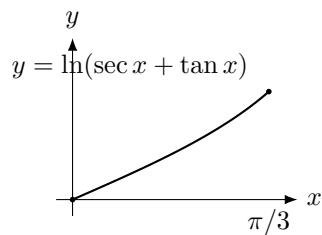
$$\begin{aligned}L &= \int_0^{2\pi/3} 3 \, dt \\ &= [3t]_0^{2\pi/3} = 2\pi.\end{aligned}$$

$L = 2\pi$

Question 4

Find the exact length of

$$y = \ln(\sec x + \tan x), \quad 0 \leq x \leq \frac{\pi}{3}.$$



Question 4 – Solution

Step 1: Differentiate.

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Thus

$$L = \int_0^{\pi/3} \sqrt{1 + \sec^2 x} \, dx.$$

Step 2: Substitute $u = \tan x$. Then $du = \sec^2 x \, dx$ and the bounds become 0 and $\sqrt{3}$:

$$L = \int_0^{\sqrt{3}} \frac{\sqrt{u^2 + 2}}{1 + u^2} \, du.$$

Step 3: Substitute $u = \sqrt{2} \sinh t$. Then

$$du = \sqrt{2} \cosh t \, dt, \quad \sqrt{u^2 + 2} = \sqrt{2} \cosh t.$$

The upper bound is $T = \operatorname{arsinh} \sqrt{3/2}$, and

$$\begin{aligned} L &= \int_0^T \frac{2 \cosh^2 t}{1 + 2 \sinh^2 t} \, dt \\ &= \int_0^T (1 + \operatorname{sech}(2t)) \, dt \\ &= \left[t + \frac{1}{2} \arctan(\sinh 2t) \right]_0^T. \end{aligned}$$

Since $\sinh(2T) = \sqrt{15}$,

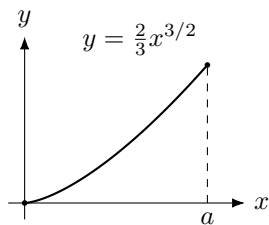
$$L = \operatorname{arsinh} \sqrt{\frac{3}{2}} + \frac{1}{2} \arctan \sqrt{15}$$

Question 5

Find $a > 0$ so that

$$y = \frac{2}{3}x^{3/2}, \quad 0 \leq x \leq a,$$

has length $14/3$.



Question 5 – Solution**Step 1: Differentiate.**

$$y' = \sqrt{x}.$$

Step 2: Write and evaluate the length.

$$\begin{aligned} L &= \int_0^a \sqrt{1+x} \, dx \\ &= \left[\frac{2}{3} (1+x)^{3/2} \right]_0^a \\ &= \frac{2}{3} \left((1+a)^{3/2} - 1 \right). \end{aligned}$$

Step 3: Set the length equal to $14/3$ and solve.

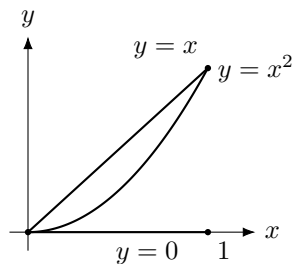
$$\begin{aligned} \frac{2}{3} \left((1+a)^{3/2} - 1 \right) &= \frac{14}{3}, \\ (1+a)^{3/2} - 1 &= 7, \\ (1+a)^{3/2} &= 8, \\ 1+a &= 8^{2/3} = 4, \\ a &= 3. \end{aligned}$$

$a = 3$

Question 6

Rank the arc lengths on $0 \leq x \leq 1$:

$$y = 0, \quad y = x, \quad y = x^2.$$



Question 6 – Solution

Step 1: Find the first two lengths.

$$L_0 = \int_0^1 \sqrt{1+0^2} \, dx = 1,$$
$$L_x = \int_0^1 \sqrt{1+1^2} \, dx = \sqrt{2} \approx 1.414.$$

Step 2: Set up the third length. For $y = x^2$, $y' = 2x$, so

$$L_{x^2} = \int_0^1 \sqrt{1+4x^2} \, dx.$$

Use the standard antiderivative

$$\int \sqrt{1+4x^2} \, dx = \frac{x}{2} \sqrt{1+4x^2} + \frac{1}{4} \operatorname{arsinh}(2x).$$

Therefore,

$$\begin{aligned} L_{x^2} &= \left[\frac{x}{2} \sqrt{1+4x^2} + \frac{1}{4} \operatorname{arsinh}(2x) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \operatorname{arsinh} 2 \\ &\approx 1.479. \end{aligned}$$

Step 3: Rank the values.

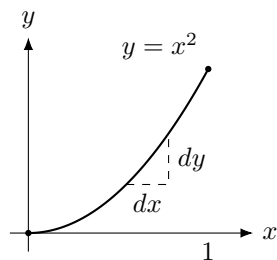
$$\boxed{L_0 < L_x < L_{x^2}}$$

Question 7

A student writes

$$L = \int_0^1 (1 + y') \, dx$$

for $y = x^2$. Explain the error and find the correct length.



Question 7 – Solution

Step 1: Correct the geometric formula. A small arc segment is the hypotenuse of a right triangle:

$$ds^2 = dx^2 + dy^2.$$

Since $dy = y' dx$,

$$ds = \sqrt{dx^2 + (y' dx)^2} = \sqrt{1 + (y')^2} dx.$$

The student's expression adds horizontal and vertical changes instead of using the Pythagorean theorem.

Step 2: Differentiate and set up the length.

$$y' = 2x, \quad L = \int_0^1 \sqrt{1 + 4x^2} dx.$$

Step 3: Evaluate.

$$\begin{aligned} L &= \left[\frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \ln(2x + \sqrt{1 + 4x^2}) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}). \end{aligned}$$

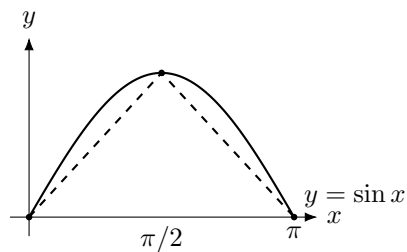
$$L = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5})$$

Question 8

Use two straight chords through $x = 0, \pi/2, \pi$ to estimate the length of

$$y = \sin x, \quad 0 \leq x \leq \pi.$$

Explain why the estimate is too small.



Question 8 – Solution**Step 1: List the three points.**

$$(0, 0), \quad \left(\frac{\pi}{2}, 1\right), \quad (\pi, 0).$$

Step 2: Find the first chord length.

$$\begin{aligned} d_1 &= \sqrt{\left(\frac{\pi}{2} - 0\right)^2 + (1 - 0)^2} \\ &= \sqrt{\frac{\pi^2}{4} + 1}. \end{aligned}$$

By symmetry, $d_2 = d_1$. Thus

$$L_{\text{chord}} = 2d_1 = 2\sqrt{\frac{\pi^2}{4} + 1} = \sqrt{\pi^2 + 4}.$$

Step 3: Compare with the true length. The true arc length is

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx.$$

A curved arc between two points is longer than its straight chord. Since the sine curve is not straight on either half,

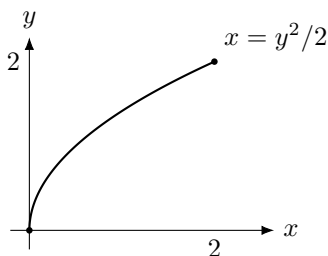
$L_{\text{chord}} = \sqrt{\pi^2 + 4}, \quad L > L_{\text{chord}}$

Question 9

For

$$x = \frac{y^2}{2}, \quad 0 \leq y \leq 2,$$

find the length using y . Then write the equivalent x -integral without evaluating it.



Question 9 – Solution

Step 1: Differentiate with respect to y .

$$\frac{dx}{dy} = y.$$

Step 2: Set up and evaluate the y -integral.

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^2 \sqrt{1 + y^2} dy \\ &= \left[\frac{y}{2} \sqrt{1 + y^2} + \frac{1}{2} \operatorname{arsinh} y \right]_0^2 \\ &= \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2. \end{aligned}$$

$$L = \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2$$

Step 3: Rewrite using x . Since $y = \sqrt{2x}$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}.$$

The x -values run from 0 to 2, so

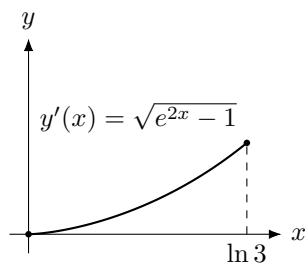
$$L = \int_0^2 \sqrt{1 + \frac{1}{2x}} dx.$$

Question 10

A curve satisfies

$$y'(x) = \sqrt{e^{2x} - 1}, \quad 0 \leq x \leq \ln 3.$$

Find its length. A formula for y is not needed.



Question 10 – Solution

Step 1: Substitute the derivative into the arc-length formula.

$$\begin{aligned}1 + (y')^2 &= 1 + \left(\sqrt{e^{2x} - 1}\right)^2 \\&= 1 + e^{2x} - 1 = e^{2x}.\end{aligned}$$

Step 2: Simplify the square root. Since $e^x > 0$,

$$\sqrt{1 + (y')^2} = \sqrt{e^{2x}} = e^x.$$

Step 3: Integrate and evaluate.

$$\begin{aligned}L &= \int_0^{\ln 3} e^x \, dx \\&= [e^x]_0^{\ln 3} \\&= e^{\ln 3} - e^0 = 3 - 1 = 2.\end{aligned}$$

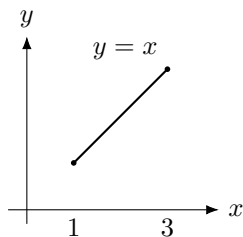
$$\boxed{L = 2}$$

Question 1

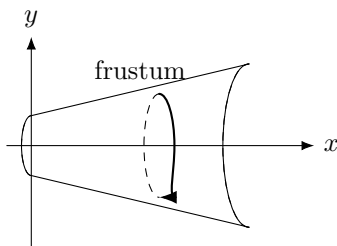
Curve: $y = x$ on $1 \leq x \leq 3$.

Axis of rotation: the x -axis.

Task: Find the lateral surface area, then verify the result with the frustum formula.



Question 1 – Solution



Step 1: Differentiate and find ds .

$$y' = 1, \quad ds = \sqrt{1 + (y')^2} dx = \sqrt{2} dx.$$

Step 2: Apply the surface-area formula.

$$\begin{aligned} S &= 2\pi \int_1^3 y ds \\ &= 2\pi \int_1^3 x\sqrt{2} dx \\ &= 2\pi\sqrt{2} \left[\frac{x^2}{2} \right]_1^3 \\ &= \pi\sqrt{2}(9 - 1) = 8\pi\sqrt{2}. \end{aligned}$$

Step 3: Verify with the frustum formula. The radii are $r_1 = 1$ and $r_2 = 3$. The slant height is

$$s = \sqrt{(3 - 1)^2 + (3 - 1)^2} = 2\sqrt{2}.$$

Therefore,

$$S = \pi(r_1 + r_2)s = \pi(1 + 3)(2\sqrt{2}) = 8\pi\sqrt{2}.$$

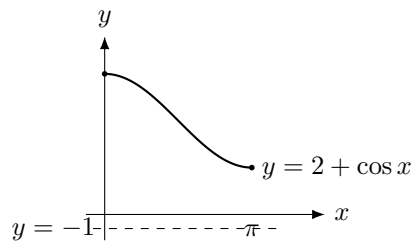
$$\boxed{S = 8\pi\sqrt{2}}$$

Question 2

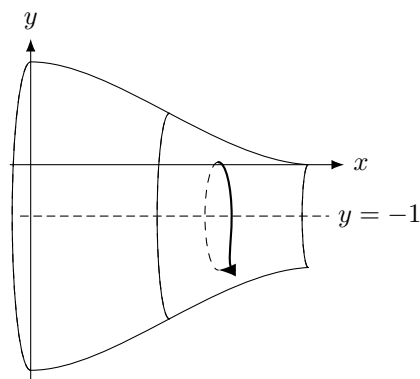
Curve: $y = 2 + \cos x$ on $0 \leq x \leq \pi$.

Axis of rotation: $y = -1$.

Task: Set up, but do not evaluate, the surface-area integral.



Question 2 – Solution



Step 1: Find the radius. The radius is the vertical distance from the curve to the axis $y = -1$:

$$r = (2 + \cos x) - (-1) = 3 + \cos x.$$

Step 2: Find the arc-length factor.

$$y' = -\sin x, \quad ds = \sqrt{1 + \sin^2 x} \, dx.$$

Step 3: Substitute into $S = 2\pi \int r \, ds$.

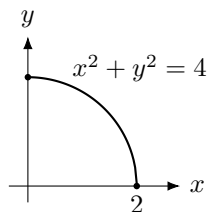
$$S = 2\pi \int_0^\pi (3 + \cos x) \sqrt{1 + \sin^2 x} \, dx$$

Question 3

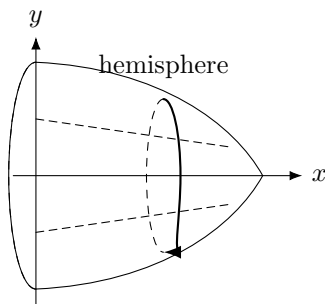
Curve: $y = \sqrt{4 - x^2}$ on $0 \leq x \leq 2$.

Axis of rotation: the x -axis.

Task: Identify the resulting surface and find its area.



Question 3 – Solution



Step 1: Identify the surface. The curve is the upper-right quarter of the circle $x^2 + y^2 = 4$. Rotating it about the x -axis produces a hemisphere of radius (2).

Step 2: Differentiate.

$$y' = -\frac{x}{\sqrt{4-x^2}}.$$

Step 3: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1 + \frac{x^2}{4-x^2}} \\ &= \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}.\end{aligned}$$

Step 4: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_0^2 y \sqrt{1+(y')^2} \, dx \\ &= 2\pi \int_0^2 \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} \, dx \\ &= 4\pi \int_0^2 dx = 8\pi.\end{aligned}$$

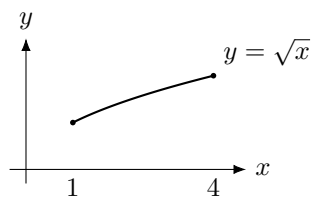
hemisphere, $S = 8\pi$

Question 4

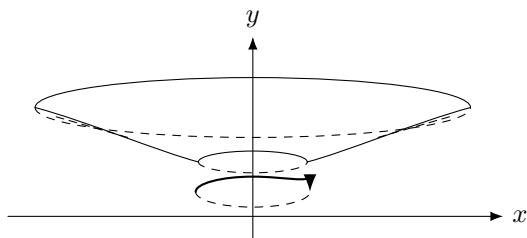
Curve: $y = \sqrt{x}$ on $1 \leq x \leq 4$.

Axis of rotation: the y -axis.

Task: Find the exact surface area.



Question 4 – Solution



Step 1: Find the radius and derivative. Rotation about the y -axis gives radius $r = x$, and

$$y' = \frac{1}{2\sqrt{x}}.$$

Step 2: Set up and simplify.

$$\begin{aligned} S &= 2\pi \int_1^4 x \sqrt{1 + \frac{1}{4x}} \, dx \\ &= \pi \int_1^4 \sqrt{x(4x+1)} \, dx. \end{aligned}$$

Step 3: Substitute $u = 8x + 1$. Then

$$dx = \frac{du}{8}, \quad \sqrt{x(4x+1)} = \frac{1}{4} \sqrt{u^2 - 1}.$$

Thus

$$S = \frac{\pi}{32} \int_9^{33} \sqrt{u^2 - 1} \, du.$$

Step 4: Integrate and return to x .

$$\int \sqrt{u^2 - 1} \, du = \frac{1}{2} \left(u \sqrt{u^2 - 1} - \ln(u + \sqrt{u^2 - 1}) \right).$$

Therefore,

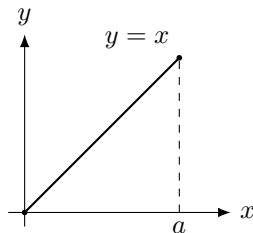
$$S = \pi \left[\frac{(8x+1)\sqrt{x(4x+1)}}{16} - \frac{1}{64} \ln \left(8x+1 + 4\sqrt{x(4x+1)} \right) \right]_1^4$$

Question 5

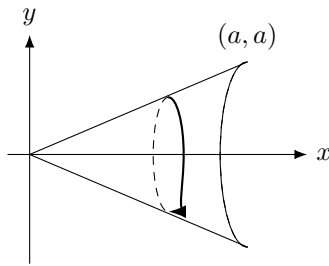
Curve: $y = x$ on $0 \leq x \leq a$, where $a > 0$.

Axis of rotation: the x -axis.

Task: Find a so that the generated surface area is $9\pi\sqrt{2}$.



Question 5 – Solution



Step 1: Find the arc-length factor.

$$y' = 1, \quad \sqrt{1 + (y')^2} = \sqrt{2}.$$

Step 2: Write and evaluate the area.

$$\begin{aligned} S &= 2\pi \int_0^a y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_0^a x \sqrt{2} dx \\ &= 2\pi \sqrt{2} \left[\frac{x^2}{2} \right]_0^a = \pi \sqrt{2} a^2. \end{aligned}$$

Step 3: Set the area equal to $9\pi\sqrt{2}$ and solve.

$$\begin{aligned} \pi \sqrt{2} a^2 &= 9\pi \sqrt{2}, \\ a^2 &= 9, \\ a &= 3 \end{aligned}$$

because $a > 0$.

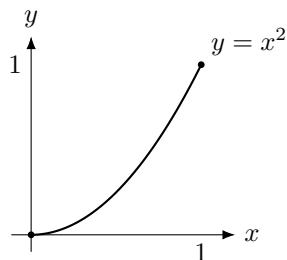
$$\boxed{a = 3}$$

Question 6

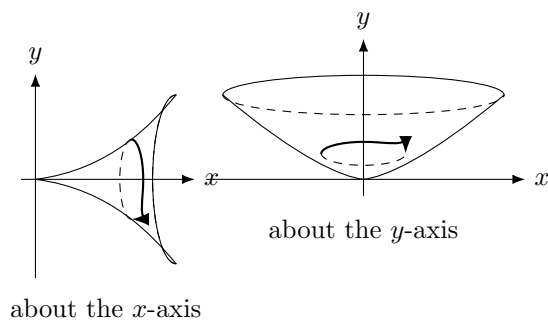
Curve: $y = x^2$ on $0 \leq x \leq 1$.

Axes of rotation: first the x -axis, then the y -axis.

Task: Write both surface-area integrals and determine which is larger without evaluating either integral.



Question 6 – Solution



Step 1: Find the common arc-length factor.

$$y' = 2x, \quad ds = \sqrt{1 + 4x^2} \, dx.$$

Step 2: Rotate about the x -axis. The radius is $y = x^2$:

$$S_x = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx.$$

Step 3: Rotate about the y -axis. The radius is x :

$$S_y = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx.$$

Step 4: Compare the integrands. For $0 < x < 1$,

$$x > x^2.$$

Multiplying by the positive common factor $2\pi\sqrt{1 + 4x^2}$ preserves the inequality. Therefore,

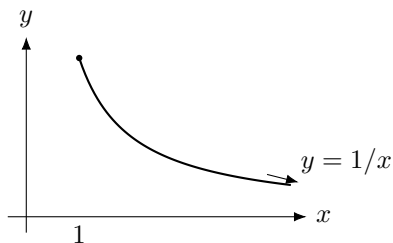
$$\boxed{S_y > S_x}.$$

Question 7

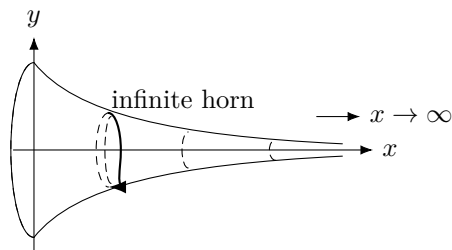
Curve: $y = 1/x$ on $x \geq 1$.

Axis of rotation: the x -axis.

Task: Determine whether the generated surface has finite area.



Question 7 – Solution



Step 1: Differentiate and set up the improper integral.

$$y' = -\frac{1}{x^2}.$$

Thus

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Step 2: Find a lower comparison. Since

$$\sqrt{1 + x^{-4}} > 1,$$

we have

$$\frac{1}{x} \sqrt{1 + x^{-4}} > \frac{1}{x}.$$

Step 3: Test the comparison integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

By direct comparison, the surface-area integral diverges.

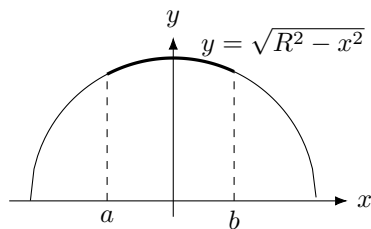
$S = \infty$ (infinite surface area)

Question 8

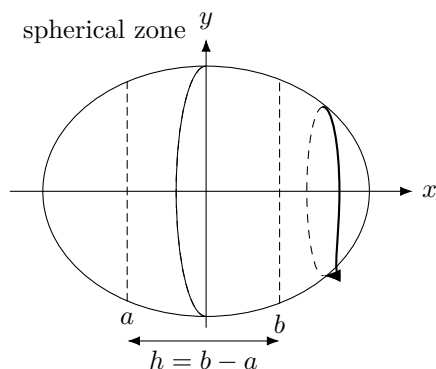
Curve: $y = \sqrt{R^2 - x^2}$ on $a \leq x \leq b$, where $R > 0$ and $-R \leq a < b \leq R$.

Axis of rotation: the x -axis.

Task: For fixed R , show that the surface area depends only on the zone height $h = b - a$.



Question 8 – Solution



Step 1: Differentiate.

$$y' = -\frac{x}{\sqrt{R^2 - x^2}}.$$

Step 2: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{R^2 - x^2}} \\ &= \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.\end{aligned}$$

Step 3: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_a^b y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_a^b \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}} dx \\ &= 2\pi R \int_a^b dx \\ &= 2\pi R(b - a) = 2\pi Rh.\end{aligned}$$

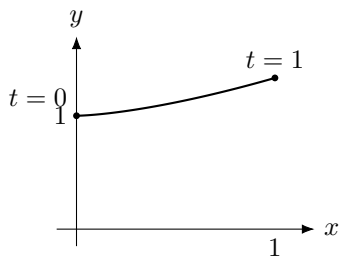
$$\boxed{S = 2\pi R(b - a) = 2\pi Rh}$$

Question 9

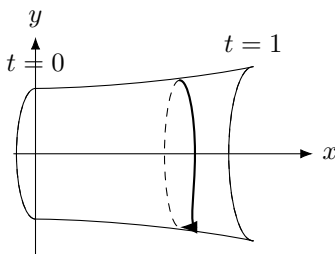
Parametric curve: $x = t^2$, $y = 1 + t^3/3$, for $0 \leq t \leq 1$.

Axis of rotation: the x -axis.

Task: Set up, but do not evaluate, the surface-area integral.



Question 9 – Solution



Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2.$$

Step 2: Find the parametric arc-length factor.

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \sqrt{4t^2 + t^4} dt \\ &= t\sqrt{4 + t^2} dt \end{aligned}$$

because $t \geq 0$.

Step 3: Use radius $y = 1 + t^3/3$.

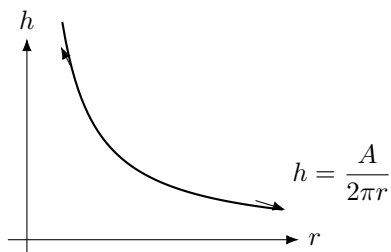
$$\begin{aligned} S &= 2\pi \int_0^1 y \, ds \\ &= 2\pi \int_0^1 \left(1 + \frac{t^3}{3}\right) t\sqrt{4 + t^2} \, dt \\ &= 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt. \end{aligned}$$

$$\boxed{S = 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt}$$

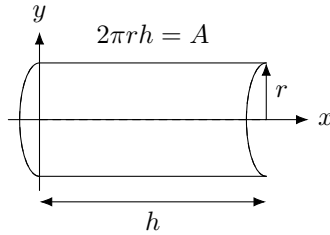
Question 10

Given: A cylinder has radius r , height h , and fixed lateral area A .

Tasks: Solve for h , find the total area including both circular ends, and then analyze the limit as $r \rightarrow 0^+$.



Question 10 – Solution



Step 1: Solve the lateral-area equation for h .

$$2\pi r h = A,$$

$$h = \frac{A}{2\pi r}.$$

Step 2: Add the two circular ends. Each end has area πr^2 , so

$$S_{\text{total}} = A + 2\pi r^2.$$

Step 3: Take the limit as $r \rightarrow 0^+$.

$$\lim_{r \rightarrow 0^+} S_{\text{total}} = \lim_{r \rightarrow 0^+} (A + 2\pi r^2) = A,$$

$$\lim_{r \rightarrow 0^+} h = \lim_{r \rightarrow 0^+} \frac{A}{2\pi r} = \infty.$$

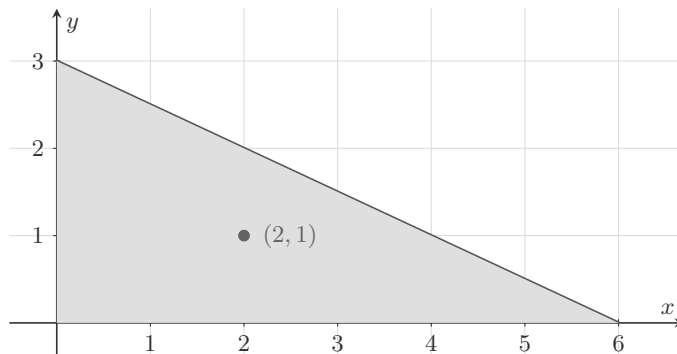
$$\boxed{S_{\text{total}} \rightarrow A, \quad h \rightarrow \infty}$$

Question 1

A triangular sign occupies $x \geq 0, y \geq 0, x + 2y \leq 6$. Find its centroid using integrals.



Question 1 – Solution



Step 1: Write the upper boundary as a function of x . From $x + 2y = 6$,

$$y = f(x) = 3 - \frac{x}{2}, \quad 0 \leq x \leq 6.$$

Step 2: Find the area (and hence the mass for unit density).

$$\begin{aligned} A &= \int_0^6 f(x) dx = \int_0^6 \left(3 - \frac{x}{2}\right) dx \\ &= \left[3x - \frac{x^2}{4}\right]_0^6 = 18 - 9 = 9. \end{aligned}$$

Step 3: Find the moment about the y -axis.

$$\begin{aligned} M_y &= \int_0^6 x f(x) dx = \int_0^6 \left(3x - \frac{x^2}{2}\right) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{6}\right]_0^6 = 54 - 36 = 18. \end{aligned}$$

Step 4: Find the moment about the x -axis. Each vertical strip has centroid height $f(x)/2$, so

$$M_x = \frac{1}{2} \int_0^6 f(x)^2 dx = \frac{1}{2} \int_0^6 \left(3 - \frac{x}{2}\right)^2 dx = 9.$$

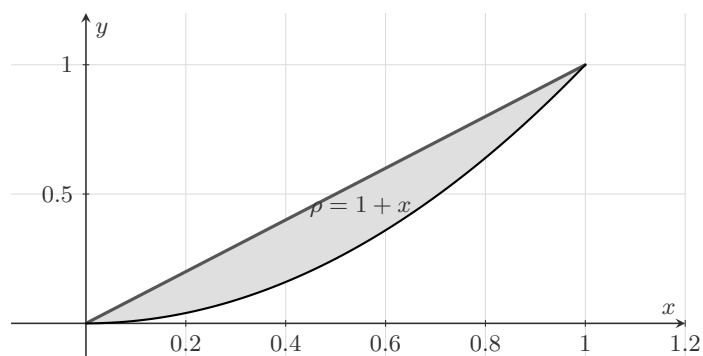
Step 5: Divide each moment by the area.

$$\bar{x} = \frac{M_y}{A} = \frac{18}{9} = 2, \quad \bar{y} = \frac{M_x}{A} = \frac{9}{9} = 1.$$

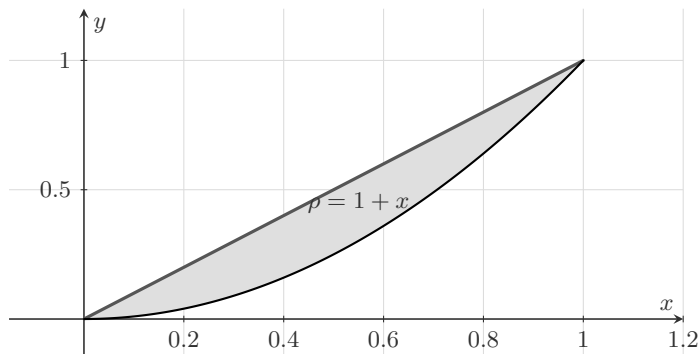
$$\boxed{(\bar{x}, \bar{y}) = (2, 1)}$$

Question 2

A lamina lies between $y = x$ and $y = x^2$, $0 \leq x \leq 1$, with density $\rho(x) = 1 + x$. Find its mass and $\bar{x}$.



Question 2 – Solution



Step 1: Identify the strip dimensions. On $0 \leq x \leq 1$, the upper curve is $y = x$ and the lower curve is $y = x^2$. Thus

$$dA = (x - x^2) dx, \quad dm = \rho(x) dA = (1 + x)(x - x^2) dx.$$

Step 2: Find the mass. Since $(1 + x)(x - x^2) = x - x^3$,

$$m = \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}.$$

Step 3: Find the moment about the y -axis. The moment arm is x , so

$$\begin{aligned} M_y &= \int_0^1 x dm = \int_0^1 x(x - x^3) dx \\ &= \int_0^1 (x^2 - x^4) dx = \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{2}{15}. \end{aligned}$$

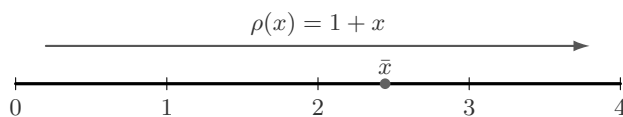
Step 4: Compute the x -coordinate of the center of mass.

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{2}{15}}{\frac{1}{4}} = \frac{8}{15}.$$

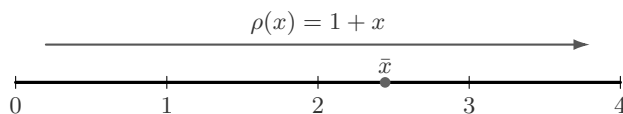
$m = \frac{1}{4}, \quad \bar{x} = \frac{8}{15}$

Question 3

A 4-m rod on $0 \leq x \leq 4$ has density $\rho(x) = k(1 + x)$. Its mass is (12) kg. Find k and its balance point.



Question 3 – Solution



Step 1: Use the total mass to determine k .

$$\begin{aligned} 12 &= \int_0^4 \rho(x) dx = k \int_0^4 (1 + x) dx \\ &= k \left[x + \frac{x^2}{2} \right]_0^4 = k(4 + 8) = 12k. \end{aligned}$$

Therefore, $k = 1$, so $\rho(x) = 1 + x$.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_0 &= \int_0^4 x\rho(x) dx = \int_0^4 (x + x^2) dx \\ &= \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_0^4 = 8 + \frac{64}{3} = \frac{88}{3}. \end{aligned}$$

Step 3: Divide the moment by the mass.

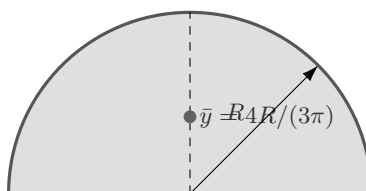
$$\bar{x} = \frac{M_0}{m} = \frac{\frac{88}{3}}{12} = \frac{22}{9} \text{ m}.$$

The balance point lies to the right of the midpoint because the density increases with x .

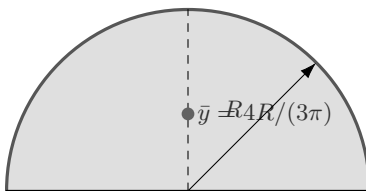
$k = 1, \quad \bar{x} = \frac{22}{9} \text{ m}$

Question 4

A uniform semicircular lamina $x^2 + y^2 \leq R^2, y \geq 0$ has $\bar{x} = 0$. Derive $\bar{y}$.



Question 4 – Solution



Step 1: Use symmetry. The lamina is symmetric about the y -axis, so

$$\bar{x} = 0.$$

Step 2: Find the area. The region is one-half of a disk of radius R :

$$A = \frac{1}{2}\pi R^2.$$

Step 3: Write the upper boundary. For $-R \leq x \leq R$,

$$f(x) = \sqrt{R^2 - x^2}.$$

A vertical strip of height $f(x)$ has centroid height $f(x)/2$.

Step 4: Find the moment about the x -axis.

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-R}^R f(x)^2 dx = \frac{1}{2} \int_{-R}^R (R^2 - x^2) dx \\ &= \frac{1}{2} \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R = \frac{2R^3}{3}. \end{aligned}$$

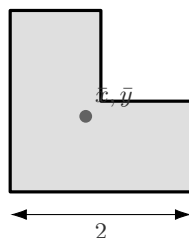
Step 5: Divide the moment by the area.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2} = \frac{4R}{3\pi}.$$

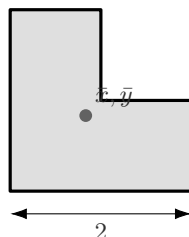
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4R}{3\pi}\right)}$$

Question 5

A uniform square $0 \leq x, y \leq 2$ has the upper-right unit square removed. Find the centroid of the remaining L-shape.



Question 5 – Solution



Step 1: Treat the removed square as negative area. The full 2×2 square has

$$A_1 = 4, \quad (x_1, y_1) = (1, 1).$$

The removed 1×1 square has

$$A_2 = 1, \quad (x_2, y_2) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Step 2: Find the remaining area.

$$A = A_1 - A_2 = 4 - 1 = 3.$$

Step 3: Subtract the moments about the y -axis.

$$M_y = A_1x_1 - A_2x_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

Step 4: Subtract the moments about the x -axis.

$$M_x = A_1y_1 - A_2y_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

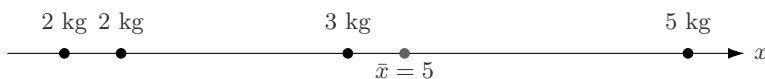
Step 5: Divide by the remaining area.

$$\bar{x} = \frac{M_y}{A} = \frac{5}{6}, \quad \bar{y} = \frac{M_x}{A} = \frac{5}{6}.$$

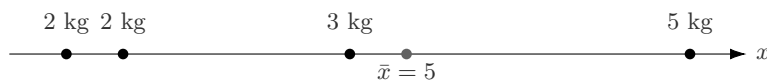
$$(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{6}\right)$$

Question 6

Masses (2,3,5) kg sit at $x = 0, 4, 10$. Where must a (2)-kg mass be placed so the center is $x = 5$?



Question 6 – Solution



Step 1: Let a be the location of the new mass. The four masses are 2, 3, 5, and 2 kg, so the total mass is

$$m = 2 + 3 + 5 + 2 = 12 \text{ kg.}$$

Step 2: Write the center-of-mass equation. For point masses on a line,

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}.$$

Requiring $\bar{x} = 5$ gives

$$5 = \frac{2(0) + 3(4) + 5(10) + 2a}{12}.$$

Step 3: Solve for a .

$$60 = 0 + 12 + 50 + 2a,$$

$$60 = 62 + 2a,$$

$$2a = -2,$$

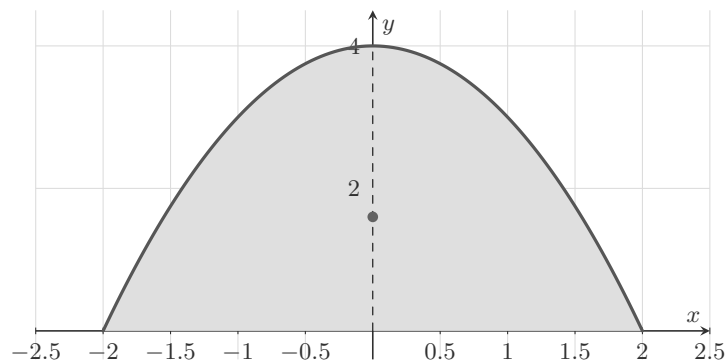
$$a = -1.$$

Step 4: Check. The total moment is $62 + 2(-1) = 60$, and $60/12 = 5$, as required.

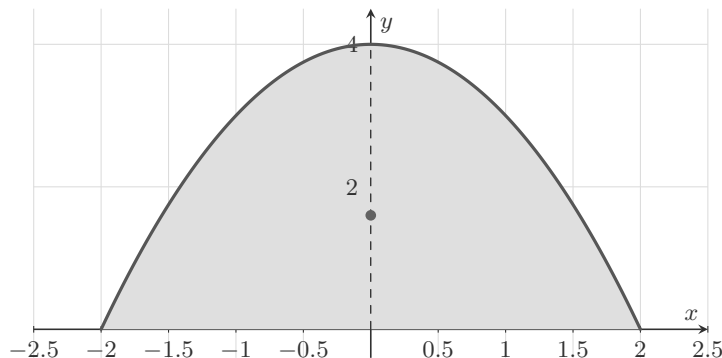
Place the 2-kg mass at $x = -1$.

Question 7

A plate under $y = 4 - x^2$, $-2 \leq x \leq 2$, has uniform density. Use symmetry before integration; find its centroid.



Question 7 – Solution



Step 1: Use symmetry before integrating. Both the region and its uniform density are symmetric about the y -axis. Therefore,

$$\bar{x} = 0.$$

Step 2: Find the area.

$$\begin{aligned} A &= \int_{-2}^2 (4 - x^2) dx = 2 \int_0^2 (4 - x^2) dx \\ &= 2 \left[4x - \frac{x^3}{3} \right]_0^2 = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}. \end{aligned}$$

Step 3: Find the moment about the x -axis. For a vertical strip under $f(x) = 4 - x^2$,

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-2}^2 f(x)^2 dx = \int_0^2 (4 - x^2)^2 dx \\ &= \int_0^2 (16 - 8x^2 + x^4) dx \\ &= \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2 = \frac{256}{15}. \end{aligned}$$

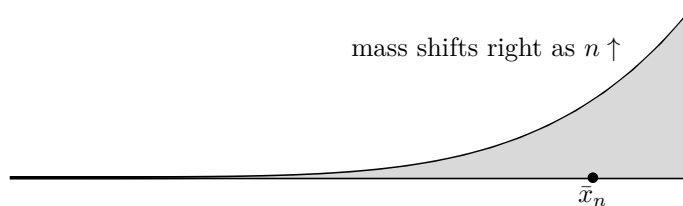
Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{256}{15}}{\frac{32}{3}} = \frac{8}{5}.$$

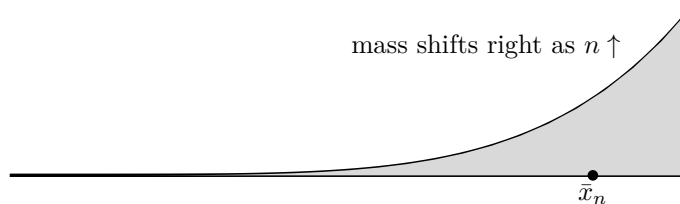
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)}$$

Question 8

A rod on $[0, 1]$ has density $\rho_n(x) = (n+1)x^n$, for integers $n \geq 0$. Find $\bar{x}_n$ and its limiting location as $n \rightarrow \infty$.



Question 8 – Solution



Step 1: Find the total mass. For $n \geq 0$,

$$\begin{aligned} m_n &= \int_0^1 \rho_n(x) dx = \int_0^1 (n+1)x^n dx \\ &= (n+1) \left[\frac{x^{n+1}}{n+1} \right]_0^1 = 1. \end{aligned}$$

Thus the density is normalized to have total mass 1.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_n &= \int_0^1 x \rho_n(x) dx = (n+1) \int_0^1 x^{n+1} dx \\ &= (n+1) \left[\frac{x^{n+2}}{n+2} \right]_0^1 = \frac{n+1}{n+2}. \end{aligned}$$

Step 3: Compute the center of mass.

$$\bar{x}_n = \frac{M_n}{m_n} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2}.$$

Step 4: Take the limit.

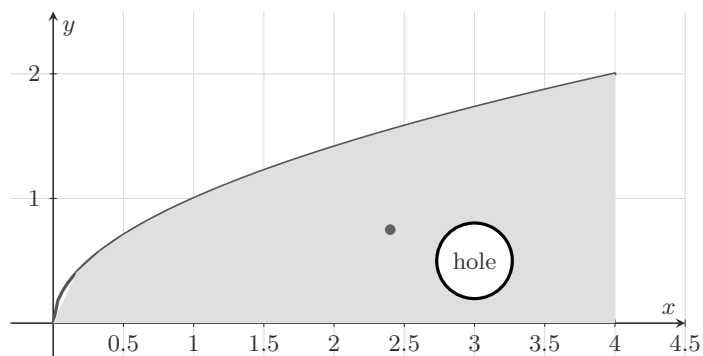
$$\lim_{n \rightarrow \infty} \bar{x}_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+2} = 1.$$

The density becomes increasingly concentrated near the right endpoint.

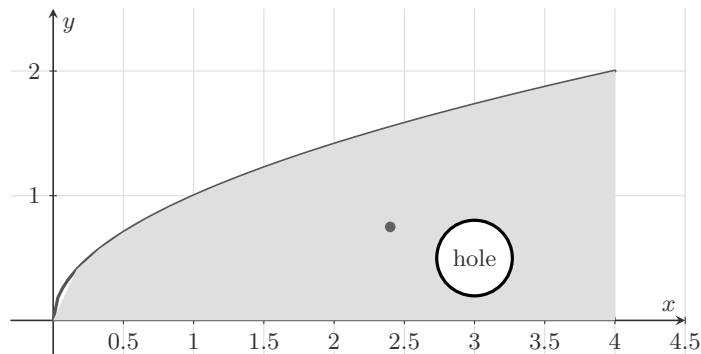
$$\bar{x}_n = \frac{n+1}{n+2}, \quad \bar{x}_n \rightarrow 1$$

Question 9

A lamina is bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$. A circular hole of radius $(1/2)$, centered at $((3, 1/2))$, is removed. Find the new centroid using subtraction.



Question 9 – Solution



Step 1: Find the area of the original region.

$$A_0 = \int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{16}{3}.$$

Step 2: Find the original moments.

$$M_{y,0} = \int_0^4 x\sqrt{x} \, dx = \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{64}{5},$$

$$M_{x,0} = \frac{1}{2} \int_0^4 (\sqrt{x})^2 \, dx = \frac{1}{2} \int_0^4 x \, dx = 4.$$

Hence the original centroid is

$$(\bar{x}_0, \bar{y}_0) = \left(\frac{M_{y,0}}{A_0}, \frac{M_{x,0}}{A_0} \right) = \left(\frac{12}{5}, \frac{3}{4} \right).$$

Step 3: Record the hole's area and moments. The hole has radius $1/2$ and centroid $(3, 1/2)$, so

$$A_h = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4}, \quad M_{y,h} = 3 \left(\frac{\pi}{4} \right) = \frac{3\pi}{4}, \quad M_{x,h} = \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}.$$

Step 4: Subtract the hole.

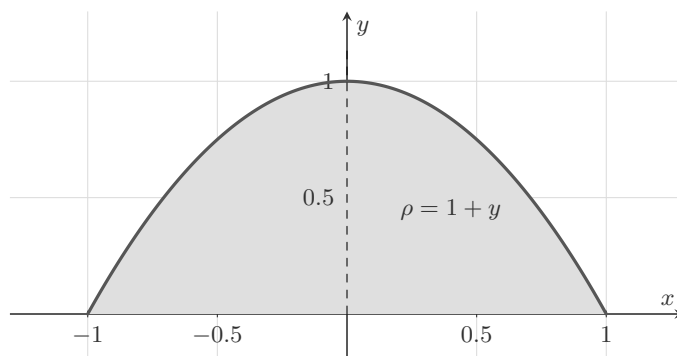
$$A = \frac{16}{3} - \frac{\pi}{4}, \quad M_y = \frac{64}{5} - \frac{3\pi}{4}, \quad M_x = 4 - \frac{\pi}{8}.$$

Step 5: Divide the remaining moments by the remaining area.

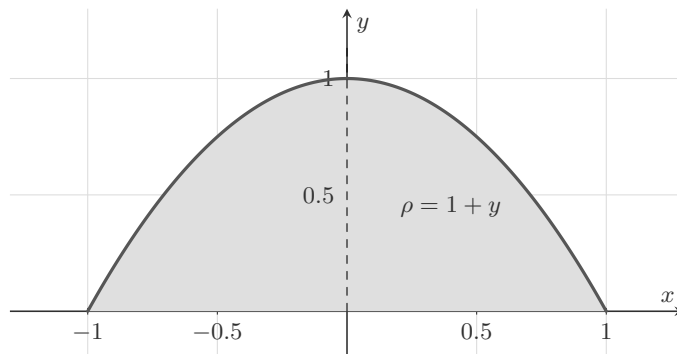
$$\boxed{\bar{x} = \frac{\frac{64}{5} - \frac{3\pi}{4}}{\frac{16}{3} - \frac{\pi}{4}}, \quad \bar{y} = \frac{4 - \frac{\pi}{8}}{\frac{16}{3} - \frac{\pi}{4}}}$$

Question 10

A symmetric lamina has density $\rho(x, y) = 1 + y$ on $-1 \leq x \leq 1, 0 \leq y \leq 1 - x^2$. State $\bar{x}$, then set up exact integrals for m and $\bar{y}$.



Question 10 – Solution



Step 1: Use symmetry. The region is symmetric about the y -axis, and $\rho(x, y) = 1 + y$ is unchanged when x is replaced by $-x$. Therefore,

$$\boxed{\bar{x} = 0}.$$

Step 2: Set up and evaluate the mass. Let $h = 1 - x^2$. Then

$$\begin{aligned} m &= \int_{-1}^1 \int_0^{1-x^2} (1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[h + \frac{h^2}{2} \right] dx \\ &= \int_{-1}^1 \left[(1-x^2) + \frac{1}{2}(1-x^2)^2 \right] dx = \frac{28}{15}. \end{aligned}$$

Step 3: Set up and evaluate the moment about the x -axis.

$$\begin{aligned} M_x &= \int_{-1}^1 \int_0^{1-x^2} y(1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[\frac{h^2}{2} + \frac{h^3}{3} \right] dx \\ &= \frac{1}{2} \int_{-1}^1 (1-x^2)^2 dx + \frac{1}{3} \int_{-1}^1 (1-x^2)^3 dx \\ &= \frac{8}{15} + \frac{32}{105} = \frac{88}{105}. \end{aligned}$$

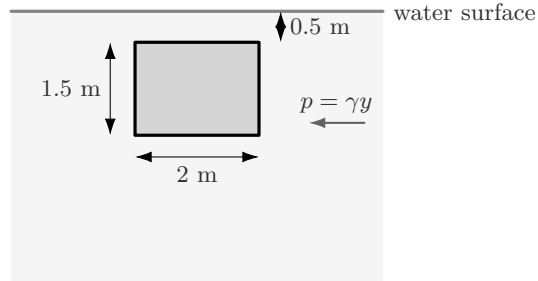
Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{88}{105}}{\frac{28}{15}} = \frac{22}{49}.$$

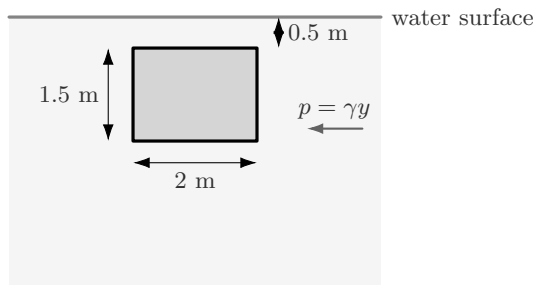
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{22}{49} \right)}$$

Question 1

A vertical aquarium window is (2) m wide and (1.5) m high; its top is (0.5) m below freshwater. Find the force ($\gamma = 9800 \text{ N/m}^3$).



Question 1 – Solution



Step 1: Choose the depth variable. Let y be the depth below the water surface, measured in meters. The window extends from

$$y = 0.5 \quad \text{to} \quad y = 0.5 + 1.5 = 2.$$

Step 2: Write the pressure at depth y . Freshwater has weight density $\gamma = 9800 \text{ N/m}^3$, so

$$p(y) = \gamma y = 9800y \quad \text{N/m}^2.$$

Step 3: Find the area of a horizontal strip. The window is 2 m wide. A strip of thickness dy has area

$$dA = 2 \, dy.$$

Therefore,

$$dF = p(y) \, dA = (9800y)(2 \, dy).$$

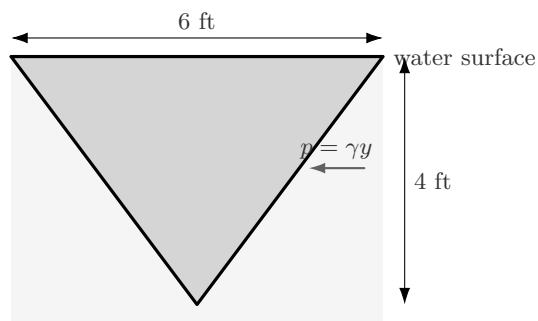
Step 4: Integrate over the window.

$$\begin{aligned} F &= 9800 \int_{0.5}^2 2y \, dy = 9800 [y^2]_{0.5}^2 \\ &= 9800(4 - 0.25) = 9800(3.75) = 36,750 \text{ N}. \end{aligned}$$

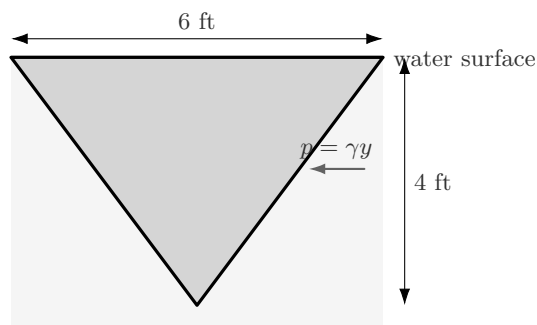
$$\boxed{F = 36,750 \text{ N}}$$

Question 2

An inverted triangular gate has top width (6) ft at the surface and vertex (4) ft deep. Find the force ($\gamma = 62.4 \text{ lb/ft}^3$).



Question 2 – Solution



Step 1: Choose the depth variable. Let y be depth below the water surface. The gate extends from $y = 0$ to $y = 4$ ft.

Step 2: Find the width of the gate at depth y . The width decreases linearly from 6 ft at $y = 0$ to 0 ft at $y = 4$. Its slope is

$$\frac{0 - 6}{4 - 0} = -\frac{3}{2},$$

so

$$w(y) = 6 - \frac{3}{2}y.$$

Step 3: Write the force on a horizontal strip. At depth y ,

$$p(y) = 62.4y \text{ lb/ft}^2, \quad dA = w(y) dy.$$

Thus

$$dF = 62.4y \left(6 - \frac{3}{2}y \right) dy.$$

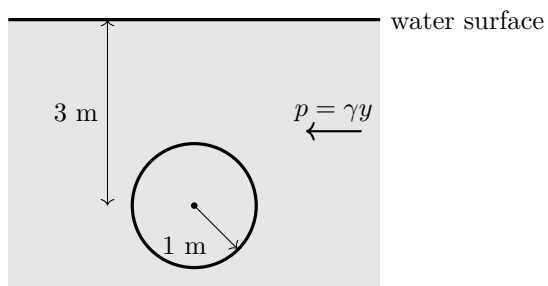
Step 4: Integrate and evaluate.

$$\begin{aligned} F &= 62.4 \int_0^4 y \left(6 - \frac{3}{2}y \right) dy \\ &= 62.4 \int_0^4 \left(6y - \frac{3}{2}y^2 \right) dy \\ &= 62.4 \left[3y^2 - \frac{1}{2}y^3 \right]_0^4 \\ &= 62.4(48 - 32) = 62.4(16) = 998.4 \text{ lb.} \end{aligned}$$

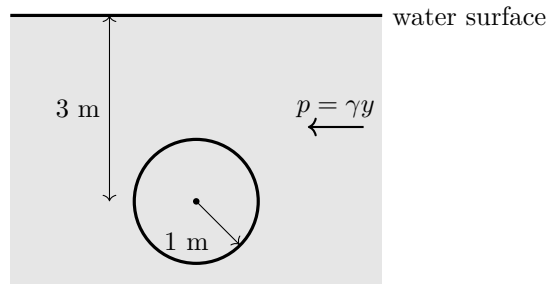
$$\boxed{F = 998.4 \text{ lb}}$$

Question 3

A vertical circular porthole of radius 1 m has center 3 m below the water surface. Use water weight density 9800 N/m^3 . Find the force without integrating strips.



Question 3 – Solution



Step 1: Use the centroid form of the hydrostatic-force formula. For a submerged vertical plane region,

$$F = \gamma A \bar{y},$$

where A is its area and $\bar{y}$ is the depth of its centroid.

Step 2: Find the area of the porthole. The radius is 1 m, so

$$A = \pi r^2 = \pi(1)^2 = \pi \text{ m}^2.$$

Step 3: Identify the centroid depth. The centroid of a circle is its center, which is 3 m below the surface. Therefore,

$$\bar{y} = 3 \text{ m}.$$

Step 4: Substitute.

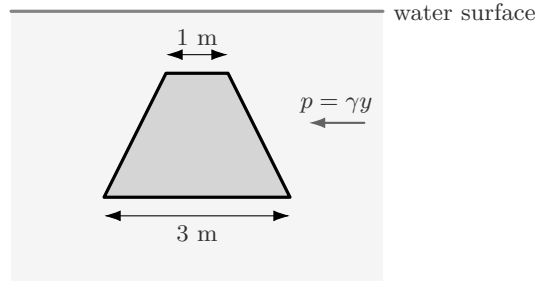
$$\begin{aligned} F &= (9800)(\pi)(3) \\ &= 29,400\pi \text{ N}. \end{aligned}$$

$$\boxed{F = 29,400\pi \text{ N}}$$

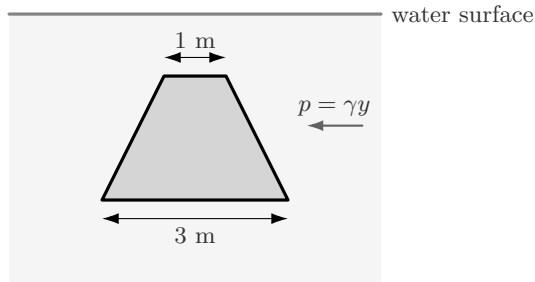
No strip integration is needed because the area and centroid depth are known.

Question 4

Oil has weight density 8000 N/m^3 . A trapezoidal plate is (1) m wide at its top, (3) m wide at its bottom, (2) m high, with top depth (1) m. Find the force.



Question 4 – Solution



Step 1: Choose the depth variable. Let y be depth below the oil surface. The plate begins at $y = 1$ and ends at $y = 3$ m.

Step 2: Find the width at depth y . The width grows linearly from 1 m at $y = 1$ to 3 m at $y = 3$. The slope is

$$\frac{3 - 1}{3 - 1} = 1,$$

so the linear width function is

$$w(y) = y, \quad 1 \leq y \leq 3.$$

Step 3: Write the force on a horizontal strip.

$$p(y) = 8000y, \quad dA = w(y) dy = y dy.$$

Hence

$$dF = p(y) dA = 8000y^2 dy.$$

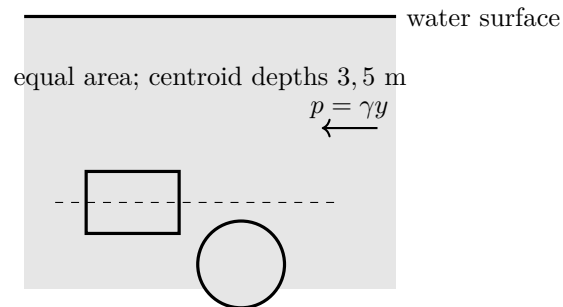
Step 4: Integrate and evaluate.

$$\begin{aligned} F &= 8000 \int_1^3 y^2 dy = 8000 \left[\frac{y^3}{3} \right]_1^3 \\ &= \frac{8000}{3} (27 - 1) = \frac{208000}{3} \text{ N}. \end{aligned}$$

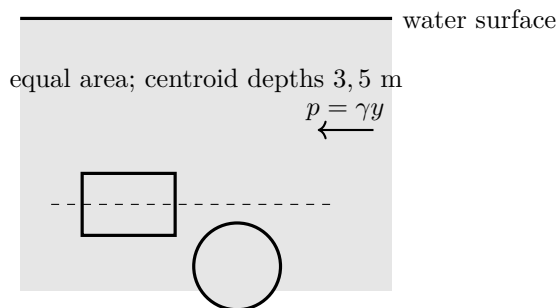
$$F = \frac{208000}{3} \text{ N} \approx 69,333.3 \text{ N}$$

Question 5

Two equal-area vertical plates are fully submerged in the same liquid of constant weight density, with centroid depths 3 m and 5 m. Compare their hydrostatic forces without knowing their shapes.



Question 5 – Solution



Step 1: Use the centroid formula. For any vertical plane region submerged in a liquid of constant weight density,

$$F = \gamma A \bar{y}.$$

Step 2: Write the force on each plate. Let the common area be A . If Plate 1 has centroid depth 3 m and Plate 2 has centroid depth 5 m, then

$$F_1 = \gamma A(3), \quad F_2 = \gamma A(5).$$

Step 3: Compare the forces.

$$\frac{F_2}{F_1} = \frac{\gamma A(5)}{\gamma A(3)} = \frac{5}{3}.$$

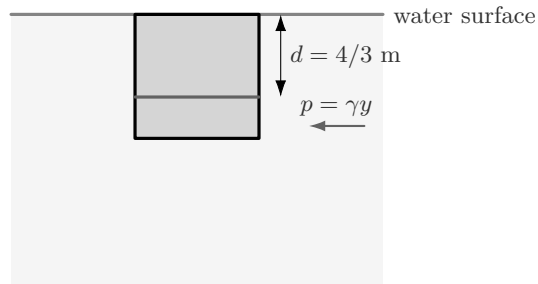
The shape does not affect this comparison because both plates have the same area; only the centroid depth differs.

$$F_2 = \frac{5}{3} F_1$$

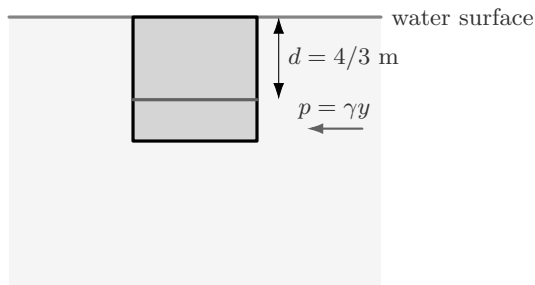
Thus the deeper plate experiences $66\frac{2}{3}\%$ more force.

Question 6

A 2×2 m square gate has its top at the surface. Find the depth d of the center of pressure, where $d = M/F$.



Question 6 – Solution



Step 1: Set up the hydrostatic force. Let y be depth below the surface. The gate has constant width 2 m and extends from $y = 0$ to $y = 2$. Thus

$$dF = (9800y)(2 dy).$$

Therefore,

$$\begin{aligned} F &= 9800 \int_0^2 2y dy = 19600 \left[\frac{y^2}{2} \right]_0^2 \\ &= 39,200 \text{ N.} \end{aligned}$$

Step 2: Find the moment of the pressure force about the surface. The moment arm of the strip force is y , so

$$\begin{aligned} M &= \int_0^2 y dF = 9800 \int_0^2 2y^2 dy \\ &= 19600 \left[\frac{y^3}{3} \right]_0^2 = \frac{156,800}{3} \text{ N m.} \end{aligned}$$

Step 3: Divide moment by total force.

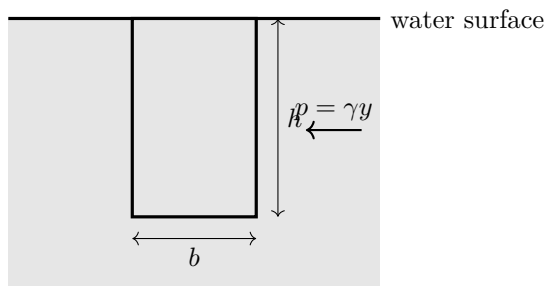
$$d = \frac{M}{F} = \frac{\frac{156800}{3}}{39200} = \frac{4}{3} \text{ m.}$$

$$d = \frac{4}{3} \text{ m below the surface}$$

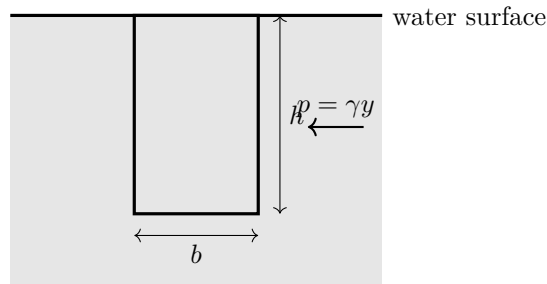
The center of pressure lies below the geometric center at depth 1 m because pressure increases with depth.

Question 7

A rectangular dam face is b meters wide and extends from the surface to depth h . Derive $F(h)$. If h doubles, by what factor does F change?



Question 7 – Solution



Step 1: Choose the depth variable. Let y measure depth below the surface. The dam face extends from $y = 0$ to $y = h$.

Step 2: Write the force on a horizontal strip. The pressure is $p(y) = \gamma y$, and a strip of width b and thickness dy has area $dA = b dy$. Therefore,

$$dF = p(y) dA = \gamma b y dy.$$

Step 3: Integrate from the surface to depth h .

$$\begin{aligned} F(h) &= \gamma b \int_0^h y dy \\ &= \gamma b \left[\frac{y^2}{2} \right]_0^h = \frac{1}{2} \gamma b h^2. \end{aligned}$$

$$F(h) = \frac{1}{2} \gamma b h^2$$

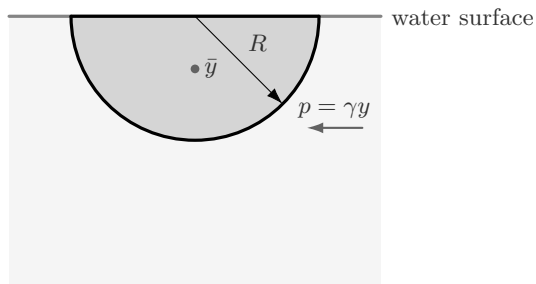
Step 4: Determine the effect of doubling the depth.

$$F(2h) = \frac{1}{2} \gamma b (2h)^2 = 4 \left(\frac{1}{2} \gamma b h^2 \right) = 4F(h).$$

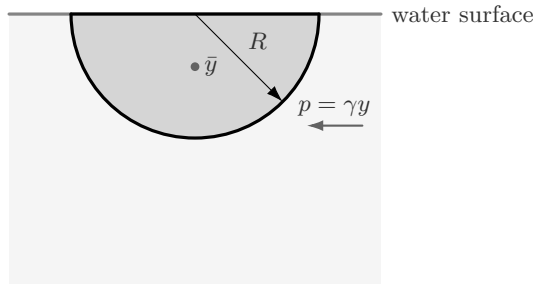
Doubling h multiplies the force by 4.

Question 8

A semicircular gate of radius R has its diameter along the water surface and lies below it. Find the force using centroid depth $4R/(3\pi)$.



Question 8 – Solution



Step 1: Use the centroid form of hydrostatic force. For a submerged vertical plane region,

$$F = \gamma A \bar{y}.$$

Step 2: Find the area of the semicircle.

$$A = \frac{1}{2}\pi R^2.$$

Step 3: Use the given centroid depth. Because the diameter lies on the water surface and the semicircle lies below it,

$$\bar{y} = \frac{4R}{3\pi}.$$

Step 4: Substitute and simplify.

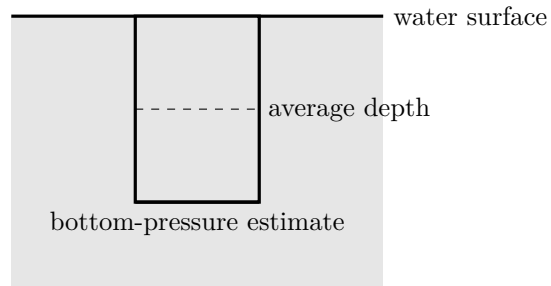
$$\begin{aligned} F &= \gamma \left(\frac{1}{2}\pi R^2 \right) \left(\frac{4R}{3\pi} \right) \\ &= \gamma \left(\frac{4R^3}{6} \right) = \frac{2}{3}\gamma R^3. \end{aligned}$$

$$\boxed{F = \frac{2}{3}\gamma R^3}$$

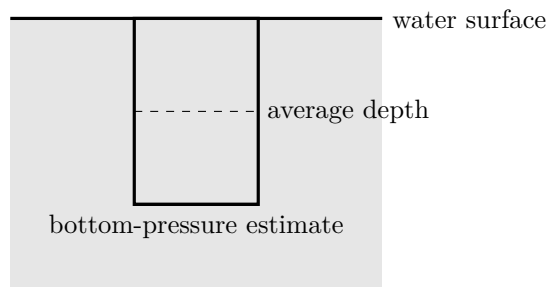
The factors of π cancel because the semicircle's area contains π while its centroid depth contains $1/\pi$.

Question 9

A student uses constant pressure at the bottom of a 3-m-tall vertical rectangular gate. Explain the resulting bias and find the ratio of this estimate to the true force when the top is at the surface.



Question 9 – Solution



Step 1: Write the student's estimate. The student applies the bottom pressure $p = \gamma(3)$ uniformly over the entire gate of area A . Thus

$$F_{\text{est}} = 3\gamma A.$$

Step 2: Find the true force. For a rectangular gate extending from the surface to depth 3 m, the centroid is at the average depth

$$\bar{y} = \frac{0 + 3}{2} = \frac{3}{2} \text{ m.}$$

Therefore,

$$F_{\text{true}} = \gamma A \bar{y} = \frac{3}{2} \gamma A.$$

Step 3: Compare the estimate with the true force.

$$\frac{F_{\text{est}}}{F_{\text{true}}} = \frac{3\gamma A}{\frac{3}{2}\gamma A} = 2.$$

Thus the estimate is twice the true value:

$$F_{\text{est}} = 2F_{\text{true}}.$$

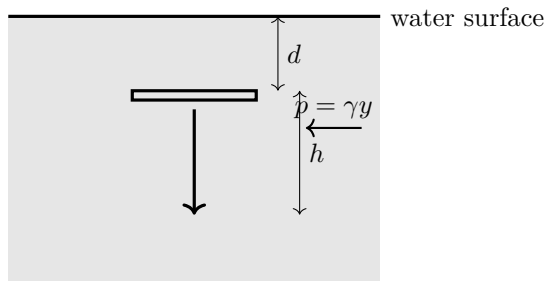
Step 4: Express the bias as a percent error.

$$\frac{F_{\text{est}} - F_{\text{true}}}{F_{\text{true}}} \times 100\% = (2 - 1)100\% = 100\%.$$

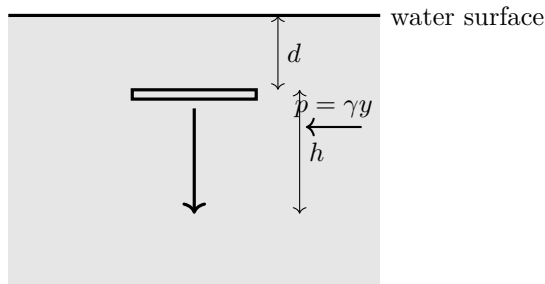
The bottom-pressure estimate is 100% too high.

Question 10

A horizontal hatch of constant area $A > 0$ is slowly lowered from depth d to $d + h$, where $d, h > 0$. Assume an upward resisting hydrostatic force $F(y) = \gamma Ay$ at depth y , with constant weight density $\gamma > 0$. Find the work done against this force, ignoring other forces.



Question 10 – Solution



Step 1: State the force model. Assume the hydrostatic force resisting the downward motion has magnitude

$$F(y) = p(y)A = \gamma yA,$$

where y is the hatch depth. The area A is constant.

Step 2: Write a differential amount of work. Moving the hatch through a small downward distance dy requires

$$dW = F(y) dy = \gamma Ay dy.$$

Step 3: Integrate over the change in depth. The hatch moves from $y = d$ to $y = d + h$, so

$$\begin{aligned} W &= \int_d^{d+h} \gamma Ay dy \\ &= \gamma A \left[\frac{y^2}{2} \right]_d^{d+h} \\ &= \frac{\gamma A}{2} ((d+h)^2 - d^2). \end{aligned}$$

Step 4: Simplify.

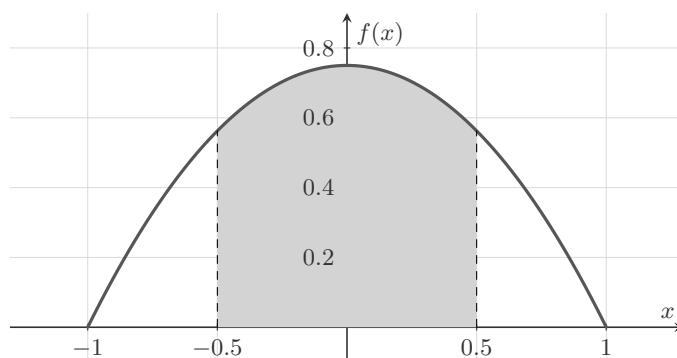
$$\begin{aligned} W &= \frac{\gamma A}{2} (d^2 + 2dh + h^2 - d^2) \\ &= \gamma A \left(dh + \frac{h^2}{2} \right). \end{aligned}$$

$$\boxed{W = \gamma A \left(dh + \frac{h^2}{2} \right)}$$

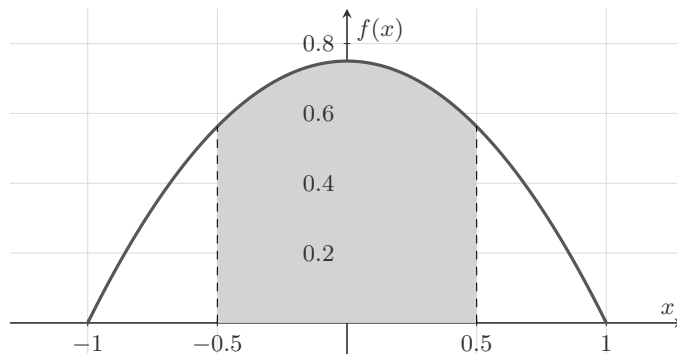
This is the external work under the stated resisting-force assumption.

Question 1

A sensor error has density $f(x) = c(1 - x^2)$, $-1 \leq x \leq 1$. Find c and $P(|X| < 1/2)$.



Question 1 – Solution



Step 1: Use the normalization condition. A probability density must have total area 1:

$$1 = c \int_{-1}^1 (1 - x^2) dx.$$

Step 2: Evaluate the normalization integral.

$$\begin{aligned} \int_{-1}^1 (1 - x^2) dx &= \left[x - \frac{x^3}{3} \right]_{-1}^1 \\ &= \frac{2}{3} - \left(-\frac{2}{3} \right) = \frac{4}{3}. \end{aligned}$$

Thus

$$1 = c \left(\frac{4}{3} \right), \quad \boxed{c = \frac{3}{4}}.$$

Step 3: Translate the event.

$$|X| < \frac{1}{2} \iff -\frac{1}{2} < X < \frac{1}{2}.$$

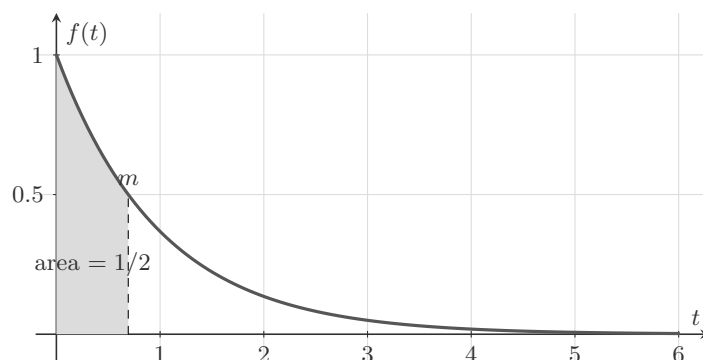
Step 4: Integrate the density over this interval.

$$\begin{aligned} P\left(|X| < \frac{1}{2}\right) &= \frac{3}{4} \int_{-1/2}^{1/2} (1 - x^2) dx \\ &= \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{-1/2}^{1/2} \\ &= \frac{3}{4} \left(\frac{11}{12} \right) = \frac{11}{16}. \end{aligned}$$

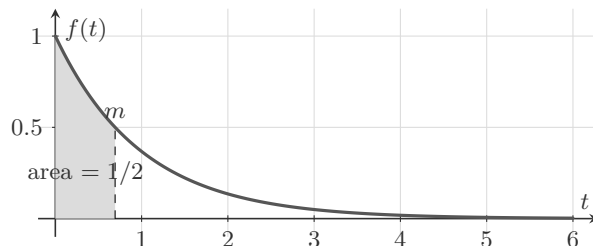
$$\boxed{P\left(|X| < \frac{1}{2}\right) = \frac{11}{16}}$$

Question 2

A waiting time has $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$. Find the median in terms of λ .



Question 2 – Solution



Step 1: Use the definition of a median. The median m satisfies

$$P(T \leq m) = \frac{1}{2}.$$

Because $T \geq 0$,

$$\int_0^m \lambda e^{-\lambda t} dt = \frac{1}{2}.$$

Step 2: Evaluate the integral.

$$\begin{aligned} \int_0^m \lambda e^{-\lambda t} dt &= [-e^{-\lambda t}]_0^m \\ &= 1 - e^{-\lambda m}. \end{aligned}$$

Therefore,

$$1 - e^{-\lambda m} = \frac{1}{2}.$$

Step 3: Solve for m .

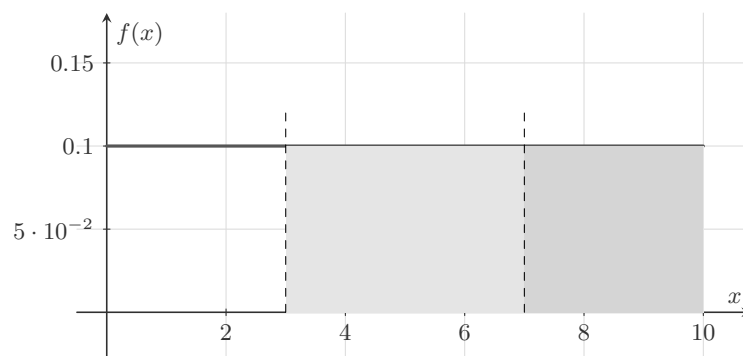
$$\begin{aligned} e^{-\lambda m} &= \frac{1}{2}, \\ -\lambda m &= \ln\left(\frac{1}{2}\right) = -\ln 2, \\ m &= \frac{\ln 2}{\lambda}. \end{aligned}$$

$$\boxed{m = \frac{\ln 2}{\lambda}}$$

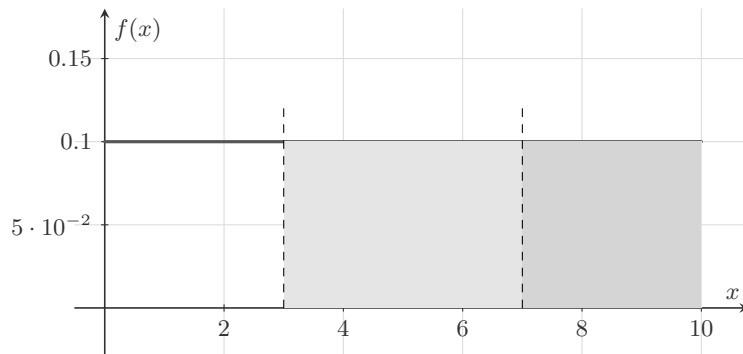
Since $\lambda > 0$, the median is positive as expected.

Question 3

A random point is chosen uniformly on $[0, 10]$. Given $X > 3$, find $P(X > 7 \mid X > 3)$.



Question 3 – Solution



Step 1: Identify the two events. Let

$$A = \{X > 7\}, \quad B = \{X > 3\}.$$

Because $A \subset B$, we have $A \cap B = A$.

Step 2: Apply conditional probability.

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

Step 3: Use uniform interval lengths. On $[0, 10]$, probability equals interval length divided by 10:

$$P(X > 7) = \frac{10 - 7}{10} = \frac{3}{10}, \quad P(X > 3) = \frac{10 - 3}{10} = \frac{7}{10}.$$

Step 4: Form the ratio.

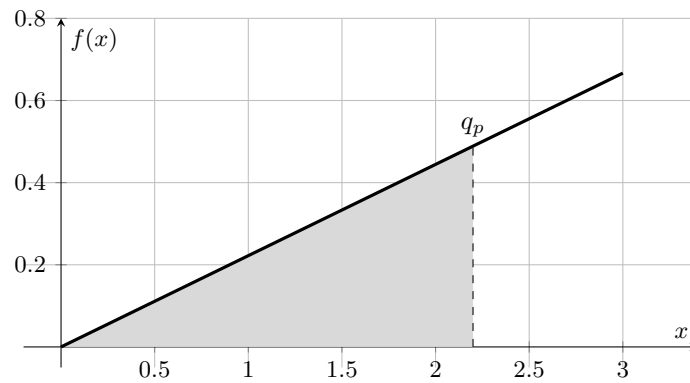
$$P(X > 7 | X > 3) = \frac{\frac{3}{10}}{\frac{7}{10}} = \frac{3}{7}.$$

$$\boxed{P(X > 7 | X > 3) = \frac{3}{7}}$$

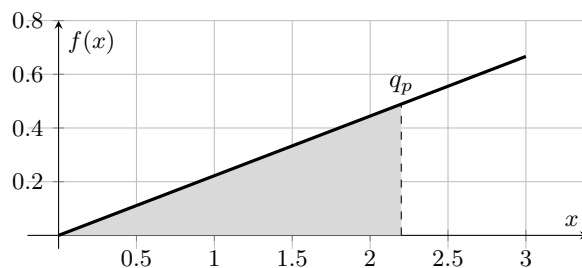
Equivalently, after conditioning on $X > 3$, the available interval is $(3, 10]$, and the favorable portion $(7, 10]$ has length 3 out of 7.

Question 4

A density is $f(x) = kx$ on $[0, a]$, where $a > 0$, and is zero elsewhere. Determine k , the CDF, and the p -quantile.



Question 4 – Solution



Step 1: Normalize the density.

$$1 = \int_0^a kx \, dx = k \left[\frac{x^2}{2} \right]_0^a = \frac{ka^2}{2}.$$

Hence

$$\boxed{k = \frac{2}{a^2}}.$$

Step 2: Build the CDF on the support. For $0 \leq x \leq a$,

$$\begin{aligned} F(x) &= \int_0^x \frac{2t}{a^2} \, dt \\ &= \frac{2}{a^2} \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{a^2}. \end{aligned}$$

Thus the complete CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{a^2}, & 0 \leq x \leq a, \\ 1, & x > a. \end{cases}$$

Step 3: Solve the quantile equation. The p -quantile q_p satisfies $F(q_p) = p$:

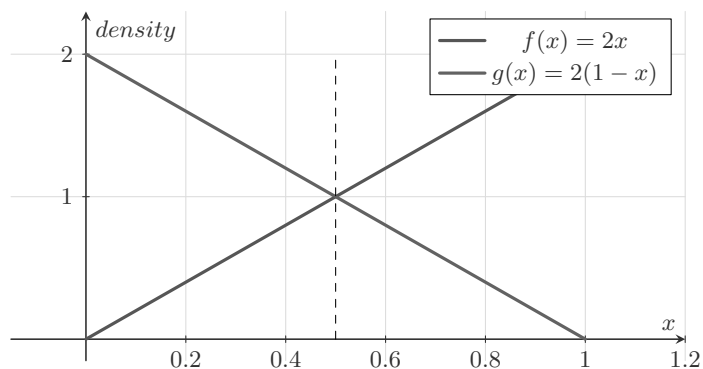
$$\frac{q_p^2}{a^2} = p.$$

Since $q_p \geq 0$,

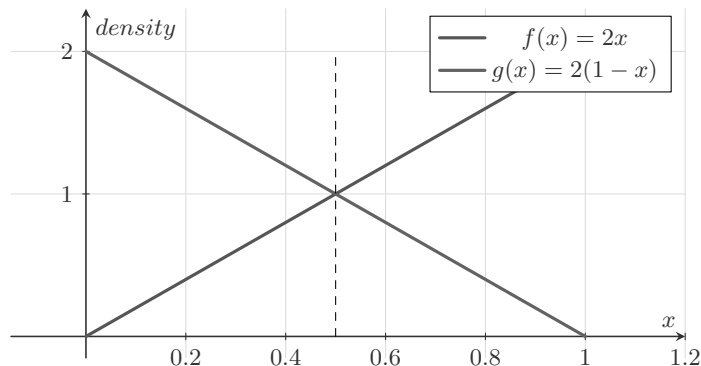
$$\boxed{q_p = a\sqrt{p}}, \quad 0 \leq p \leq 1.$$

Question 5

Two densities on $[0, 1]$ are $f(x) = 2x$ and $g(x) = 2(1 - x)$. Compare their means without integration by identifying a reflection.



Question 5 – Solution



Step 1: Identify the reflection.

$$f(1-x) = 2(1-x) = g(x).$$

Therefore, the graph of g is the reflection of the graph of f across $x = 1/2$.

Step 2: Find the mean of f geometrically. The area under $f(x) = 2x$ is a right triangle with vertices

$$(0,0), \quad (1,0), \quad (1,2).$$

The x -coordinate of a triangle's centroid is the average of its vertex x -coordinates:

$$\mu_f = \frac{0 + 1 + 1}{3} = \frac{2}{3}.$$

Step 3: Reflect the mean. Reflection across $x = 1/2$ sends a location x to $1-x$. Hence

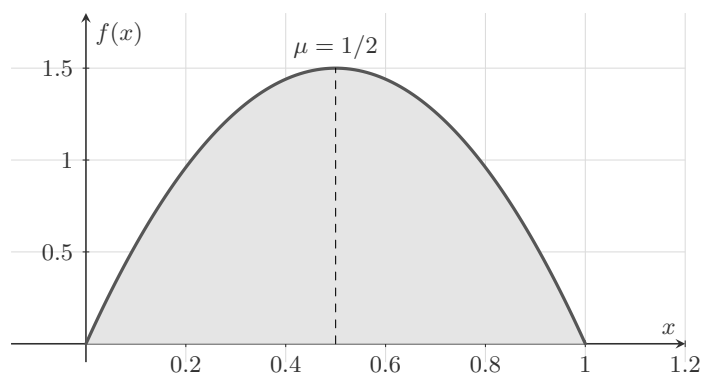
$$\mu_g = 1 - \mu_f = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\boxed{\mu_f = \frac{2}{3}, \quad \mu_g = \frac{1}{3}}$$

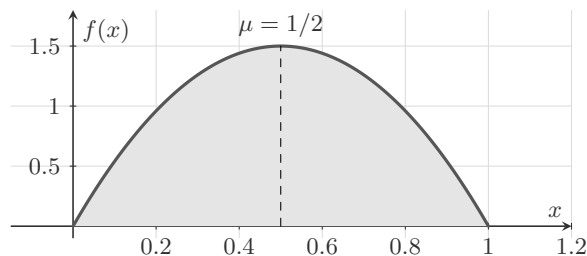
As a check, the two reflected means add to 1.

Question 6

For $f(x) = 6x(1 - x)$, $0 \leq x \leq 1$, find $\text{Var}(X)$. Use symmetry for the mean.



Question 6 – Solution



Step 1: Use symmetry to find the mean. The density satisfies

$$f(1-x) = 6(1-x)x = f(x),$$

so it is symmetric about $x = 1/2$. Therefore,

$$\mu = E[X] = \frac{1}{2}.$$

Step 2: Compute the second moment.

$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx \\ &= \int_0^1 x^2 (6x - 6x^2) dx \\ &= \int_0^1 (6x^3 - 6x^4) dx \\ &= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5 \right]_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}. \end{aligned}$$

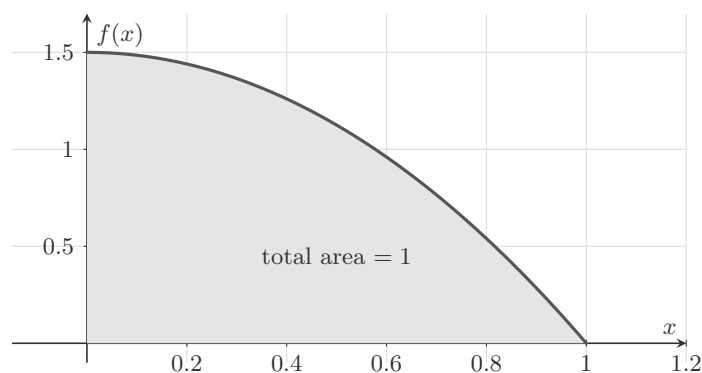
Step 3: Apply the variance formula.

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{3}{10} - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{10} - \frac{1}{4} = \frac{1}{20}. \end{aligned}$$

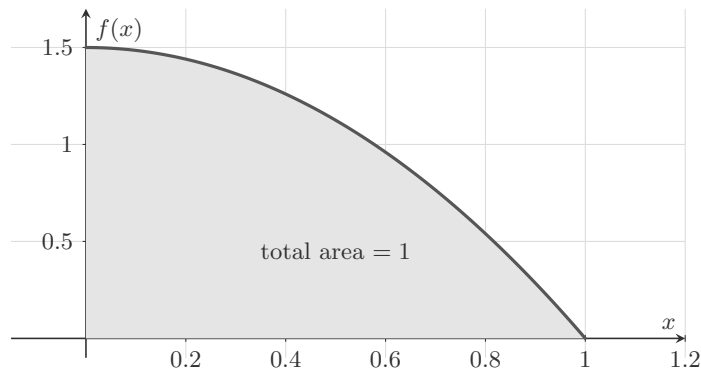
$$\text{Var}(X) = \frac{1}{20}$$

Question 7

A proposed density is $f(x) = \frac{3}{2}(1 - x^2)$ on $[0, 1]$. Is it valid? If not, repair only the constant.



Question 7 – Solution



Step 1: Check nonnegativity. For $0 \leq x \leq 1$,

$$1 - x^2 \geq 0,$$

so

$$f(x) = \frac{3}{2}(1 - x^2) \geq 0.$$

Step 2: Check the total area.

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{3}{2} \int_0^1 (1 - x^2) dx \\ &= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_0^1 \\ &= \frac{3}{2} \left(1 - \frac{1}{3} \right) = \frac{3}{2} \left(\frac{2}{3} \right) = 1. \end{aligned}$$

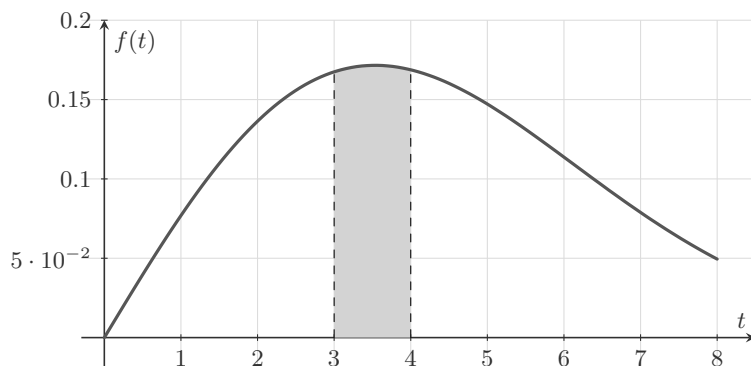
Step 3: State the conclusion. The function is nonnegative on its support and integrates to 1. Therefore it already satisfies both requirements for a probability density.

The proposed density is valid; no repair is needed.

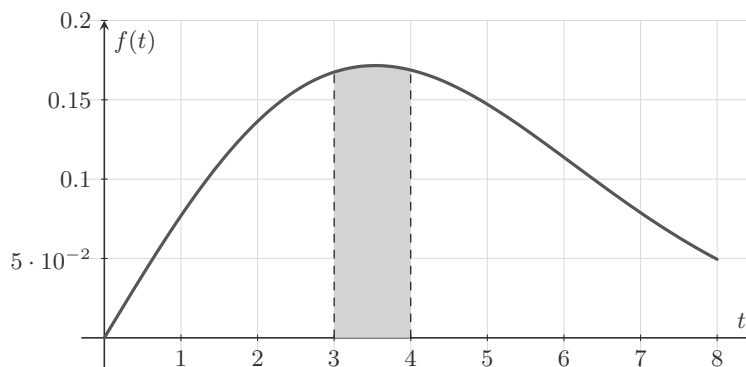
If the constant had been unknown, normalization would give $c(2/3) = 1$, hence $c = 3/2$.

Question 8

A lifetime satisfies $P(X > t) = e^{-(t/5)^2}$, $t \geq 0$. Find its density and $P(3 < X < 4)$.



Question 8 – Solution



Step 1: Convert the survival function to the CDF. For $t \geq 0$,

$$P(X > t) = e^{-(t/5)^2} = e^{-t^2/25}.$$

Therefore,

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-t^2/25}.$$

Step 2: Differentiate to obtain the density.

$$\begin{aligned} f(t) &= F'(t) \\ &= -\frac{d}{dt} \left(e^{-t^2/25} \right) \\ &= \frac{2t}{25} e^{-t^2/25}, \quad t \geq 0. \end{aligned}$$

Also, $f(t) = 0$ for $t < 0$.

Step 3: Compute the interval probability. Using the CDF,

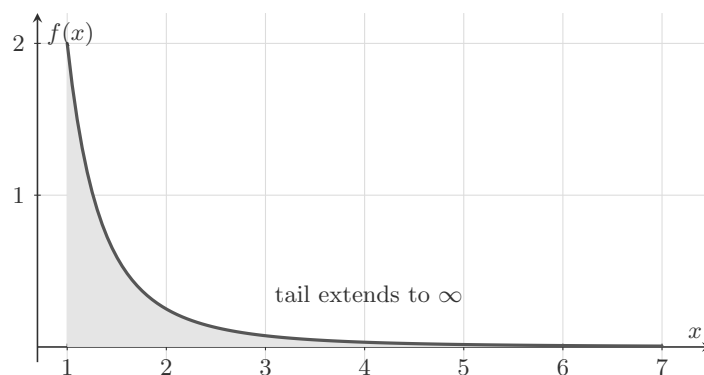
$$\begin{aligned} P(3 < X < 4) &= F(4) - F(3) \\ &= \left(1 - e^{-16/25} \right) - \left(1 - e^{-9/25} \right) \\ &= e^{-9/25} - e^{-16/25}. \end{aligned}$$

$$P(3 < X < 4) = e^{-9/25} - e^{-16/25}$$

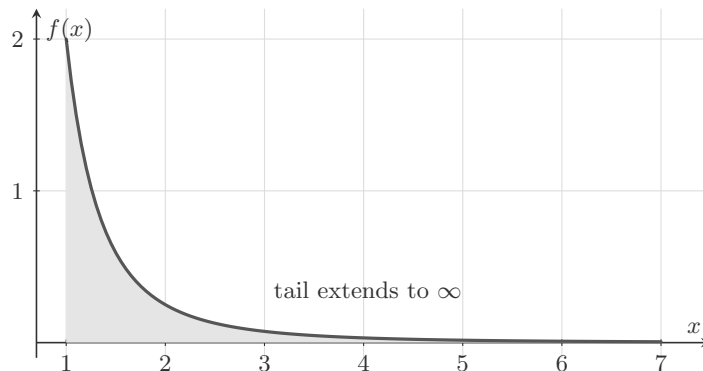
The same result follows by integrating $f(t)$ from 3 to 4.

Question 9

Choose c so $f(x) = c/x^3$, $x \geq 1$, is a density. Determine whether $E[X]$ and $E[X^2]$ are finite.



Question 9 – Solution



Step 1: Normalize the density.

$$\begin{aligned} 1 &= c \int_1^{\infty} x^{-3} dx \\ &= c \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{c}{2}. \end{aligned}$$

Thus

$$\boxed{c = 2}.$$

Step 2: Test the first moment.

$$\begin{aligned} E[X] &= \int_1^{\infty} x \frac{2}{x^3} dx = 2 \int_1^{\infty} x^{-2} dx \\ &= 2 \left[-\frac{1}{x} \right]_1^{\infty} = 2. \end{aligned}$$

Therefore, the mean is finite.

Step 3: Test the second moment.

$$E[X^2] = \int_1^{\infty} x^2 \frac{2}{x^3} dx = 2 \int_1^{\infty} \frac{dx}{x}.$$

For an upper cutoff b ,

$$2 \int_1^b \frac{dx}{x} = 2 \ln b \rightarrow \infty.$$

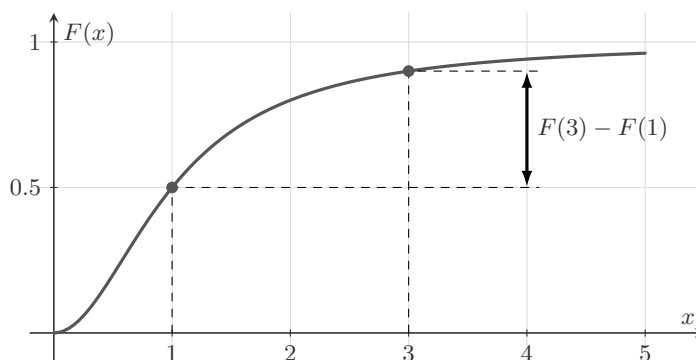
Thus the second moment diverges.

$$\boxed{E[X] = 2, \quad E[X^2] = \infty}$$

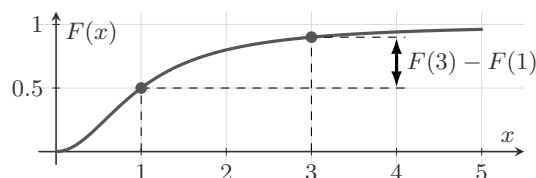
Consequently, $\text{Var}(X)$ is also infinite.

Question 10

A continuous random variable has CDF $F(x) = 0$ for $x < 0$, $F(x) = x^2/(1+x^2)$ for $x \geq 0$. Find the density and $P(1 < X < 3)$.



Question 10 – Solution



Step 1: Differentiate the CDF on each interval. For $x < 0$, $F(x) = 0$, so $f(x) = 0$. For $x > 0$,

$$\begin{aligned} f(x) &= \frac{d}{dx} \left(\frac{x^2}{1+x^2} \right) \\ &= \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} \\ &= \frac{2x}{(1+x^2)^2}. \end{aligned}$$

The CDF is continuous at $x = 0$, so there is no point mass there. Hence

$$f(x) = \begin{cases} 0, & x < 0, \\ \frac{2x}{(1+x^2)^2}, & x \geq 0. \end{cases}$$

Step 2: Use the CDF to compute the interval probability. For a continuous random variable,

$$P(1 < X < 3) = F(3) - F(1).$$

Now

$$F(3) = \frac{3^2}{1+3^2} = \frac{9}{10}, \quad F(1) = \frac{1^2}{1+1^2} = \frac{1}{2}.$$

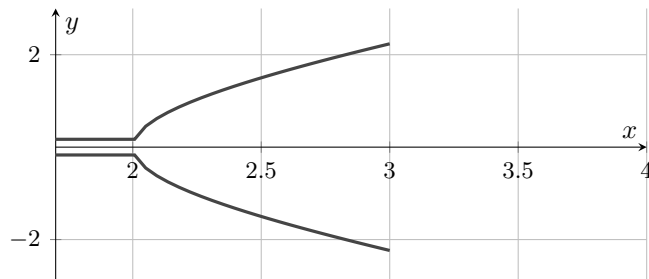
Step 3: Subtract.

$$P(1 < X < 3) = \frac{9}{10} - \frac{1}{2} = \frac{9}{10} - \frac{5}{10} = \frac{4}{10} = \frac{2}{5}.$$

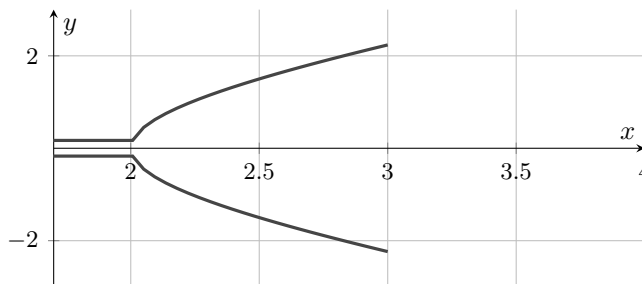
$$P(1 < X < 3) = \frac{2}{5}$$

Exercise 1**Problem**

A mechanism has $x = t + t^{-1}$ and $y = t - t^{-1}$, $t > 0$. Identify its Cartesian curve and explain which part the mechanism can reach.



Exercise 1: Solution



Solution

1. Square both parametric equations:

$$x^2 = \left(t + \frac{1}{t}\right)^2 = t^2 + 2 + \frac{1}{t^2}, \quad y^2 = \left(t - \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2}.$$

2. Subtract the second equation from the first:

$$x^2 - y^2 = 4.$$

Thus the Cartesian curve is the hyperbola

$$\boxed{\frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

3. Because $t > 0$, the arithmetic–geometric mean inequality gives

$$x = t + \frac{1}{t} \geq 2.$$

Therefore, the parametrization reaches only the right-hand branch of the hyperbola.

4. To determine the direction of travel, note that

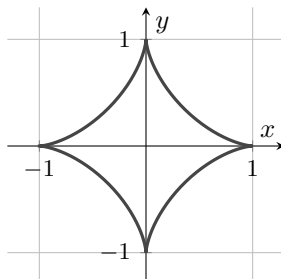
$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 1 + \frac{1}{t^2} > 0.$$

As t increases from 0 to 1, the point moves from the lower right toward $(2, 0)$. As t increases beyond 1, it moves from $(2, 0)$ toward the upper right.

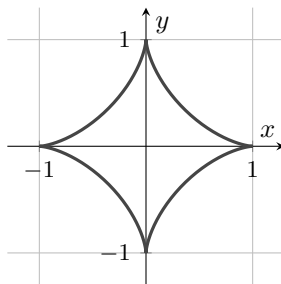
5. Hence the mechanism traces the entire right branch exactly once, moving upward as t increases.

Exercise 2**Problem**

The curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$, looks almost circular. Eliminate t and identify the four points where its shape is least circle-like.



Exercise 2: Solution



Solution

1. Begin with

$$x = \cos^3 t, \quad y = \sin^3 t.$$

Taking each coordinate to the power $2/3$ gives

$$x^{2/3} = \cos^2 t, \quad y^{2/3} = \sin^2 t.$$

Here $x^{2/3}$ means $(\sqrt[3]{x})^2$, so the calculation is valid even when a coordinate is negative.

2. Add the two equations and use
- $\cos^2 t + \sin^2 t = 1$
- :

$$\boxed{x^{2/3} + y^{2/3} = 1}.$$

This curve is an *astroid*.

3. The curve is least circle-like where it has cusps. These occur when one trigonometric coordinate is 0 and the other is ± 1 .
4. Evaluating the parametrization at the corresponding parameter values gives

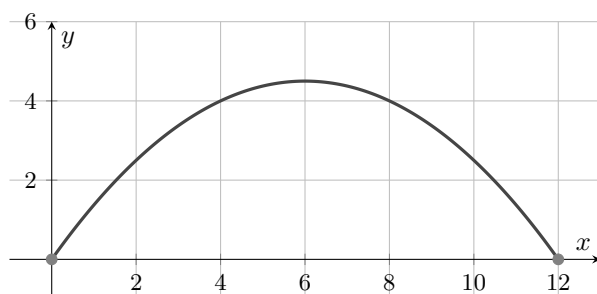
t	(x, y)
0	(1, 0)
$\pi/2$	(0, 1)
π	(-1, 0)
$3\pi/2$	(0, -1)

5. Therefore, the four sharp points are

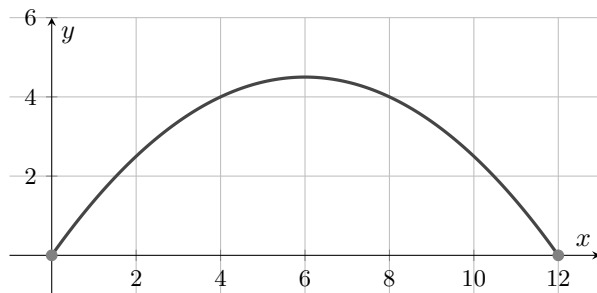
$$\boxed{(1, 0), (0, 1), (-1, 0), (0, -1)}.$$

Exercise 3**Problem**

A fountain droplet follows $x = 6t$ and $y = 9t - \frac{9}{2}t^2$. Eliminate time, find the landing point, and state why the parameter is more informative than the Cartesian equation.



Exercise 3: Solution



Solution

1. Solve the horizontal equation for the parameter:

$$x = 6t \implies t = \frac{x}{6}.$$

2. Substitute $t = x/6$ into the vertical equation:

$$\begin{aligned} y &= 9\left(\frac{x}{6}\right) - \frac{9}{2}\left(\frac{x}{6}\right)^2 \\ &= \frac{3}{2}x - \frac{1}{8}x^2. \end{aligned}$$

Thus the Cartesian path is the downward-opening parabola

$$\boxed{y = \frac{3}{2}x - \frac{1}{8}x^2}.$$

3. The droplet lands when it returns to ground level, so set $y = 0$:

$$0 = \frac{3}{2}x - \frac{1}{8}x^2 = \frac{x}{8}(12 - x).$$

The solutions are $x = 0$ and $x = 12$.

4. The value $x = 0$ is the launch point. The nonzero intercept is therefore the landing point

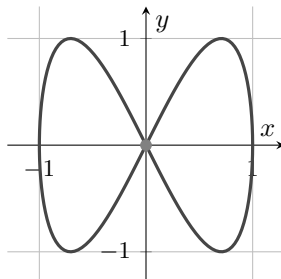
$$\boxed{(12, 0)}.$$

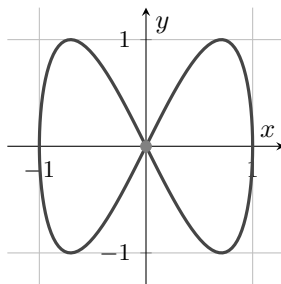
It occurs at $t = 12/6 = 2$.

5. The Cartesian equation describes only the geometric path. The parameter t also records time, so it identifies the droplet's position and direction of motion at each instant.

Exercise 4**Problem**

For $x = \cos t$, $y = \sin(2t)$, $0 \leq t \leq 2\pi$, locate the self-intersection and determine how many times it is visited.



Exercise 4: Solution**Solution**

1. A self-intersection occurs when two different parameter values produce the same point. The graph suggests checking the origin.
2. For the point to lie at the origin, first require $x = 0$:

$$\cos t = 0.$$

On $0 \leq t \leq 2\pi$, this gives

$$t = \frac{\pi}{2} \quad \text{or} \quad t = \frac{3\pi}{2}.$$

3. Check the y -coordinate at both values:

$$y\left(\frac{\pi}{2}\right) = \sin \pi = 0, \quad y\left(\frac{3\pi}{2}\right) = \sin 3\pi = 0.$$

Hence both parameter values produce $(0, 0)$.

4. The two visits correspond to different directions. Since

$$\frac{dy}{dx} = \frac{2 \cos(2t)}{-\sin t},$$

the slopes are 2 at $t = \pi/2$ and -2 at $t = 3\pi/2$.

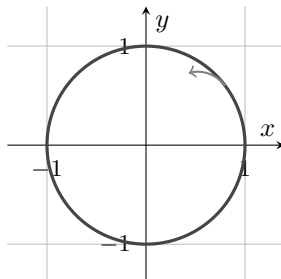
5. Therefore, the curve crosses itself at

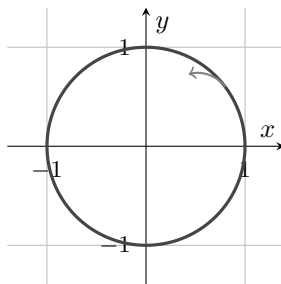
$$\boxed{(0, 0)},$$

and this point is visited exactly twice on the stated interval.

Exercise 5**Problem**

Two students parametrize the unit circle by $(\cos t, \sin t)$ and $(\cos 2t, -\sin 2t)$ on $0 \leq t \leq 2\pi$. Compare orientation and number of traversals.



Exercise 5: Solution**Solution**

1. For the first parametrization,

$$(x, y) = (\cos t, \sin t),$$

the polar angle is t . As t increases from 0 to 2π , the angle increases through one full revolution.

2. It begins at $(1, 0)$, then moves through $(0, 1)$, $(-1, 0)$, and $(0, -1)$ before returning to $(1, 0)$. Therefore, it traces the circle counterclockwise exactly once.

3. Rewrite the second parametrization as

$$(x, y) = (\cos(2t), -\sin(2t)) = (\cos(-2t), \sin(-2t)).$$

Its polar angle is therefore $-2t$.

4. As t increases from 0 to 2π , the angle decreases from 0 to -4π . A decreasing angle produces clockwise motion, and a total angular change of 4π represents two complete revolutions.

5. Thus

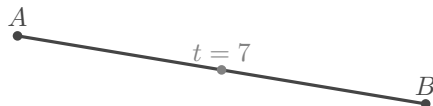
$(\cos t, \sin t)$ travels counterclockwise once,

whereas

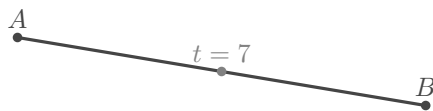
$(\cos 2t, -\sin 2t)$ travels clockwise twice.

Exercise 6**Problem**

Find a parametrization of the line segment from $A = (-2, 5)$ to $B = (4, -1)$ that reaches its midpoint at $t = 7$, with $6 \leq t \leq 8$.



Exercise 6: Solution



Solution

1. A line segment from A to B can be parametrized by

$$\mathbf{r} = A + u(B - A), \quad 0 \leq u \leq 1.$$

2. The parameter interval must be changed from $6 \leq t \leq 8$ to $0 \leq u \leq 1$. The linear change of variables is

$$u = \frac{t - 6}{8 - 6} = \frac{t - 6}{2}.$$

3. Compute the displacement vector:

$$B - A = (4, -1) - (-2, 5) = (6, -6).$$

Therefore,

$$(x, y) = (-2, 5) + \frac{t - 6}{2}(6, -6).$$

4. Simplify each coordinate:

$$x = -2 + 3(t - 6) = 3t - 20,$$

$$y = 5 - 3(t - 6) = 23 - 3t.$$

Hence a suitable parametrization is

$$\boxed{x = 3t - 20, \quad y = 23 - 3t, \quad 6 \leq t \leq 8}.$$

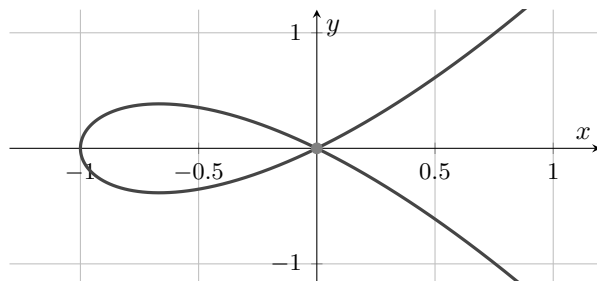
5. At $t = 6$ it gives $A = (-2, 5)$, and at $t = 8$ it gives $B = (4, -1)$. At $t = 7$ it gives

$$(x, y) = (1, 2) = \frac{A + B}{2},$$

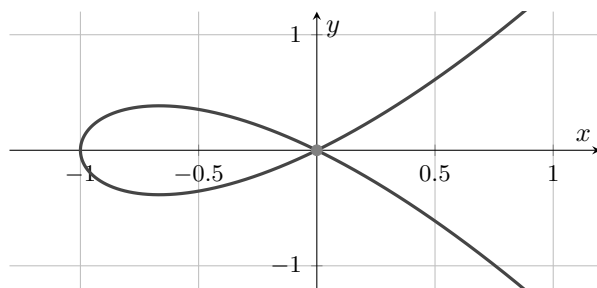
so the midpoint condition is satisfied.

Exercise 7**Problem**

The curve $x = t^2 - 1$, $y = t^3 - t$ crosses itself. Find the two parameter values at the crossing and describe the two tangent directions there.



Exercise 7: Solution



Solution

1. Factor the coordinate equations:

$$x = t^2 - 1 = (t - 1)(t + 1), \quad y = t^3 - t = t(t - 1)(t + 1).$$

2. Both coordinates equal zero when $t = 1$ and when $t = -1$:

$$\mathbf{r}(1) = (0, 0), \quad \mathbf{r}(-1) = (0, 0).$$

Because two distinct parameter values produce the same point, the self-intersection is $(0, 0)$.

3. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1.$$

Since $dx/dt \neq 0$ at $t = \pm 1$, the tangent slope is

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}.$$

4. At $t = 1$,

$$\frac{dy}{dx} = \frac{3(1)^2 - 1}{2(1)} = 1,$$

so one tangent line is $y = x$.

5. At $t = -1$,

$$\frac{dy}{dx} = \frac{3(-1)^2 - 1}{2(-1)} = -1,$$

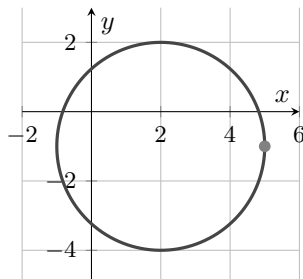
so the other tangent line is $y = -x$.

6. Therefore, the curve crosses itself at the origin with tangent directions of slopes

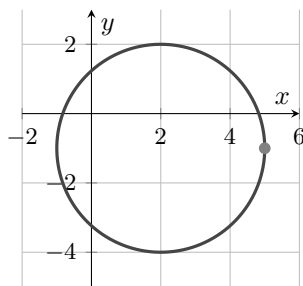
$$\boxed{1 \text{ and } -1}.$$

Exercise 8**Problem**

A moving point satisfies $x = 2 + 3 \cos t$, $y = -1 + 3 \sin t$. Without eliminating t , determine its center, radius, starting point, and initial direction.



Exercise 8: Solution



Solution

1. Compare the equations

$$x = 2 + 3 \cos t, \quad y = -1 + 3 \sin t$$

with the standard circle parametrization

$$x = h + R \cos t, \quad y = k + R \sin t.$$

2. It follows immediately that

$$(h, k) = (2, -1) \quad \text{and} \quad R = 3.$$

Thus the curve is a circle centered at $(2, -1)$ with radius 3 .

3. Evaluate the coordinates at $t = 0$:

$$x(0) = 2 + 3 \cos 0 = 5, \quad y(0) = -1 + 3 \sin 0 = -1.$$

Therefore, the starting point is $(5, -1)$.

4. Differentiate to find the velocity:

$$\mathbf{r}'(t) = (-3 \sin t, 3 \cos t).$$

At $t = 0$,

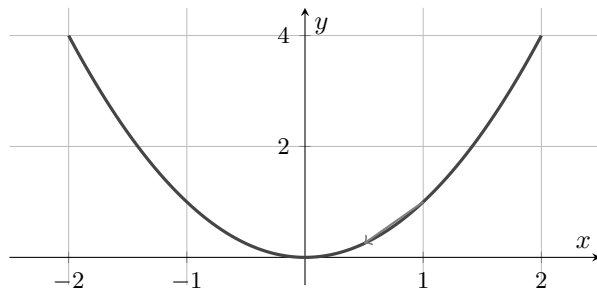
$$\mathbf{r}'(0) = (0, 3),$$

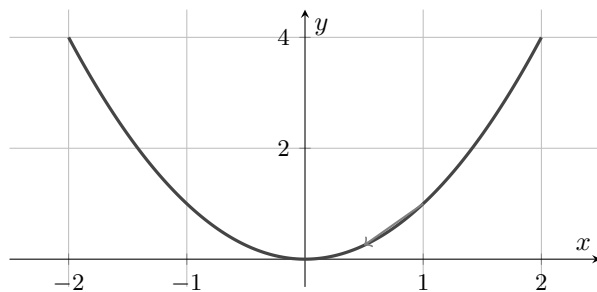
which points straight upward from the rightmost point of the circle.

5. Upward motion from the rightmost point is counterclockwise, so the initial direction is **counterclockwise**.

Exercise 9**Problem**

Can the parabola $y = x^2$ be traced from right to left while the parameter increases? Give a parametrization on $-2 \leq t \leq 2$ that does so.



Exercise 9: Solution**Solution**

1. To trace the parabola from right to left, choose the x -coordinate so that it decreases as t increases. A simple choice is

$$x = -t.$$

2. The curve must satisfy $y = x^2$. Substituting $x = -t$ gives

$$y = (-t)^2 = t^2.$$

Hence one suitable parametrization is

$$\boxed{x = -t, \quad y = t^2, \quad -2 \leq t \leq 2}.$$

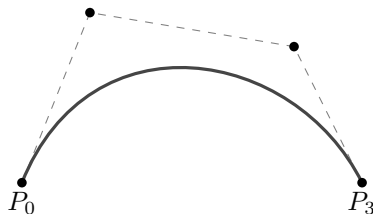
3. Verify that the curve is correct by eliminating the parameter:

$$t = -x \implies y = t^2 = (-x)^2 = x^2.$$

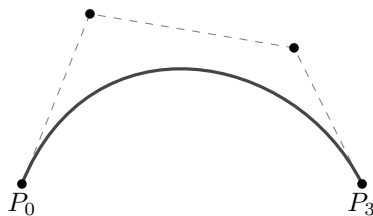
4. Verify the direction. At $t = -2$, the point is $(2, 4)$; at $t = 0$, it is $(0, 0)$; and at $t = 2$, it is $(-2, 4)$.
5. Also, $dx/dt = -1 < 0$, so x decreases continuously. Therefore, the parabola is traced from right to left as required.

Exercise 10**Problem**

A cubic Bézier path is $\mathbf{r}(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3$. Explain, without expanding, why it begins at P_0 and ends at P_3 .



Exercise 10: Solution



Solution

1. The cubic Bézier path is

$$\mathbf{r}(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3.$$

2. Evaluate the path at $t = 0$:

$$\begin{aligned} \mathbf{r}(0) &= (1-0)^3 P_0 + 3(1-0)^2(0)P_1 \\ &\quad + 3(1-0)(0)^2 P_2 + (0)^3 P_3 \\ &= P_0. \end{aligned}$$

Every term containing a factor of t vanishes, leaving only P_0 .

3. Evaluate the path at $t = 1$:

$$\begin{aligned} \mathbf{r}(1) &= (1-1)^3 P_0 + 3(1-1)^2(1)P_1 \\ &\quad + 3(1-1)(1)^2 P_2 + (1)^3 P_3 \\ &= P_3. \end{aligned}$$

Every term containing a factor of $1-t$ vanishes, leaving only P_3 .

4. Therefore,

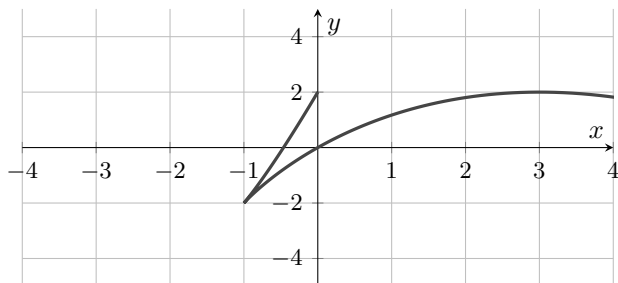
$$\boxed{\mathbf{r}(0) = P_0 \quad \text{and} \quad \mathbf{r}(1) = P_3}.$$

The path begins at the first control point and ends at the last control point.

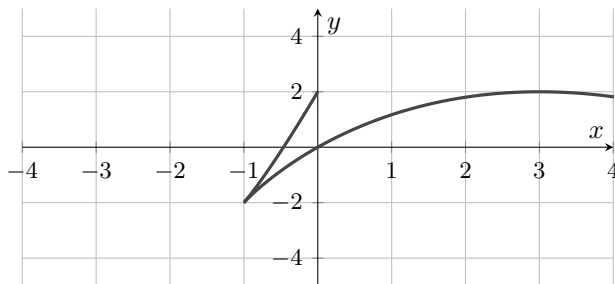
Question 1

Problem

For $x = t^2 - 2t$, $y = t^3 - 3t$, find every horizontal and vertical tangent.



Question 1 – Solution



Solution

1. Differentiate each coordinate:

$$\frac{dx}{dt} = 2t - 2 = 2(t - 1), \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1).$$

2. A horizontal tangent requires $dy/dt = 0$ while $dx/dt \neq 0$. The candidates are

$$3(t - 1)(t + 1) = 0 \implies t = -1, 1.$$

At $t = -1$, $dx/dt = -4 \neq 0$, so this value gives a horizontal tangent. The point is

$$x(-1) = 3, \quad y(-1) = 2.$$

Thus the horizontal tangent line is

$$\boxed{y = 2}.$$

3. At $t = 1$, both derivatives are zero, so the usual horizontal- or vertical-tangent tests are inconclusive. For $t \neq 1$, cancel the common factor:

$$\frac{dy}{dx} = \frac{3(t - 1)(t + 1)}{2(t - 1)} = \frac{3}{2}(t + 1).$$

Therefore,

$$\lim_{t \rightarrow 1} \frac{dy}{dx} = 3.$$

The point at $t = 1$ is $(-1, -2)$, and its tangent line is $y + 2 = 3(x + 1)$; it is neither horizontal nor vertical.

4. A vertical tangent would require $dx/dt = 0$ with $dy/dt \neq 0$. The only zero of dx/dt is $t = 1$, where dy/dt is also zero and the limiting slope is finite. Hence there are no vertical tangents.

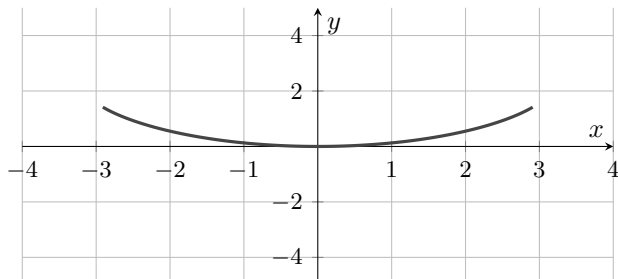
5. The complete answer is

$$\boxed{\text{horizontal tangent } y = 2 \text{ at } (3, 2); \quad \text{no vertical tangents.}}$$

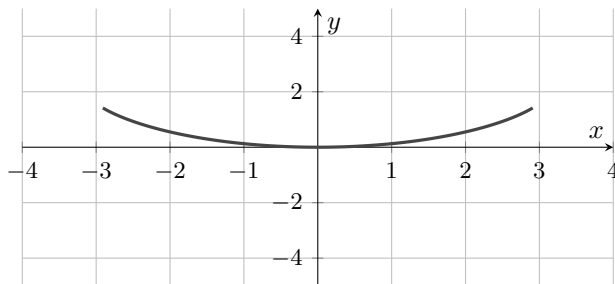
Question 2

Problem

The path $x = t + \sin t$, $y = 1 - \cos t$ models a rolling marker. Find its tangent slope at $t = \pi/2$ and interpret its sign.



Question 2 – Solution



Solution

1. Differentiate the coordinate functions:

$$\frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = \sin t.$$

2. At $t = \pi/2$,

$$\frac{dx}{dt} = 1 + \cos \frac{\pi}{2} = 1, \quad \frac{dy}{dt} = \sin \frac{\pi}{2} = 1.$$

Since $dx/dt \neq 0$, the tangent slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{1} = \boxed{1}.$$

3. The point on the path is

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2} + 1, 1\right).$$

Therefore, the tangent line is

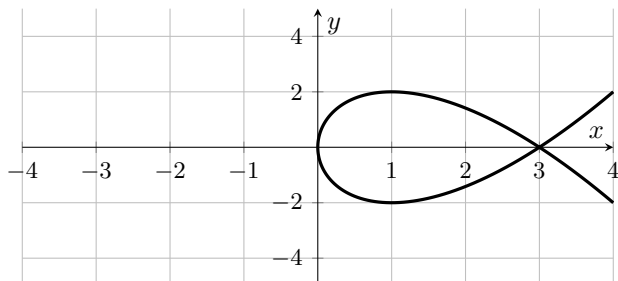
$$y - 1 = x - \left(\frac{\pi}{2} + 1\right).$$

4. Both velocity components are positive at this instant. Thus the marker is moving to the right and upward, which is consistent with the positive slope.

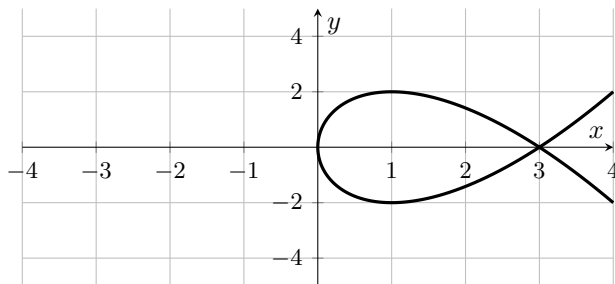
Question 3

Problem

At which points does $x = t^2$, $y = t^3 - 3t$ have tangent slope 0? Is the particle momentarily stopped there?



Question 3 – Solution



Solution

1. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1).$$

2. A tangent has slope 0 when $dy/dt = 0$ and $dx/dt \neq 0$. Solve

$$3(t-1)(t+1) = 0,$$

which gives $t = -1$ and $t = 1$.

3. Check the horizontal velocity:

$$x'(-1) = -2 \neq 0, \quad x'(1) = 2 \neq 0.$$

Thus both parameter values give genuine horizontal tangents.

4. Find the corresponding points:

$$\mathbf{r}(-1) = (1, 2), \quad \mathbf{r}(1) = (1, -2).$$

The horizontal tangent lines are therefore $y = 2$ and $y = -2$.

5. A particle is momentarily stopped only when both velocity components are zero. At $t = \pm 1$, $dy/dt = 0$ but $dx/dt = \pm 2$ is nonzero, so the particle is moving horizontally rather than stopping.

6. Hence the requested points are

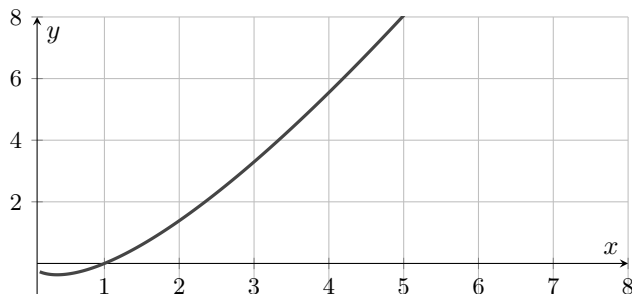
$$\boxed{(1, 2) \text{ and } (1, -2)},$$

and the particle is not momentarily stopped at either point.

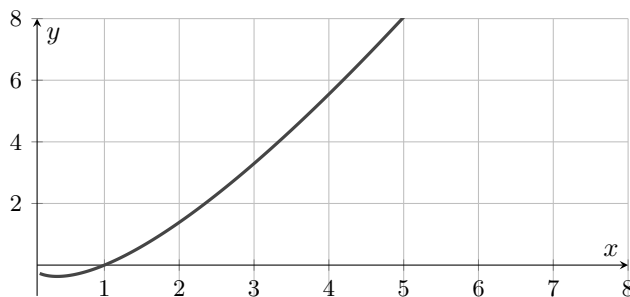
Question 4

Problem

For $x = e^t$, $y = te^t$, find the tangent line at the point whose x -coordinate is 1.



Question 4 – Solution



Solution

1. Locate the required parameter value by solving

$$x = e^t = 1.$$

Since the exponential function equals 1 only at $t = 0$, the required parameter value is $t = 0$.

2. Evaluate the y -coordinate:

$$y(0) = 0 \cdot e^0 = 0.$$

Thus the point of tangency is $(1, 0)$.

3. Differentiate both coordinates. The product rule is needed for y :

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t + te^t = e^t(1 + t).$$

4. Because $dx/dt = e^t \neq 0$, divide the derivatives:

$$\frac{dy}{dx} = \frac{e^t(1 + t)}{e^t} = 1 + t.$$

At $t = 0$, the slope is $m = 1$.

5. Use point-slope form through $(1, 0)$:

$$y - 0 = 1(x - 1).$$

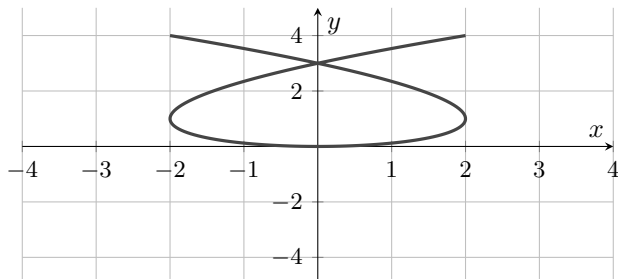
Therefore, the tangent line is

$$\boxed{y = x - 1}.$$

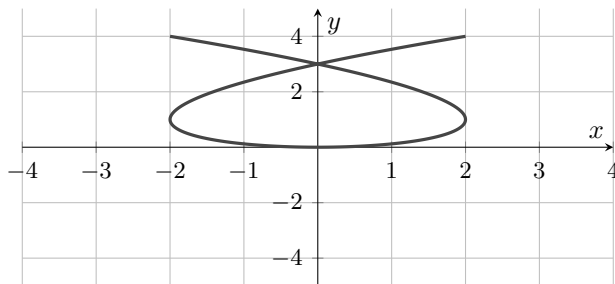
Question 5

Problem

A student claims that the two tangents to $x = t^3 - 3t$, $y = t^2$ at its self-intersection are perpendicular. Locate the crossing and test the claim.



Question 5 – Solution



Solution

- At a self-intersection, two distinct parameter values give the same y -coordinate. Since $y = t^2$, equal y -coordinates occur for parameters t and $-t$.
- Require the corresponding x -coordinates to be equal:

$$t^3 - 3t = (-t)^3 - 3(-t) = -t^3 + 3t.$$

Simplifying gives

$$2t^3 - 6t = 2t(t^2 - 3) = 0.$$

The value $t = 0$ does not give two distinct parameters. The nontrivial pair is

$$t = \pm\sqrt{3}.$$

- At either parameter value,

$$x = t(t^2 - 3) = 0, \quad y = t^2 = 3.$$

Hence the self-intersection is $\boxed{(0, 3)}$.

- Differentiate:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t.$$

At $t = \pm\sqrt{3}$, $dx/dt = 6 \neq 0$, so

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 3} = \frac{t}{3}.$$

The two slopes are

$$m_1 = \frac{1}{\sqrt{3}}, \quad m_2 = -\frac{1}{\sqrt{3}}.$$

- Two nonvertical lines are perpendicular only if the product of their slopes is -1 . Here

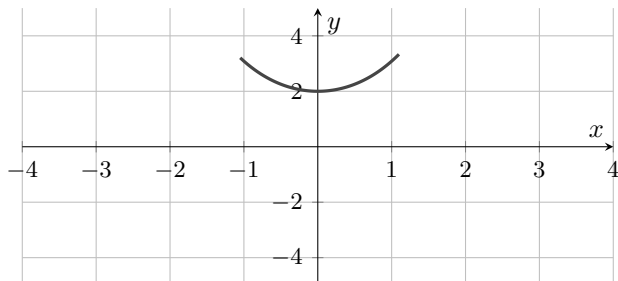
$$m_1 m_2 = -\frac{1}{3} \neq -1.$$

Therefore, the two tangents are $\boxed{\text{not perpendicular}}$.

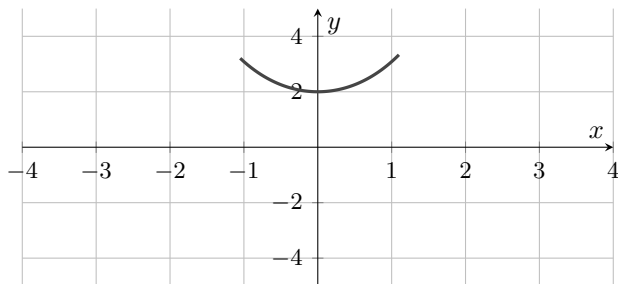
Question 6

Problem

For $x = \ln t$, $y = t + t^{-1}$, find d^2y/dx^2 at $t = 1$ and classify the local shape.



Question 6 – Solution



Solution

1. Because $x = \ln t$, the parameter domain is $t > 0$. Differentiate the coordinates:

$$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2}.$$

2. Compute the first derivative with respect to x :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - t^{-2}}{t^{-1}} = t - \frac{1}{t}.$$

3. For a parametric curve, differentiate dy/dx with respect to t and divide once more by dx/dt :

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt}.$$

Since

$$\frac{d}{dt} \left(t - \frac{1}{t} \right) = 1 + \frac{1}{t^2},$$

we obtain

$$\frac{d^2y}{dx^2} = \frac{1 + t^{-2}}{t^{-1}} = t + \frac{1}{t}.$$

4. Evaluate at $t = 1$:

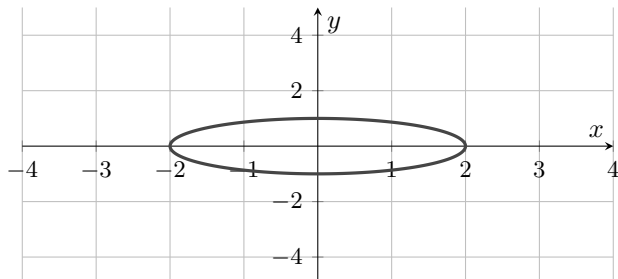
$$\left. \frac{d^2y}{dx^2} \right|_{t=1} = 1 + 1 = \boxed{2}.$$

5. Because the second derivative is positive, the curve is concave up at this point. Also, $(x(1), y(1)) = (0, 2)$ and $dy/dx = 0$, so $(0, 2)$ is a local minimum of the curve when viewed as y versus x .

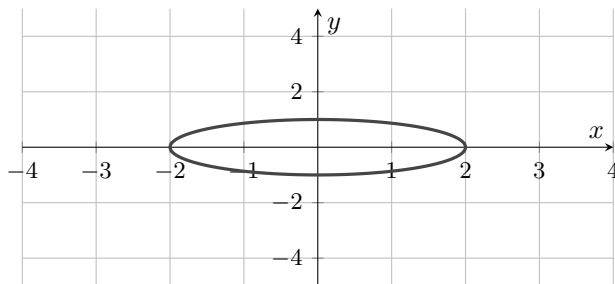
Question 7

Problem

Find the angle at which the parametric curve $x = 2 \cos t$, $y = \sin t$ crosses the line $y = x/2$ in the first quadrant.



Question 7 – Solution



Solution

1. At an intersection with $y = x/2$, substitute the parametric equations:

$$\sin t = \frac{2 \cos t}{2} = \cos t.$$

In the first quadrant, this gives $t = \pi/4$.

2. The intersection point is

$$\left(2 \cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right) = \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right).$$

3. Differentiate the curve:

$$\frac{dx}{dt} = -2 \sin t, \quad \frac{dy}{dt} = \cos t.$$

Thus its tangent slope at $t = \pi/4$ is

$$m_c = \frac{\cos t}{-2 \sin t} = -\frac{1}{2}.$$

The line $y = x/2$ has slope $m_\ell = 1/2$.

4. If ϕ is the acute angle between two lines, then

$$\tan \phi = \left| \frac{m_\ell - m_c}{1 + m_\ell m_c} \right|.$$

Therefore,

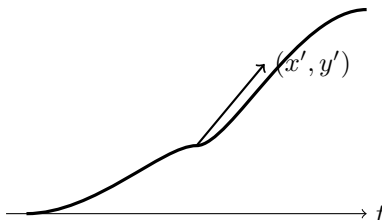
$$\tan \phi = \left| \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + (\frac{1}{2})(-\frac{1}{2})} \right| = \frac{1}{3/4} = \frac{4}{3}.$$

5. Hence the crossing angle is

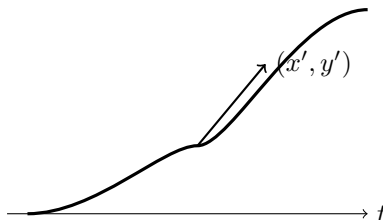
$$\phi = \arctan \left(\frac{4}{3} \right) \approx 53.13^\circ.$$

Question 8**Problem**

A particle has velocity $(x', y') = (t - 2, t^2 - 4)$. Does its path have a vertical tangent? Analyze the tangent and direction of motion at any stationary time.



Question 8 – Solution

**Solution**

1. A regular vertical tangent would require

$$x'(t) = t - 2 = 0 \quad \text{and} \quad y'(t) = t^2 - 4 \neq 0.$$

The first condition gives the only candidate, $t = 2$.

2. At $t = 2$,

$$(x'(2), y'(2)) = (0, 0).$$

Thus the particle is momentarily stopped, and the ordinary vertical-tangent test does not apply.

3. For $t \neq 2$, factor y' and simplify the slope:

$$\frac{dy}{dx} = \frac{t^2 - 4}{t - 2} = \frac{(t - 2)(t + 2)}{t - 2} = t + 2.$$

Consequently,

$$\lim_{t \rightarrow 2} \frac{dy}{dx} = 4,$$

which is finite rather than infinite. The path has a tangent of slope 4 at the stationary point, not a vertical tangent.

4. The direction also changes across $t = 2$. Just before 2, both x' and y' are negative, so the particle moves down and left. Just after 2, both are positive, so it moves up and right. At $t = 2$ itself, the velocity is zero and therefore has no instantaneous direction.

5. Hence

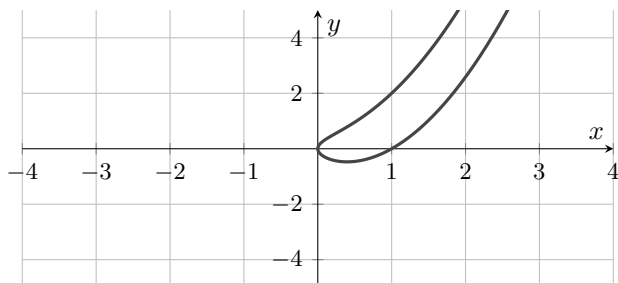
the path has no vertical tangent.

The only candidate is a stationary point where the particle reverses direction along a tangent of slope 4.

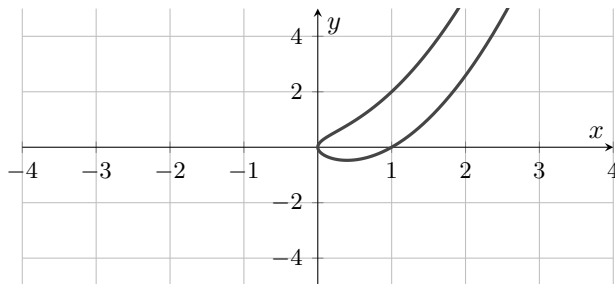
Question 9

Problem

The curve $x = t^2$, $y = t^4 - t$ passes through $(1, 0)$. Find all tangent lines there.



Question 9 – Solution



Solution

1. First find every parameter value that could produce the point $(1, 0)$. From $x = t^2 = 1$,

$$t = 1 \quad \text{or} \quad t = -1.$$

2. Check the y -coordinate for both values:

$$y(1) = 1^4 - 1 = 0, \quad y(-1) = (-1)^4 - (-1) = 2.$$

Only $t = 1$ produces $(1, 0)$, so there can be only one tangent line there.

3. Differentiate:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4t^3 - 1.$$

At $t = 1$, $dx/dt = 2 \neq 0$, and the slope is

$$\frac{dy}{dx} = \frac{4t^3 - 1}{2t} = \frac{4(1)^3 - 1}{2(1)} = \frac{3}{2}.$$

4. Apply point-slope form through $(1, 0)$:

$$y - 0 = \frac{3}{2}(x - 1).$$

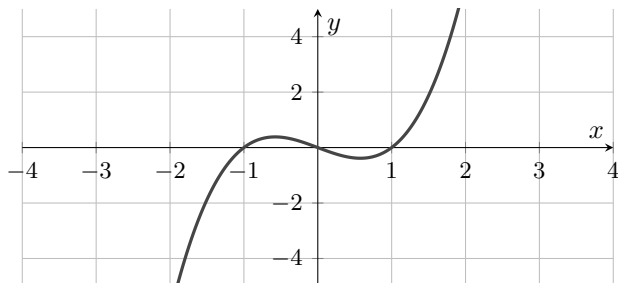
Therefore, the only tangent line at the specified point is

$$\boxed{y = \frac{3}{2}(x - 1)}.$$

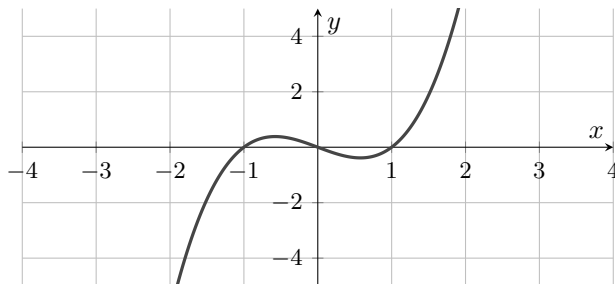
Question 10

Problem

Design a so that $x = t$, $y = t^3 - at$ meets the x -axis at $t = 1$ with slope 2. Is it possible?



Question 10 – Solution



Solution

1. To meet the x -axis at $t = 1$, the y -coordinate must be zero:

$$y(1) = 1^3 - a(1) = 1 - a = 0.$$

Therefore, the position condition requires

$$a = 1.$$

2. Now impose the slope condition. Differentiate:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3t^2 - a.$$

Since $dx/dt = 1$, the tangent slope is

$$\frac{dy}{dx} = 3t^2 - a.$$

3. At $t = 1$, requiring slope 2 gives

$$3 - a = 2 \implies a = 1.$$

Thus the position and slope conditions agree.

4. Verify the result directly. With $a = 1$,

$$y(1) = 1 - 1 = 0, \quad \left. \frac{dy}{dx} \right|_{t=1} = 3 - 1 = 2.$$

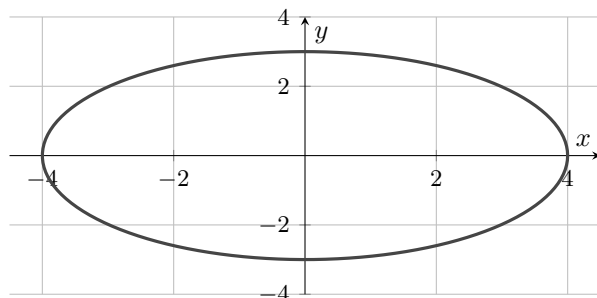
The point is $(x(1), y(1)) = (1, 0)$, and its tangent line is $y = 2(x - 1)$.

5. Hence the design is possible, and the unique value is

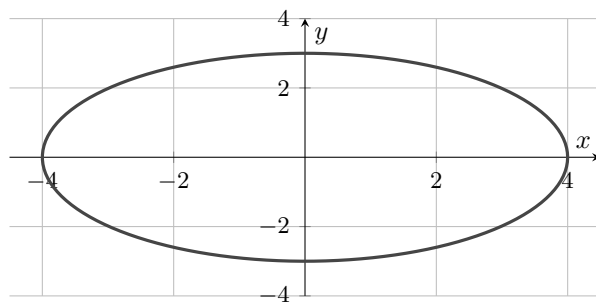
$$\boxed{a = 1}.$$

Question 1**Problem**

Use a parametric integral to find the area of the ellipse $x = 4 \cos t$, $y = 3 \sin t$.



Question 1 – Solution



Solution

1. The interval $0 \leq t \leq \pi/2$ traces the ellipse's first-quadrant arc from $(4, 0)$ to $(0, 3)$. By symmetry, the total area is four times the area in that quadrant.

2. Differentiate the horizontal coordinate:

$$\frac{dx}{dt} = -4 \sin t.$$

Because x decreases on this interval, the first-quadrant area is

$$A_1 = - \int_0^{\pi/2} y(t)x'(t) dt.$$

3. Substitute $y = 3 \sin t$ and $x' = -4 \sin t$:

$$A_1 = - \int_0^{\pi/2} (3 \sin t)(-4 \sin t) dt = 12 \int_0^{\pi/2} \sin^2 t dt.$$

4. Using $\sin^2 t = (1 - \cos 2t)/2$,

$$\int_0^{\pi/2} \sin^2 t dt = \frac{\pi}{4}.$$

Hence $A_1 = 3\pi$.

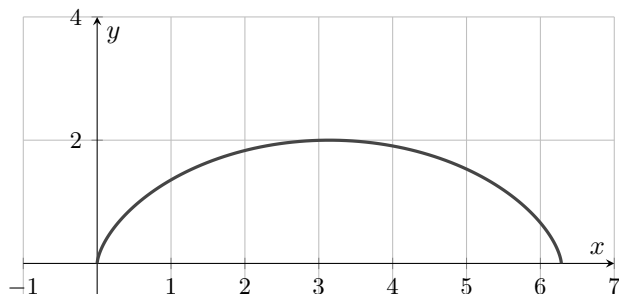
5. Multiply by four quadrants:

$$A = 4A_1 = \boxed{12\pi}.$$

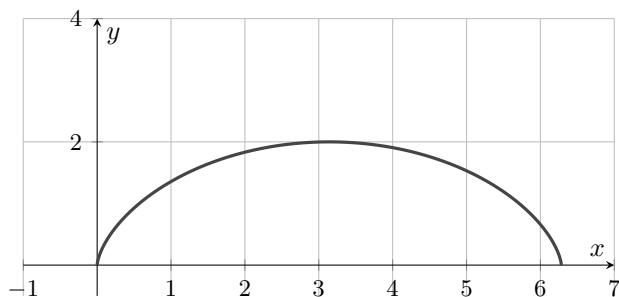
Question 2

Problem

Find the area under one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Question 2 – Solution



Solution

1. The given interval traces one arch from $(0, 0)$ to $(2\pi, 0)$. Since

$$x'(t) = 1 - \cos t \geq 0,$$

the curve moves from left to right, so the area under it is $\int y \, dx$ with the stated bounds.

2. Substitute $y(t) = 1 - \cos t$ and $dx = x'(t)dt$:

$$A = \int_0^{2\pi} y(t)x'(t) \, dt = \int_0^{2\pi} (1 - \cos t)^2 \, dt.$$

3. Expand the integrand:

$$(1 - \cos t)^2 = 1 - 2 \cos t + \cos^2 t.$$

Therefore,

$$A = \int_0^{2\pi} 1 \, dt - 2 \int_0^{2\pi} \cos t \, dt + \int_0^{2\pi} \cos^2 t \, dt.$$

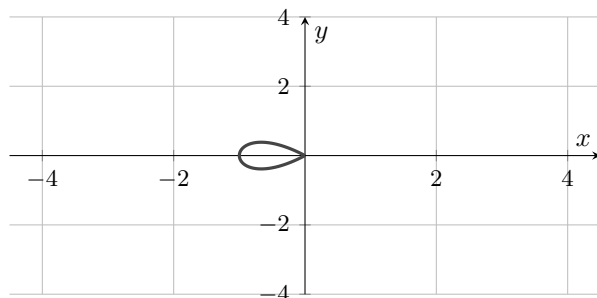
4. These three integrals equal 2π , 0, and π , respectively. Thus

$$\boxed{A = 3\pi}.$$

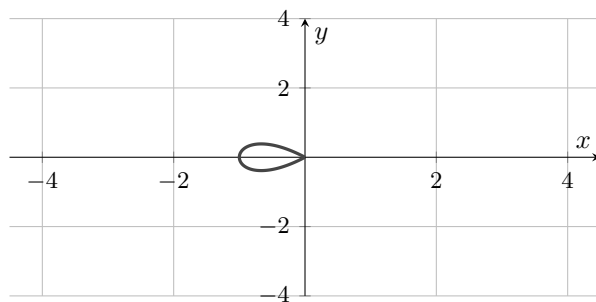
Question 3

Problem

A loop is traced by $x = t^2 - 1$, $y = t^3 - t$. Find the exact area enclosed by the loop.



Question 3 – Solution

**Solution**

1. Locate the parameter values at which the loop closes. Since

$$x = t^2 - 1, \quad y = t(t^2 - 1),$$

both $t = -1$ and $t = 1$ produce the point $(0, 0)$. The interval $-1 \leq t \leq 1$ traces the loop once.

2. Differentiate x :

$$x'(t) = 2t.$$

The signed area integral is

$$\int_{-1}^1 y(t)x'(t) dt = \int_{-1}^1 (t^3 - t)(2t) dt.$$

3. Simplify and integrate:

$$\begin{aligned} \int_{-1}^1 (2t^4 - 2t^2) dt &= \left[\frac{2}{5}t^5 - \frac{2}{3}t^3 \right]_{-1}^1 \\ &= \frac{4}{5} - \frac{4}{3} = -\frac{8}{15}. \end{aligned}$$

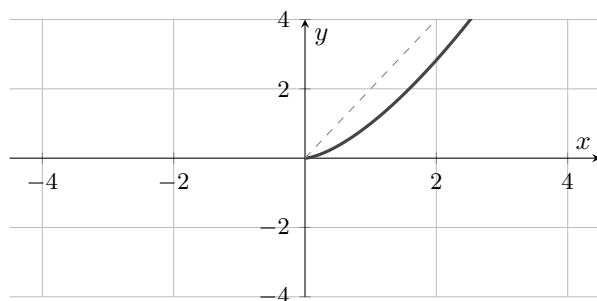
4. The negative sign records the orientation of traversal; geometric area is nonnegative. Hence

$$A = \left| -\frac{8}{15} \right| = \frac{8}{15}.$$

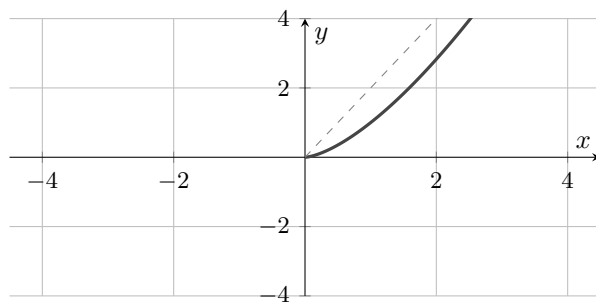
Question 4

Problem

The curve $x = t^2$, $y = t^3$ joins $(0,0)$ to $(4,8)$. Find the area between it and the chord joining those endpoints.



Question 4 – Solution



Solution

1. The endpoints correspond to $t = 0$ and $t = 2$. The chord through $(0, 0)$ and $(4, 8)$ has slope

$$m = \frac{8 - 0}{4 - 0} = 2,$$

so its equation is $y = 2x$.

2. Along the parametric curve, $x = t^2$ and $y = t^3$. The chord's height at the same x -coordinate is

$$y_{\text{chord}} = 2x = 2t^2.$$

For $0 \leq t \leq 2$, $2t^2 - t^3 = t^2(2 - t) \geq 0$, so the chord lies above the curve.

3. Since $dx/dt = 2t$, the area between them is

$$A = \int_0^2 (y_{\text{chord}} - y_{\text{curve}}) dx = \int_0^2 (2t^2 - t^3)(2t) dt.$$

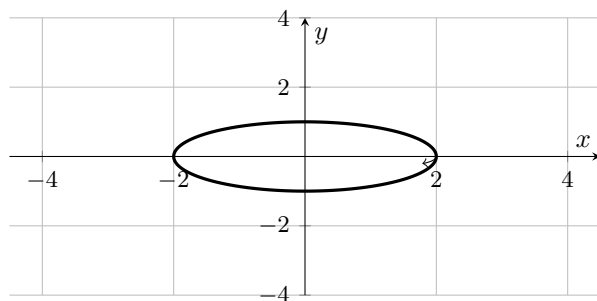
4. Evaluate:

$$\begin{aligned} A &= \int_0^2 (4t^3 - 2t^4) dt \\ &= \left[t^4 - \frac{2}{5}t^5 \right]_0^2 = 16 - \frac{64}{5} = \boxed{\frac{16}{5}}. \end{aligned}$$

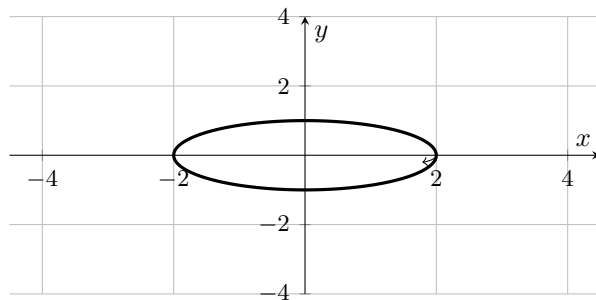
Question 5

Problem

A simple, piecewise smooth closed curve is traversed once clockwise. Explain why $\int y \, dx$ may be positive even though area cannot be negative.



Question 5 – Solution



Solution

1. A line integral around a closed curve is signed: reversing the direction of traversal reverses the sign because dx changes sign.
2. For a positively oriented, counterclockwise curve C , Green's theorem gives

$$A = - \oint_C y \, dx.$$

Thus $\oint_C y \, dx = -A$ for counterclockwise traversal.

3. If the same boundary is traversed clockwise, the line integral changes sign:

$$\oint_{-C} y \, dx = - \oint_C y \, dx = A.$$

Therefore, $\int y \, dx$ can be positive for clockwise motion; the sign describes orientation, not a negative or positive physical area by itself.

4. A direction-independent formula for geometric area is

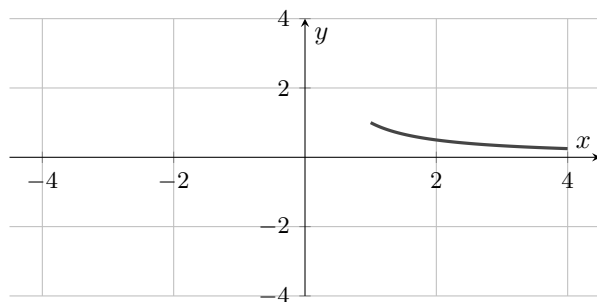
$$A = \left| \oint_C y \, dx \right|.$$

Equivalently, use $-\oint_C y \, dx$ with counterclockwise orientation or reverse the bounds when necessary.

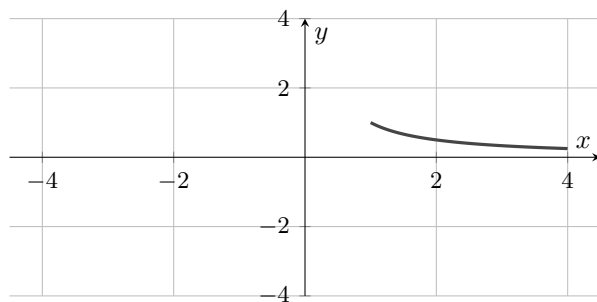
Question 6

Problem

Find the area swept under $x = e^t$, $y = e^{-t}$ from $t = 0$ to $t = \ln 4$.



Question 6 – Solution

**Solution**

1. On $0 \leq t \leq \ln 4$, $x = e^t$ increases from 1 to 4 because

$$x'(t) = e^t > 0.$$

Therefore, the requested area under the curve is directly $\int y \, dx$.

2. Substitute the parametric expressions:

$$A = \int_0^{\ln 4} y(t)x'(t) \, dt = \int_0^{\ln 4} e^{-t} e^t \, dt.$$

3. The exponential factors cancel:

$$e^{-t} e^t = 1.$$

Hence

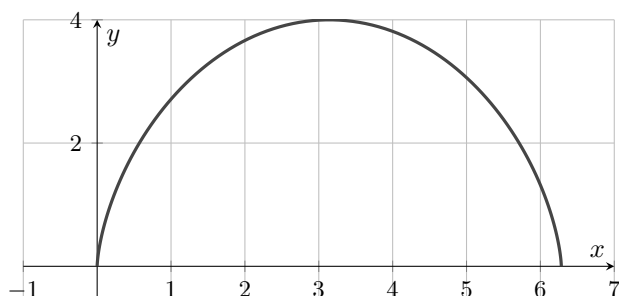
$$A = \int_0^{\ln 4} 1 \, dt = [t]_0^{\ln 4} = \boxed{\ln 4}.$$

4. As a check, eliminating t gives $y = 1/x$, and $\int_1^4 (1/x) \, dx = \ln 4$, the same result.

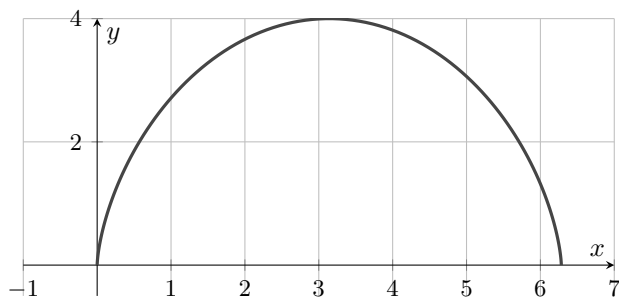
Question 7

Problem

The parametric curve $x = t - \sin t$, $y = 2 - 2 \cos t$ is a stretched cycloid. Predict, then compute, how its arch area compares with the standard cycloid.



Question 7 – Solution



Solution

1. Compared with the standard cycloid $x = t - \sin t$, $y = 1 - \cos t$, the x -coordinates are unchanged while every y -coordinate is multiplied by 2. A vertical stretch by a factor of 2 doubles area, so the predicted arch area is $2(3\pi) = 6\pi$.
2. Verify this prediction directly on $0 \leq t \leq 2\pi$. Differentiate x :

$$x'(t) = 1 - \cos t.$$

Since $x'(t) \geq 0$, the arch is traced from left to right.

3. Substitute $y = 2(1 - \cos t)$:

$$A = \int_0^{2\pi} y(t)x'(t) dt = 2 \int_0^{2\pi} (1 - \cos t)^2 dt.$$

4. From

$$\int_0^{2\pi} (1 - \cos t)^2 dt = 2\pi + \pi = 3\pi,$$

it follows that

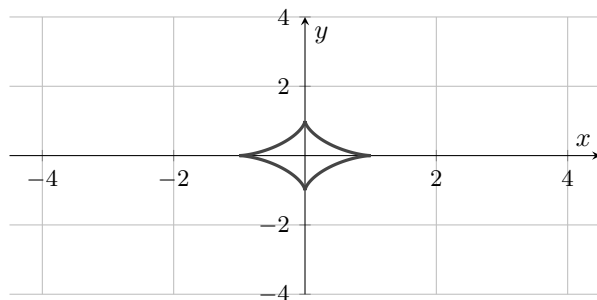
$$\boxed{A = 2(3\pi) = 6\pi}.$$

The computation agrees with the scaling prediction.

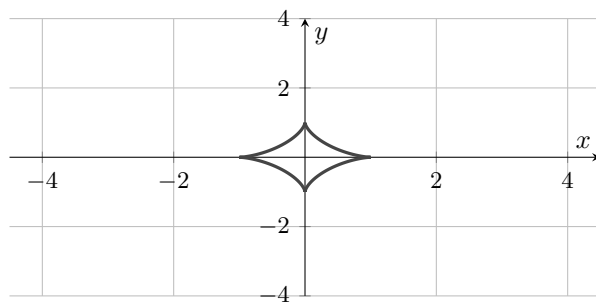
Question 8

Problem

Find the area enclosed by $x = \cos^3 t$, $y = \sin^3 t$ without first eliminating the parameter.



Question 8 – Solution

**Solution**

1. The interval $0 \leq t \leq \pi/2$ traces the first-quadrant arc from $(1, 0)$ to $(0, 1)$. By symmetry, the enclosed area is four times the first-quadrant area.
2. It is convenient to integrate with respect to y :

$$A_1 = \int x \, dy = \int_0^{\pi/2} x(t) y'(t) \, dt.$$

Since $y = \sin^3 t$,

$$y'(t) = 3 \sin^2 t \cos t.$$

3. Substitute $x = \cos^3 t$:

$$A_1 = 3 \int_0^{\pi/2} \sin^2 t \cos^4 t \, dt.$$

Using the even-power integral (or power-reduction identities),

$$\int_0^{\pi/2} \sin^2 t \cos^4 t \, dt = \frac{\pi}{32}.$$

Thus $A_1 = 3\pi/32$.

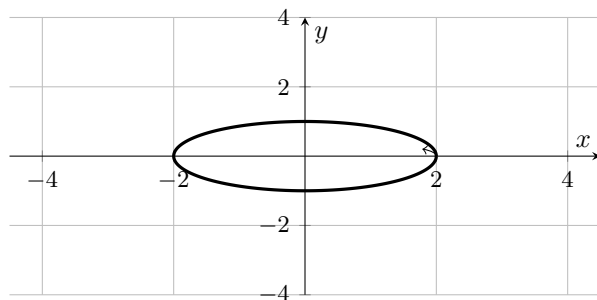
4. Multiply by four quadrants:

$$A = 4A_1 = 4 \left(\frac{3\pi}{32} \right) = \boxed{\frac{3\pi}{8}}.$$

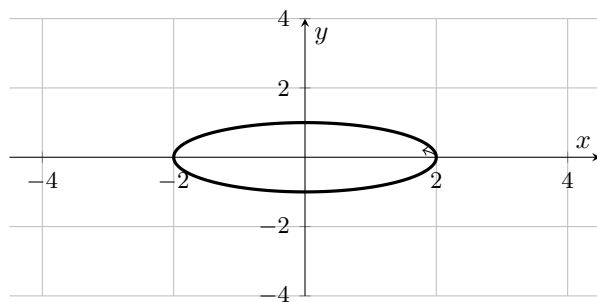
Question 9

Problem

A designer reverses a parametrization on $0 \leq t \leq 1$ by replacing t with $1 - t$. What happens to the signed area integral $\int yx' dt$?



Question 9 – Solution

**Solution**

1. Let the original parametrization be $(x(t), y(t))$ on $0 \leq t \leq 1$, with signed integral

$$I = \int_0^1 y(t)x'(t) dt.$$

2. Under the reversed parametrization, set $u = 1 - t$. The new coordinates are

$$\tilde{x}(t) = x(1 - t), \quad \tilde{y}(t) = y(1 - t).$$

By the chain rule,

$$\tilde{x}'(t) = -x'(1 - t).$$

3. The reversed signed integral is

$$\begin{aligned} \tilde{I} &= \int_0^1 \tilde{y}(t)\tilde{x}'(t) dt \\ &= - \int_0^1 y(1 - t)x'(1 - t) dt. \end{aligned}$$

Substituting $u = 1 - t$ shows that $\tilde{I} = -I$.

4. Therefore, reversing the parametrization changes the sign of the signed area integral but not its magnitude:

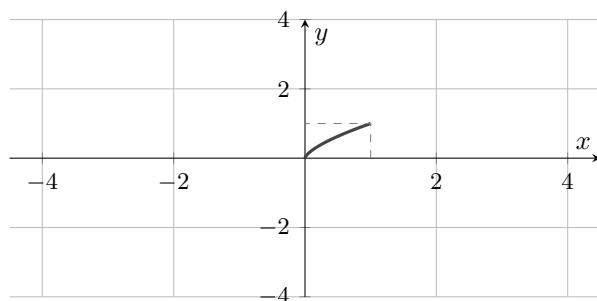
$$\boxed{\tilde{I} = -I, \quad |\tilde{I}| = |I|}.$$

The geometric area is unchanged.

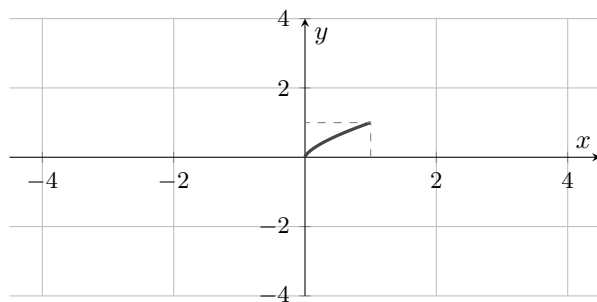
Question 10

Problem

For $x = t^3$, $y = t^2$, $0 \leq t \leq 1$, compare the area under the curve with the area to its left inside the unit square.



Question 10 – Solution

**Solution**

1. As t increases from 0 to 1, the curve runs from $(0, 0)$ to $(1, 1)$ with

$$x'(t) = 3t^2 \geq 0.$$

The area below the curve is therefore

$$A_{\text{below}} = \int_0^1 y(t)x'(t) dt.$$

2. Substitute $y = t^2$ and $x' = 3t^2$:

$$A_{\text{below}} = \int_0^1 3t^4 dt = \left[\frac{3}{5}t^5 \right]_0^1 = \boxed{\frac{3}{5}}.$$

3. The curve divides the unit square into the area below it and the area to its left (equivalently, above it). Since the square has area 1,

$$A_{\text{left}} = 1 - A_{\text{below}} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}.$$

4. This can also be checked directly with $dy = 2t dt$:

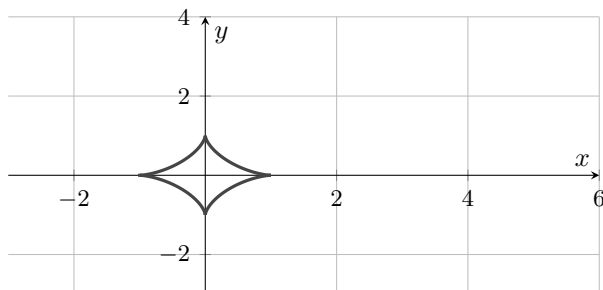
$$A_{\text{left}} = \int_0^1 x dy = \int_0^1 t^3(2t) dt = \frac{2}{5}.$$

Thus the area under the curve is larger by $1/5$.

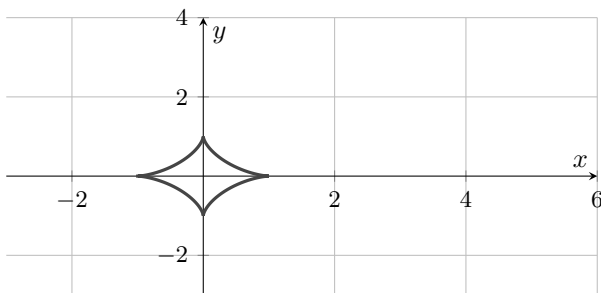
Exercise 1

Problem

Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$.



Exercise 1: Solution



Solution

1. The astroid is symmetric about both coordinate axes, so compute the length in the first quadrant, $0 \leq t \leq \pi/2$, and multiply by four.

2. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

3. Simplify the speed:

$$\begin{aligned} \sqrt{x'^2 + y'^2} &= 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} \\ &= 3\sqrt{\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} \\ &= 3|\sin t \cos t|. \end{aligned}$$

On $0 \leq t \leq \pi/2$, both factors are nonnegative, so the speed is $3 \sin t \cos t$.

4. The first-quadrant length is

$$L_1 = 3 \int_0^{\pi/2} \sin t \cos t \, dt = \frac{3}{2} [\sin^2 t]_0^{\pi/2} = \frac{3}{2}.$$

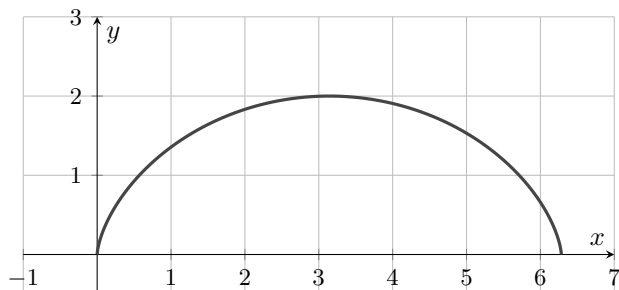
5. Therefore, the total length is

$$L = 4L_1 = 4 \left(\frac{3}{2} \right) = 6.$$

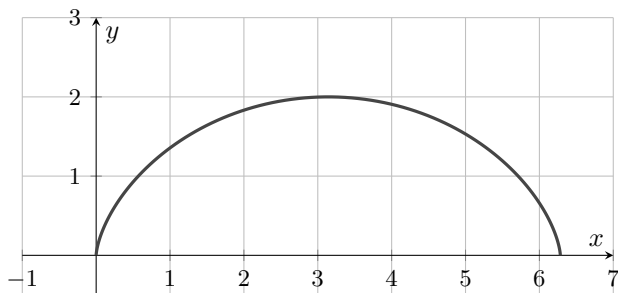
Exercise 2

Problem

Find the length of one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Exercise 2: Solution



Solution

1. Differentiate the parametric equations:

$$x'(t) = 1 - \cos t, \quad y'(t) = \sin t.$$

2. Compute and simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= (1 - \cos t)^2 + \sin^2 t \\ &= 1 - 2 \cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2 \cos t = 4 \sin^2 \left(\frac{t}{2} \right). \end{aligned}$$

Hence the speed is

$$2 \left| \sin \left(\frac{t}{2} \right) \right|.$$

3. On $0 \leq t \leq 2\pi$, we have $0 \leq t/2 \leq \pi$, so $\sin(t/2) \geq 0$. Thus

$$L = 2 \int_0^{2\pi} \sin \left(\frac{t}{2} \right) dt.$$

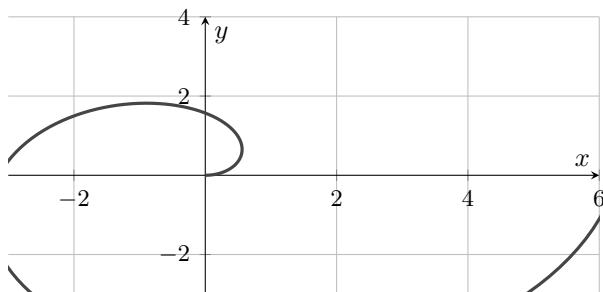
4. Evaluate the integral:

$$L = 2 \left[-2 \cos \left(\frac{t}{2} \right) \right]_0^{2\pi} = -4(\cos \pi - \cos 0) = \boxed{8}.$$

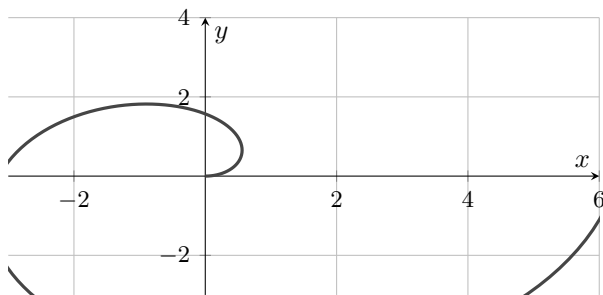
Exercise 3

Problem

A spiral-like path is $x = t \cos t$, $y = t \sin t$, $0 \leq t \leq 2\pi$. Set up its exact length and simplify the speed.



Exercise 3: Solution



Solution

1. Apply the product rule to each coordinate:

$$x'(t) = \cos t - t \sin t, \quad y'(t) = \sin t + t \cos t.$$

2. Square and add:

$$\begin{aligned} x'^2 + y'^2 &= (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 \\ &= \cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) \\ &\quad - 2t \sin t \cos t + 2t \sin t \cos t \\ &= 1 + t^2. \end{aligned}$$

The cross terms cancel exactly.

3. Therefore, the speed is $\sqrt{1+t^2}$, and the exact length is

$$L = \int_0^{2\pi} \sqrt{1+t^2} dt.$$

4. If an evaluated form is desired, use

$$\int \sqrt{1+t^2} dt = \frac{1}{2} \left(t\sqrt{1+t^2} + \operatorname{arsinh} t \right),$$

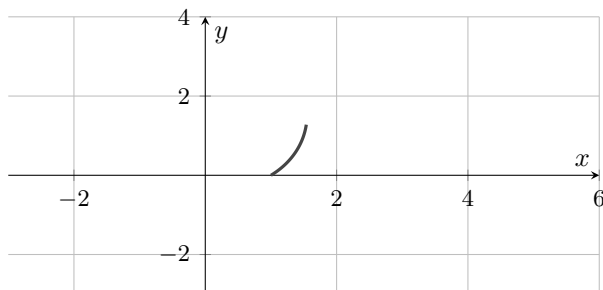
giving

$$L = \pi\sqrt{1+4\pi^2} + \frac{1}{2} \operatorname{arsinh}(2\pi).$$

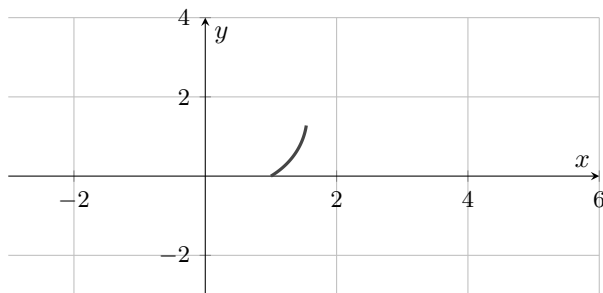
Exercise 4

Problem

Show that $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \ln 2$, has length $\sqrt{2}$.



Exercise 4: Solution



Solution

1. Differentiate using the product rule:

$$x'(t) = e^t(\cos t - \sin t), \quad y'(t) = e^t(\sin t + \cos t).$$

2. Compute the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= e^{2t} [(\cos t - \sin t)^2 + (\sin t + \cos t)^2] \\ &= e^{2t} [2\cos^2 t + 2\sin^2 t] \\ &= 2e^{2t}. \end{aligned}$$

Since $e^t > 0$, the speed is $\sqrt{2}e^t$.

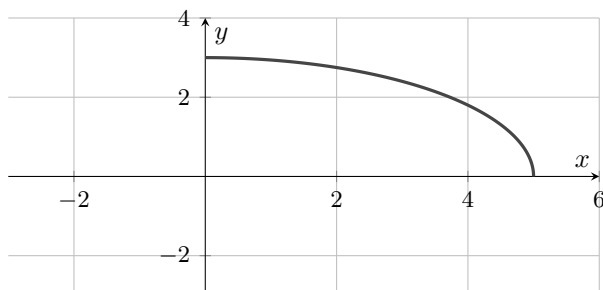
3. Set up and evaluate the length:

$$\begin{aligned} L &= \int_0^{\ln 2} \sqrt{2}e^t \, dt \\ &= \sqrt{2} [e^t]_0^{\ln 2} \\ &= \sqrt{2}(2 - 1) = \boxed{\sqrt{2}}. \end{aligned}$$

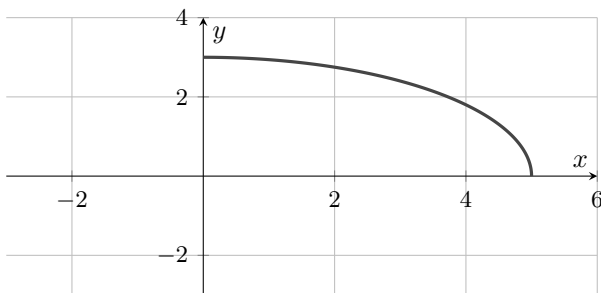
Exercise 5

Problem

A quarter ellipse is $x = 5 \cos t$, $y = 3 \sin t$, $0 \leq t \leq \pi/2$. Which simple quarter-circle lengths bound it?



Exercise 5: Solution



Solution

1. Differentiate:

$$x'(t) = -5 \sin t, \quad y'(t) = 3 \cos t.$$

Thus the speed is

$$v(t) = \sqrt{25 \sin^2 t + 9 \cos^2 t}.$$

2. Rewrite the expression under the square root in two useful ways:

$$25 \sin^2 t + 9 \cos^2 t = 9 + 16 \sin^2 t = 25 - 16 \cos^2 t.$$

Since $0 \leq \sin^2 t, \cos^2 t \leq 1$, it follows that

$$9 \leq v(t)^2 \leq 25, \quad \text{so} \quad 3 \leq v(t) \leq 5.$$

3. Integrate these bounds over an interval of width $\pi/2$:

$$3 \int_0^{\pi/2} dt \leq L \leq 5 \int_0^{\pi/2} dt.$$

Therefore,

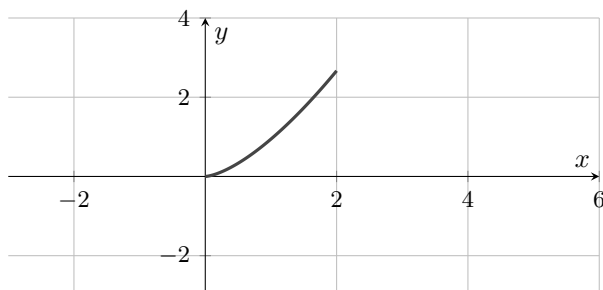
$$\boxed{\frac{3\pi}{2} \leq L \leq \frac{5\pi}{2}}.$$

These are the quarter-circumferences of circles with radii 3 and 5.

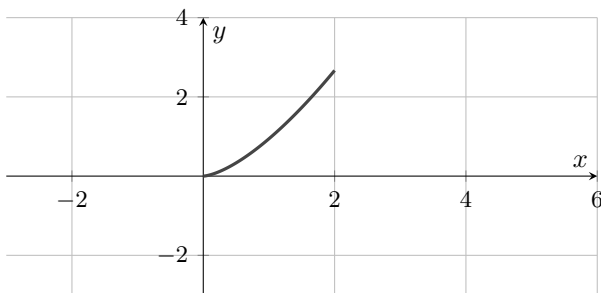
Exercise 6

Problem

Find the length of $x = t^2/2$, $y = t^3/3$, $0 \leq t \leq 2$.



Exercise 6: Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = t, \quad y'(t) = t^2.$$

2. The speed is

$$\sqrt{t^2 + t^4} = \sqrt{t^2(1 + t^2)} = |t|\sqrt{1 + t^2}.$$

Since $0 \leq t \leq 2$, $|t| = t$.

3. Therefore,

$$L = \int_0^2 t\sqrt{1 + t^2} dt.$$

Let $u = 1 + t^2$, so $du = 2t dt$. Then

$$\begin{aligned} L &= \frac{1}{2} \int_1^5 u^{1/2} du \\ &= \frac{1}{3} \left[u^{3/2} \right]_1^5 \\ &= \frac{1}{3} (5^{3/2} - 1). \end{aligned}$$

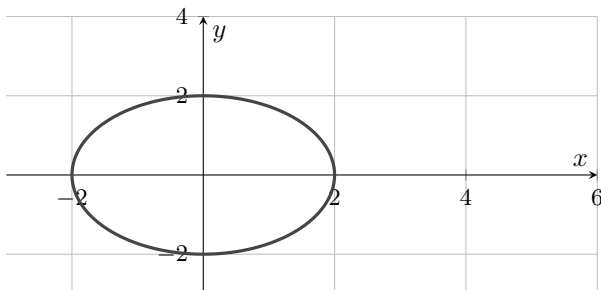
4. Hence

$$L = \frac{5\sqrt{5} - 1}{3}.$$

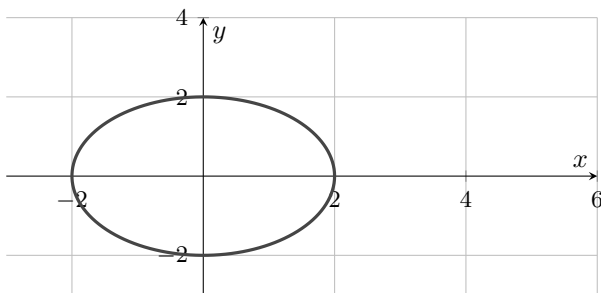
Exercise 7

Problem

A curve is reparametrized from t to $u = t^3$. Explain why its length is unchanged when both parameters increase.



Exercise 7: Solution



Solution

1. Let the original curve be $\mathbf{r}(t)$ for $a \leq t \leq b$, and suppose $u = t^3$ increases on this interval. Then

$$t = u^{1/3}, \quad \frac{dt}{du} = \frac{1}{3u^{2/3}}$$

wherever this derivative is defined.

2. By the chain rule, the velocity in the new parameter is

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \frac{dt}{du}.$$

Taking magnitudes gives

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \left\| \frac{d\mathbf{r}}{dt} \right\| \left| \frac{dt}{du} \right|.$$

3. Because both parameters increase, $dt/du \geq 0$ and $dt = (dt/du)du$. Hence

$$\left\| \frac{d\mathbf{r}}{du} \right\| du = \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$

Integrating over corresponding endpoints yields the same value in either parameter.

4. Therefore, reparametrization changes the rate at which the curve is traced but not the geometric arc length:

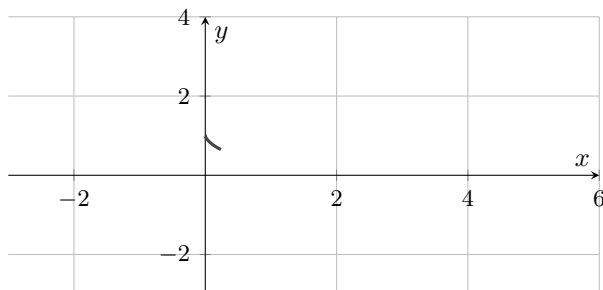
$$\boxed{L_u = L_t}.$$

A possible derivative singularity at the single point $u = 0$ does not change the value of the proper or corresponding improper integral.

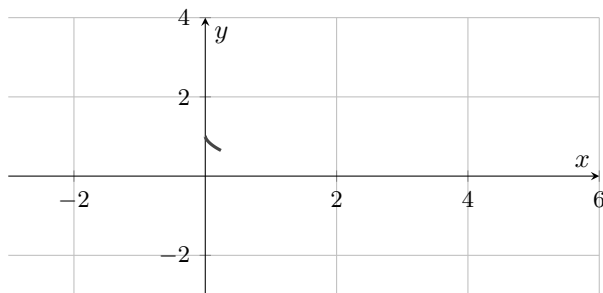
Exercise 8

Problem

Find the length of the tractable curve $x = t - \tanh t$, $y = \operatorname{sech} t$, $0 \leq t \leq 1$.



Exercise 8: Solution



Solution

1. Differentiate and use $1 - \operatorname{sech}^2 t = \tanh^2 t$:

$$x'(t) = 1 - \operatorname{sech}^2 t = \tanh^2 t,$$

$$y'(t) = -\operatorname{sech} t \tanh t.$$

2. Simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t \\ &= \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t) \\ &= \tanh^2 t. \end{aligned}$$

Thus the speed is $|\tanh t|$.

3. On $0 \leq t \leq 1$, $\tanh t \geq 0$, so

$$L = \int_0^1 \tanh t \, dt.$$

Since $d(\ln \cosh t)/dt = \tanh t$,

$$L = [\ln(\cosh t)]_0^1 = \ln(\cosh 1) - \ln 1.$$

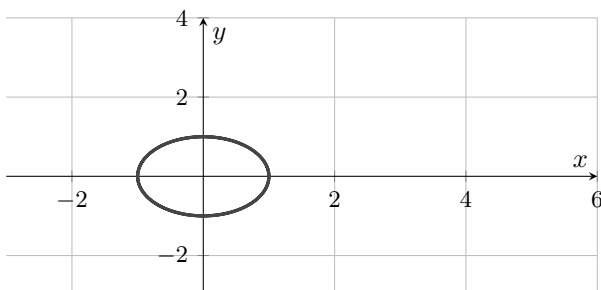
4. Therefore,

$$\boxed{L = \ln(\cosh 1)}.$$

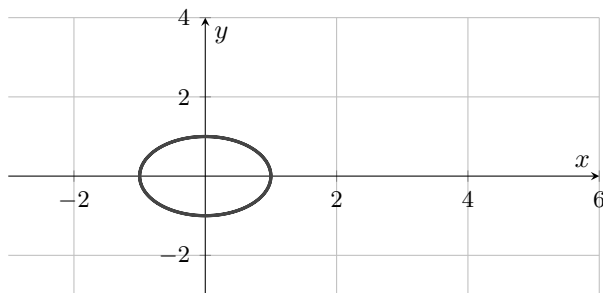
Exercise 9

Problem

A particle goes around the unit circle twice. Is its path length 2π or 4π ? Justify using $x = \cos 2t$, $y = \sin 2t$, $0 \leq t \leq 2\pi$.



Exercise 9: Solution



Solution

1. Differentiate:

$$x'(t) = -2 \sin 2t, \quad y'(t) = 2 \cos 2t.$$

2. The speed is constant:

$$\sqrt{x'^2 + y'^2} = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2.$$

3. Integrate over the full parameter interval:

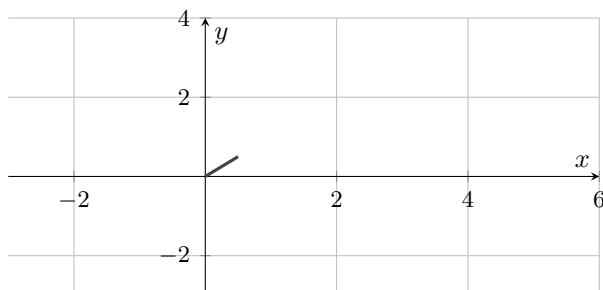
$$L = \int_0^{2\pi} 2 \, dt = \boxed{4\pi}.$$

4. The angle of the point is $2t$, which increases from 0 to 4π . Therefore, the unit circle is traversed twice. Its geometric circumference is 2π , but the particle travels that circumference twice, producing total traveled length 4π .

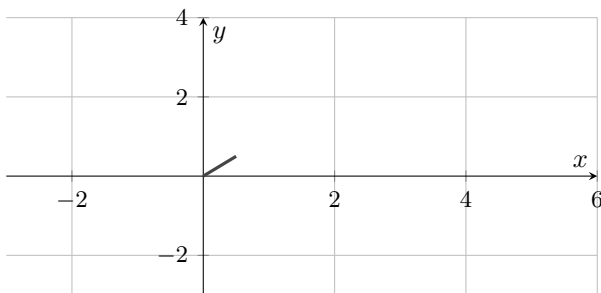
Exercise 10

Problem

Find $a > 0$ so that the line path $x = at$, $y = (1 - a)t$, $0 \leq t \leq 1$, has minimum length.



Exercise 10: Solution



Solution

1. Differentiate the line parametrization:

$$x'(t) = a, \quad y'(t) = 1 - a.$$

The speed is constant, so

$$L(a) = \int_0^1 \sqrt{a^2 + (1-a)^2} dt = \sqrt{a^2 + (1-a)^2}.$$

2. Because the square-root function is increasing, minimizing $L(a)$ is equivalent to minimizing

$$L(a)^2 = a^2 + (1-a)^2 = 2a^2 - 2a + 1.$$

Complete the square:

$$2a^2 - 2a + 1 = 2 \left(a - \frac{1}{2} \right)^2 + \frac{1}{2}.$$

3. The squared term is smallest when $a = 1/2$, which satisfies $a > 0$. Therefore,

$$\boxed{a = \frac{1}{2}}.$$

4. At this value, the minimum length is

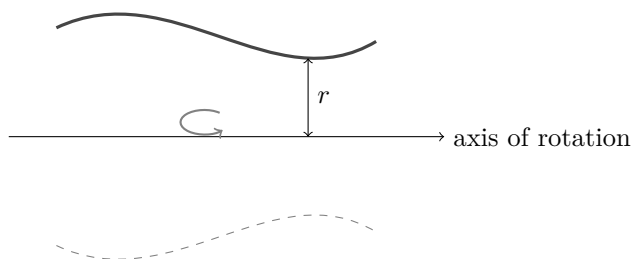
$$L_{\min} = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}}.$$

Geometrically, the endpoint $(a, 1-a)$ is then $(1/2, 1/2)$, the closest point on the line $x + y = 1$ to the origin.

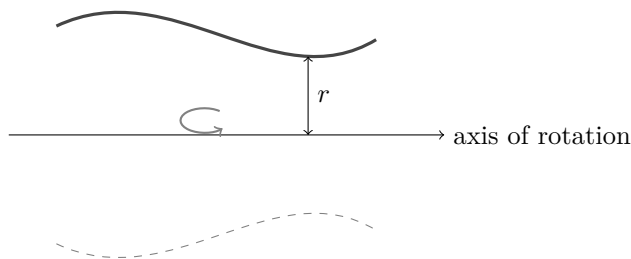
Question 1

Problem

Rotate the line segment $x = t$, $y = 1 - t$, $0 \leq t \leq 1$, about the x -axis. Find the surface area and identify the surface.



Question 1 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 1, \quad y'(t) = -1.$$

Therefore,

$$ds = \sqrt{1^2 + (-1)^2} dt = \sqrt{2} dt.$$

2. The distance from (x, y) to the x -axis is $y = 1 - t$, which is nonnegative on $0 \leq t \leq 1$.
3. Apply the surface-area formula:

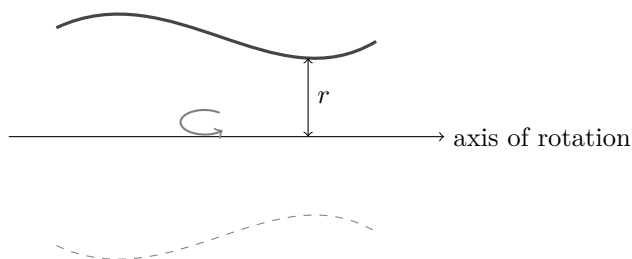
$$\begin{aligned} S &= 2\pi \int_0^1 y ds \\ &= 2\pi\sqrt{2} \int_0^1 (1-t) dt \\ &= 2\pi\sqrt{2} \left[t - \frac{t^2}{2} \right]_0^1 = \boxed{\pi\sqrt{2}}. \end{aligned}$$

4. The segment runs from $(0, 1)$ to $(1, 0)$. Rotating it about the x -axis produces a right circular cone with radius 1, height 1, and slant height $\sqrt{2}$. The result agrees with $\pi r \ell = \pi\sqrt{2}$.

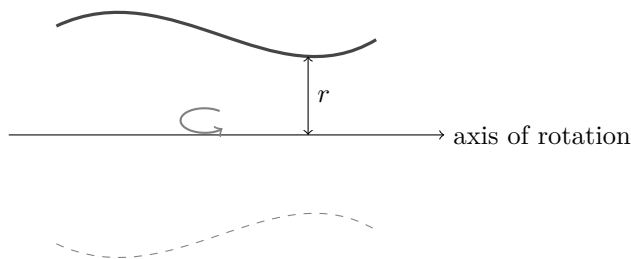
Question 2

Problem

The upper semicircle $x = R \cos t$, $y = R \sin t$, $0 \leq t \leq \pi$, rotates about the x -axis. Recover the sphere's area.



Question 2 – Solution



Solution

1. Differentiate:

$$x'(t) = -R \sin t, \quad y'(t) = R \cos t.$$

Thus

$$ds = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt = R dt.$$

2. On $0 \leq t \leq \pi$, the generating semicircle lies above the x -axis, so its radius of rotation is

$$y = R \sin t \geq 0.$$

This semicircle generates the sphere exactly once.

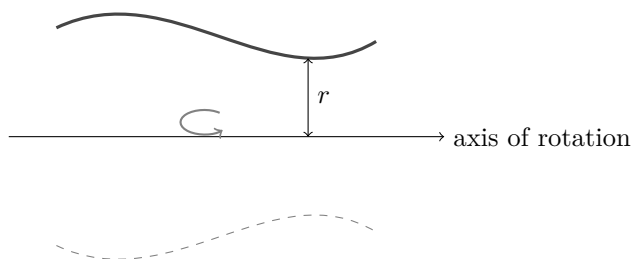
3. Substitute into $S = 2\pi \int y ds$:

$$\begin{aligned} S &= 2\pi \int_0^\pi (R \sin t)(R dt) \\ &= 2\pi R^2 [-\cos t]_0^\pi \\ &= 2\pi R^2 (2) = \boxed{4\pi R^2}. \end{aligned}$$

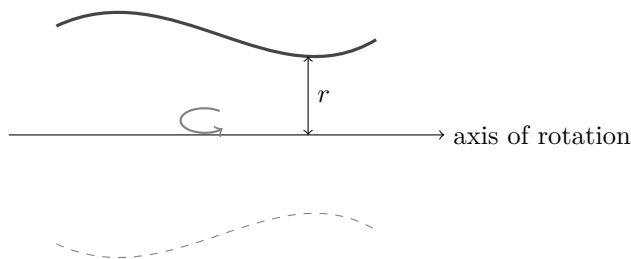
Question 3

Problem

A circle $x = 3 + \cos t$, $y = \sin t$ rotates about the y -axis. Find the torus area without citing a memorized formula.



Question 3 – Solution



Solution

1. The interval $0 \leq t \leq 2\pi$ traces the generating circle once. Differentiate:

$$x'(t) = -\sin t, \quad y'(t) = \cos t.$$

Hence

$$ds = \sqrt{\sin^2 t + \cos^2 t} dt = dt.$$

2. The circle lies between $x = 2$ and $x = 4$, entirely to the right of the y -axis. Therefore, the radius of rotation is the nonnegative distance

$$x = 3 + \cos t.$$

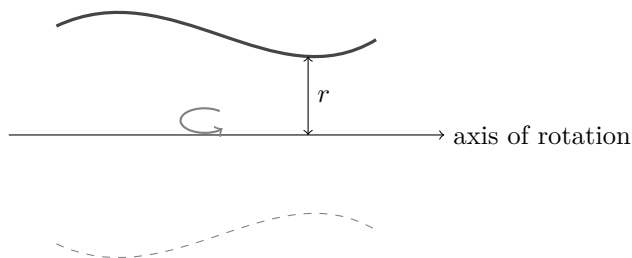
3. Apply $S = 2\pi \int x ds$:

$$\begin{aligned} S &= 2\pi \int_0^{2\pi} (3 + \cos t) dt \\ &= 2\pi [3t + \sin t]_0^{2\pi} \\ &= 2\pi(6\pi) = \boxed{12\pi^2}. \end{aligned}$$

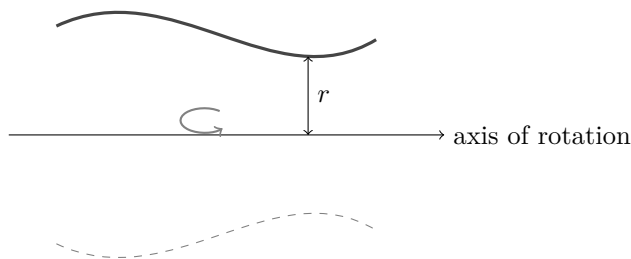
Question 4

Problem

Set up, but do not evaluate, the area formed by rotating $x = t^2$, $y = t^3$, $0 \leq t \leq 1$, about the y -axis.



Question 4 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 2t, \quad y'(t) = 3t^2.$$

2. Compute the arc-length element:

$$\begin{aligned} ds &= \sqrt{(2t)^2 + (3t^2)^2} dt \\ &= \sqrt{4t^2 + 9t^4} dt = |t|\sqrt{4 + 9t^2} dt. \end{aligned}$$

Since $0 \leq t \leq 1$, this becomes $ds = t\sqrt{4 + 9t^2} dt$.

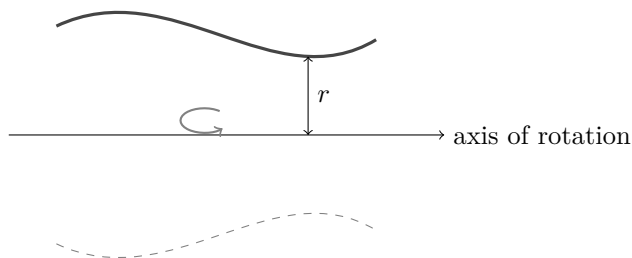
3. Rotation is about the y -axis, so the radius is the distance $x = t^2$.
4. Therefore, without evaluating,

$$\begin{aligned} S &= 2\pi \int_0^1 x(t) ds \\ &= \boxed{2\pi \int_0^1 t^3 \sqrt{4 + 9t^2} dt}. \end{aligned}$$

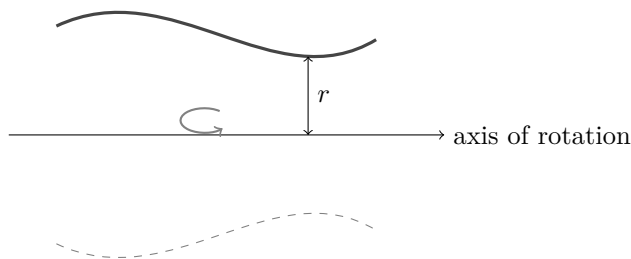
Question 5

Problem

Why is rotating $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$, about the x -axis a bad direct use of the surface formula?



Question 5 – Solution



Solution

1. The interval $0 \leq t \leq \pi$ traces the upper semicircle, while $\pi \leq t \leq 2\pi$ traces the lower semicircle.
2. When either semicircle is rotated about the x -axis, it generates the entire unit sphere. A point (x, y) and its reflection $(x, -y)$ produce the same circle of revolution because both are the same distance $|y|$ from the axis.
3. If the full interval is used with the correct radius $|y|$, then $ds = dt$ and

$$2\pi \int_0^{2\pi} |\sin t| dt = 8\pi,$$

which is twice the unit sphere's area. The formula counts the repeated sweep.

4. If one incorrectly uses signed $y = \sin t$ instead, the upper and lower contributions cancel and give zero, which is also not physical surface area.
5. Use one generating semicircle only, for example

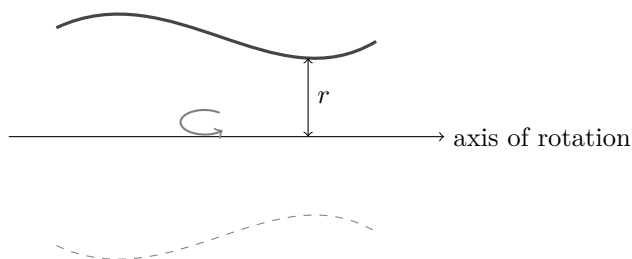
$$S = 2\pi \int_0^\pi \sin t dt = \boxed{4\pi}.$$

Thus the full-circle interval is unsuitable because it double-covers the surface.

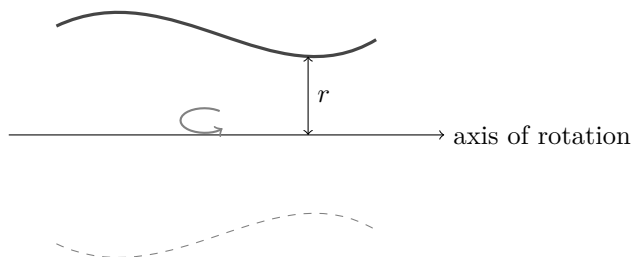
Question 6

Problem

Rotate $x = e^t$, $y = e^{-t}$, $0 \leq t \leq \ln 2$, about the x -axis. Write the exact surface-area integral in simplest form.



Question 6 – Solution



Solution

1. Differentiate:

$$x'(t) = e^t, \quad y'(t) = -e^{-t}.$$

Therefore,

$$ds = \sqrt{e^{2t} + e^{-2t}} dt.$$

2. The curve lies above the x -axis because $y = e^{-t} > 0$, so the radius of rotation is $y = e^{-t}$.
3. Multiply the radius by ds and simplify:

$$\begin{aligned} y ds &= e^{-t} \sqrt{e^{2t} + e^{-2t}} dt \\ &= \sqrt{e^{-2t}(e^{2t} + e^{-2t})} dt \\ &= \sqrt{1 + e^{-4t}} dt. \end{aligned}$$

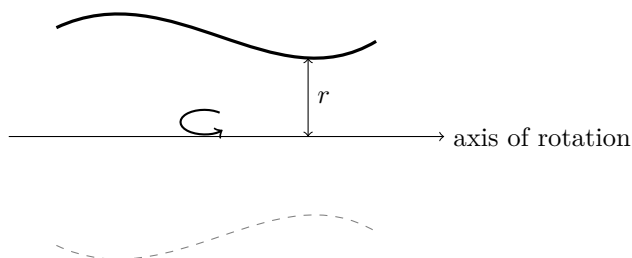
4. Hence the requested exact integral is

$$S = 2\pi \int_0^{\ln 2} \sqrt{1 + e^{-4t}} dt.$$

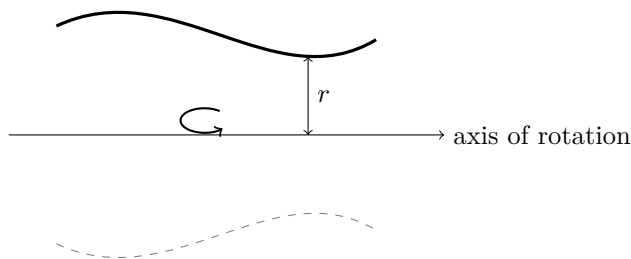
Question 7

Problem

A cylinder is generated parametrically by rotating the horizontal segment $x = t, y = h, 0 \leq t \leq L$, about the x -axis, where $h, L > 0$. Derive its lateral area.



Question 7 – Solution



Solution

1. Differentiate the horizontal segment:

$$x'(t) = 1, \quad y'(t) = 0.$$

Hence

$$ds = \sqrt{1^2 + 0^2} dt = dt.$$

2. Every point of the segment is the constant distance h from the x -axis, assuming $h \geq 0$. Thus the radius of rotation is h .
3. Apply the surface-area formula:

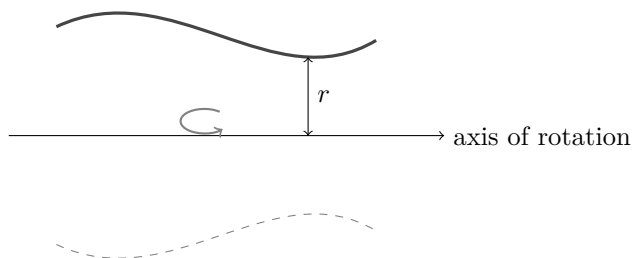
$$\begin{aligned} S &= 2\pi \int_0^L h dt \\ &= 2\pi h[t]_0^L = \boxed{2\pi hL}. \end{aligned}$$

4. This is circumference times length: $(2\pi h)L$, which is precisely the lateral area of a cylinder of radius h and length L .

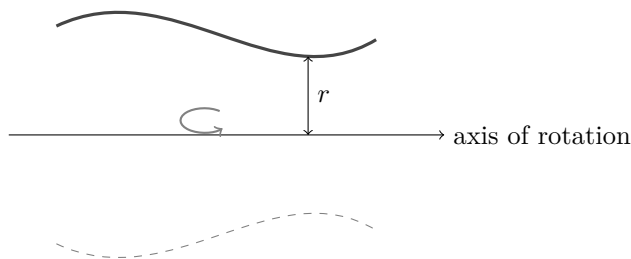
Question 8

Problem

A curve crosses the axis of rotation. What precaution is needed in $S = 2\pi \int y \, ds$?



Question 8 – Solution



Solution

1. In $S = 2\pi \int y \, ds$, the factor multiplying 2π must be the radius of rotation. A radius is a distance and therefore cannot be negative.
2. When the axis is the x -axis, the distance from $(x(t), y(t))$ to that axis is

$$r(t) = |y(t)|,$$

not signed $y(t)$.

3. The correct general formula is therefore

$$S = 2\pi \int_a^b |y(t)| \sqrt{x'(t)^2 + y'(t)^2} \, dt.$$

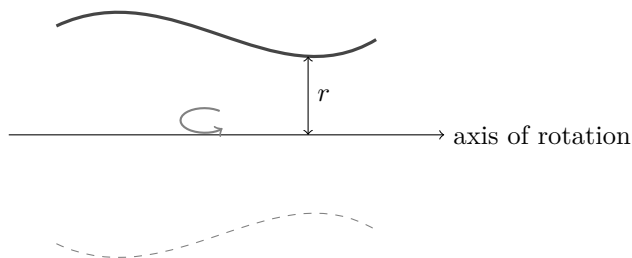
Equivalently, split the integral at every parameter value where $y(t) = 0$ and use the appropriate sign on each subinterval.

4. Without the absolute value, contributions below the axis would be negative and could cancel contributions above the axis. Also check whether portions on opposite sides of the axis generate the same physical surface; if they do, restrict the interval to prevent double-counting.

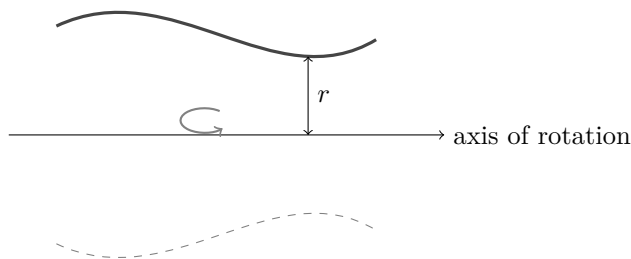
Question 9

Problem

Rotate the astroid quadrant $x = \cos^3 t, y = \sin^3 t, 0 \leq t \leq \pi/2$, about the x -axis. Set up a simplified integral.



Question 9 – Solution



Solution

1. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

2. As in the astroid arc-length calculation,

$$ds = 3 |\sin t \cos t| dt.$$

Both factors are nonnegative on $0 \leq t \leq \pi/2$, so

$$ds = 3 \sin t \cos t dt.$$

3. The radius about the x -axis is $y = \sin^3 t$. Therefore,

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} y ds \\ &= 6\pi \int_0^{\pi/2} \sin^4 t \cos t dt. \end{aligned}$$

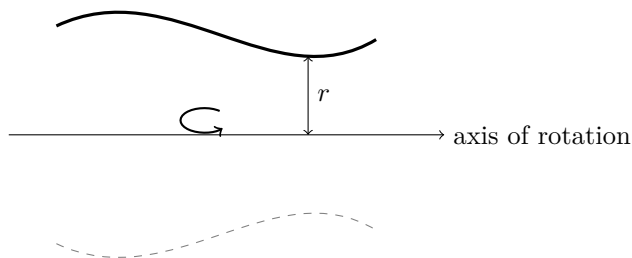
4. Let $u = \sin t$, so $du = \cos t dt$ and $0 \leq u \leq 1$:

$$S = 6\pi \int_0^1 u^4 du = 6\pi \left[\frac{u^5}{5} \right]_0^1 = \boxed{\frac{6\pi}{5}}.$$

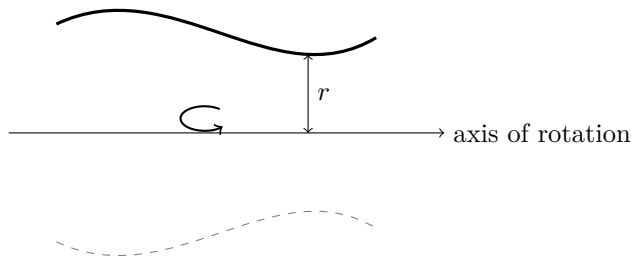
Question 10

Problem

Two curves have the same length and the same arc-length-weighted average distance from an axis, and each sweeps its surface exactly once. Must their surfaces of revolution have equal area?



Question 10 – Solution



Solution

- Let $r(s)$ denote the nonnegative distance from a point on a generating curve to the axis, where s is arc length. Its arc-length-weighted average distance is

$$r_{\text{avg}} = \frac{1}{L} \int_C r \, ds.$$

- Multiply by L :

$$\int_C r \, ds = L r_{\text{avg}}.$$

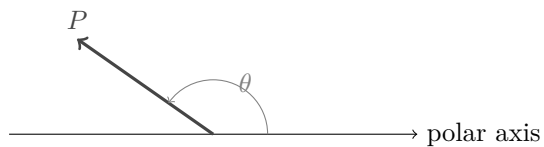
The surface area of revolution is therefore

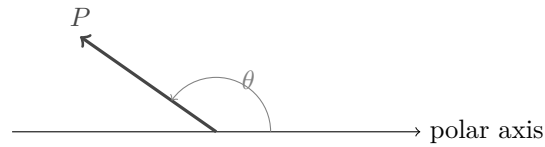
$$S = 2\pi \int_C r \, ds = 2\pi L r_{\text{avg}}.$$

- If two curves have the same length L and the same arc-length-weighted average radius, this formula gives the same value of S for both.
- Thus the answer is ☐ yes, provided each generating curve sweeps its surface exactly once. If a parametrization retraces the curve or distinct pieces generate the same surface, the integral counts multiplicity and the comparison requires removing that double-counting first.

Exercise 1**Problem**

The point has polar coordinates $(-4, \pi/3)$. Give two equivalent representations with positive radius and locate its quadrant.



Exercise 1: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

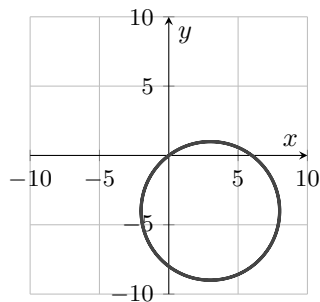
Equivalent polar coordinates satisfy

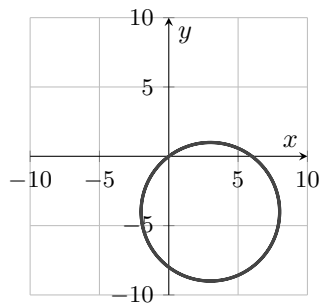
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Adding π while changing the radius sign gives $(4, 4\pi/3)$; also $(4, -2\pi/3)$.
4. It lies in Quadrant III.

Exercise 2**Problem**

Convert $r = 6 \cos \theta - 8 \sin \theta$ to Cartesian form and identify the curve.



Exercise 2: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

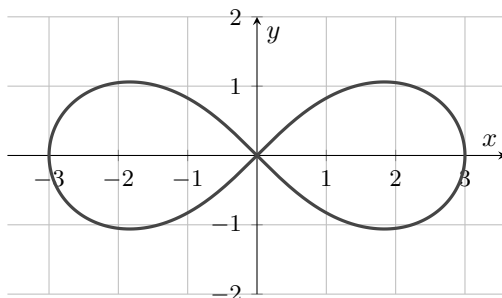
Equivalent polar coordinates satisfy

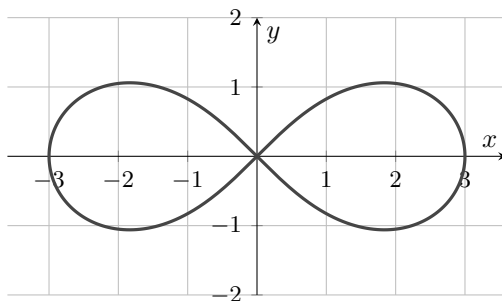
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. $r^2 = 6x - 8y$, so $(x - 3)^2 + (y + 4)^2 = 25$: a circle centered at $(3, -4)$ with radius 5.

Exercise 3**Problem**

Convert $r^2 = 9 \cos 2\theta$ to Cartesian form without using inverse trigonometric functions.



Exercise 3: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

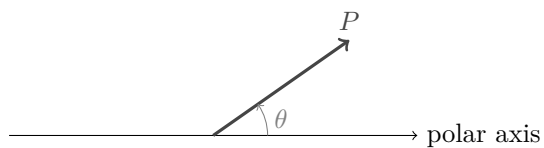
Equivalent polar coordinates satisfy

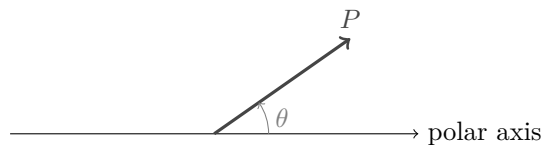
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Since $\cos 2\theta = (x^2 - y^2)/r^2$, $r^4 = 9(x^2 - y^2)$.
4. Thus $\boxed{(x^2 + y^2)^2 = 9(x^2 - y^2)}$.

Exercise 4**Problem**

Determine all intersections of $r = 2$ and $r = 4 \cos \theta$.



Exercise 4: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

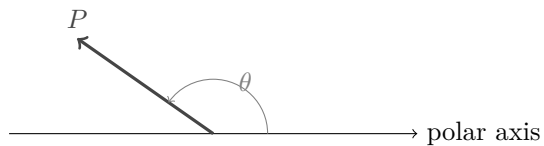
Equivalent polar coordinates satisfy

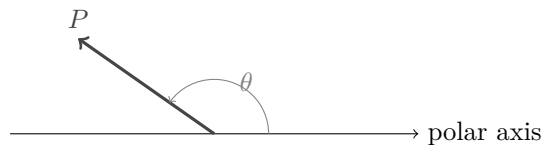
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. $2 = 4 \cos \theta$ gives $\theta = \pm\pi/3$.
4. The points are $(1, \pm\sqrt{3})$.

Exercise 5**Problem**

A polar curve is unchanged when θ is replaced by $-\theta$. What symmetry does this reveal, and why?



Exercise 5: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

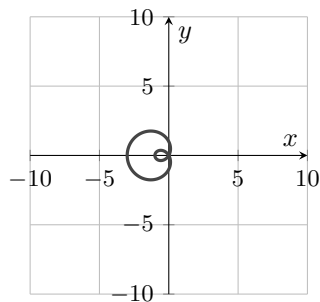
Equivalent polar coordinates satisfy

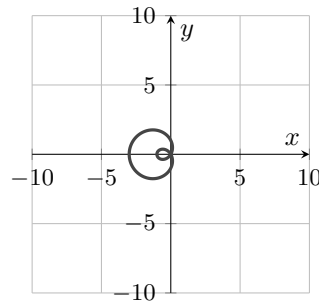
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The points at angles θ and $-\theta$ reflect across the polar axis, so the curve is symmetric about the x -axis.

Exercise 6**Problem**

Sketch mentally $r = 1 - 2 \cos \theta$. Why does it have an inner loop?



Exercise 6: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

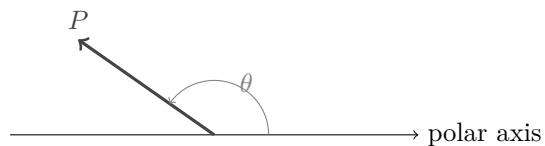
Equivalent polar coordinates satisfy

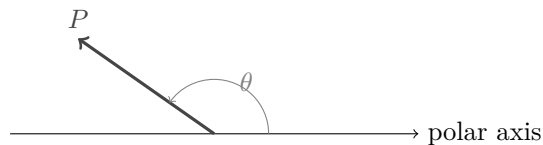
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The radius becomes negative when $\cos \theta > 1/2$, i.e. near $\theta = 0$.
4. Negative radii plot in the opposite direction, producing the inner loop.

Exercise 7**Problem**

Find the Cartesian equation corresponding to $\theta = 2\pi/3$ and explain why r must be unrestricted.



Exercise 7: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

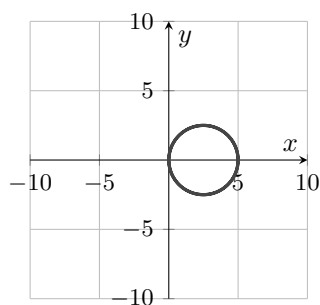
Equivalent polar coordinates satisfy

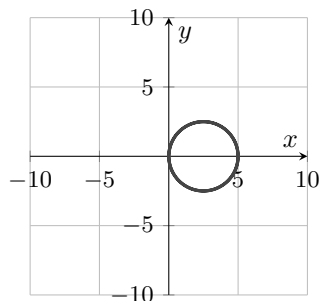
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The line has slope $\tan(2\pi/3) = -\sqrt{3}$, so $y = -\sqrt{3}x$.
4. Allowing negative r supplies the opposite ray and therefore the entire line.

Exercise 8**Problem**

Which description better reveals the geometry of $x^2 + y^2 = 5x$: Cartesian or polar? Convert it and explain.



Exercise 8: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

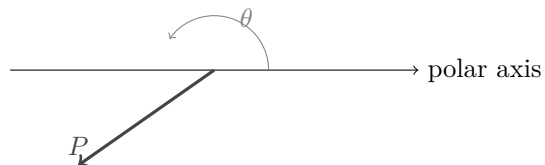
Equivalent polar coordinates satisfy

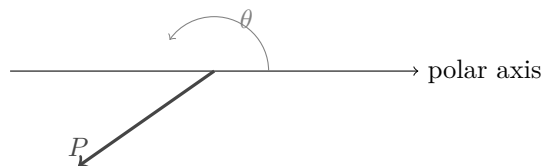
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Polar form is $r = 5 \cos \theta$, immediately revealing a circle through the pole with diameter 5 along the polar axis.

Exercise 9**Problem**

Find the smallest positive angle between the rays representing $(3, 17\pi/12)$ and $(2, -\pi/4)$.



Exercise 9: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

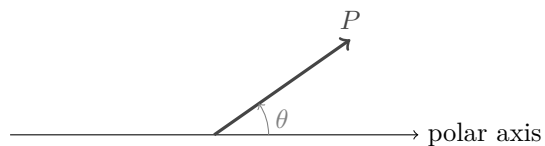
Equivalent polar coordinates satisfy

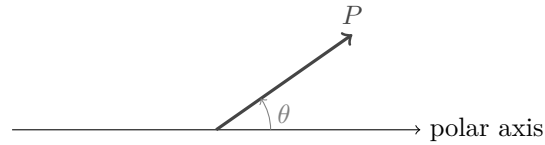
$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. Reduce angles to $17\pi/12$ and $21\pi/12$.
4. Their smaller difference is $\boxed{\pi/3}$.

Exercise 10**Problem**

A navigation system reports $(r, \theta) = (5, \text{atan2}(4, -3))$. Convert to Cartesian coordinates.



Exercise 10: Solution**Solution**

1. Use the polar–Cartesian relationships

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

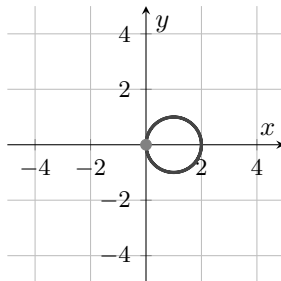
Equivalent polar coordinates satisfy

$$(r, \theta) = (r, \theta + 2k\pi) = (-r, \theta + (2k + 1)\pi).$$

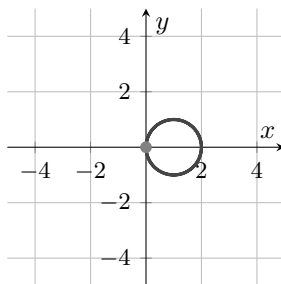
2. Apply the identity that matches the requested conversion, symmetry test, or intersection, and then check the resulting point or curve in the original polar equation.
3. The angle is in Quadrant II with cosine $-3/5$ and sine $4/5$.
4. Hence $(x, y) = \boxed{(-3, 4)}$.

Question 1**Problem**

Find the tangent line to $r = 2 \cos \theta$ at $\theta = \pi/4$.



Question 1 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

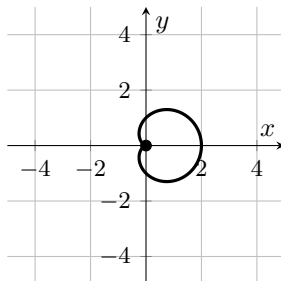
3. The point is $(1, 1)$.

4. With $r' = -2 \sin \theta$, the polar slope formula gives 0, so the tangent is $y = 1$.

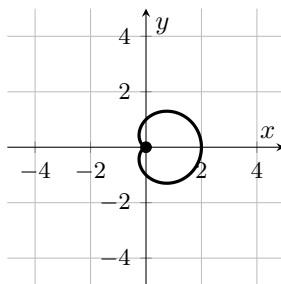
Question 2

Problem

Find all horizontal tangents of the cardioid $r = 1 + \cos \theta$ away from the pole.



Question 2 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

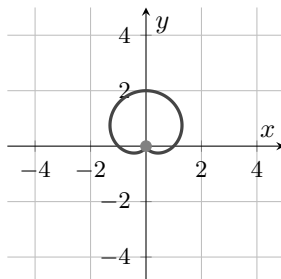
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

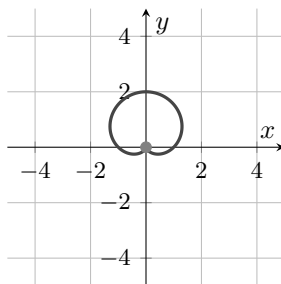
3. The numerator $r' \sin \theta + r \cos \theta = \cos \theta + \cos 2\theta$ vanishes.
4. This gives $\cos \theta = 1/2$ or -1 , and excluding the pole leaves $\theta = \pi/3, 5\pi/3$. At these angles $dx/d\theta \neq 0$, and the points are $(3/4, \pm 3\sqrt{3}/4)$. The tangent lines are $y = \pm 3\sqrt{3}/4$.

Question 3**Problem**

A student claims that $\theta = 0$ gives the rightmost point and a vertical tangent of $r = 1 + \sin \theta$. Test both parts of the claim.



Question 3 – Solution

**Solution**

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

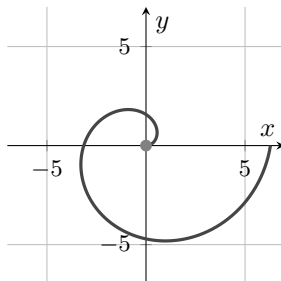
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

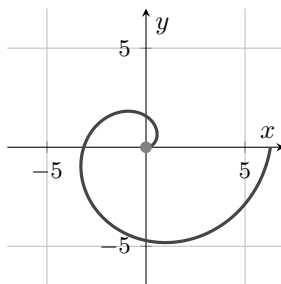
3. At $\theta = 0$, the point is $(1, 0)$ and the slope is 1, so the tangent is not vertical.
4. Also $x'(0) = 1 > 0$, so x is still increasing; the point is not rightmost.
5. Both parts are false.

Question 4**Problem**

The spiral $r = \theta$ reaches $\theta = \pi$. Find its tangent slope there.



Question 4 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

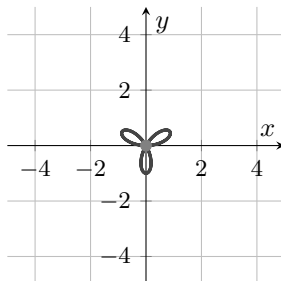
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. With $r' = 1$, the slope is $(\sin \pi + \pi \cos \pi)/(\cos \pi - \pi \sin \pi) = \boxed{\pi}$.

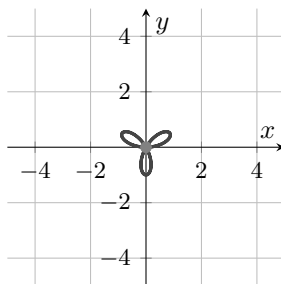
Question 5

Problem

Find the tangent directions at the pole for the rose $r = \sin 3\theta$.



Question 5 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

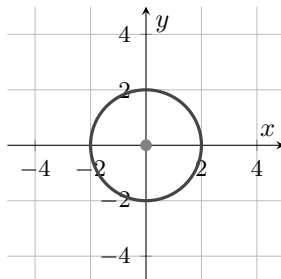
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The pole occurs when $3\theta = k\pi$, so the tangent rays satisfy $\theta = k\pi/3$.
4. The distinct unoriented lines are

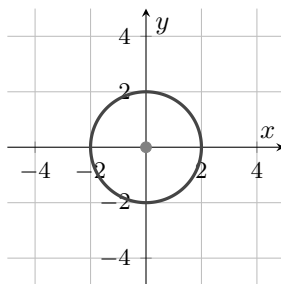
$$\theta = 0, \quad \theta = \pi/3, \quad \theta = 2\pi/3.$$

Question 6**Problem**

At an intersection of $r = 2$ and $r = 4 \cos \theta$ in the upper half-plane, find the acute angle between the two curves.



Question 6 – Solution

**Solution**

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

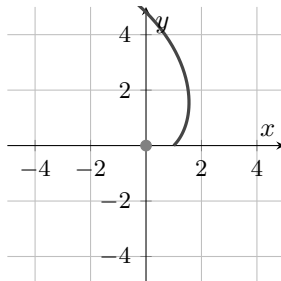
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $\theta = \pi/3$, the circle $r = 2$ has tangent direction $5\pi/6$.
4. The second circle has slope $1/\sqrt{3}$, hence tangent direction $\pi/6$.
5. The smaller angle between the tangent lines is $\boxed{\pi/3}$.

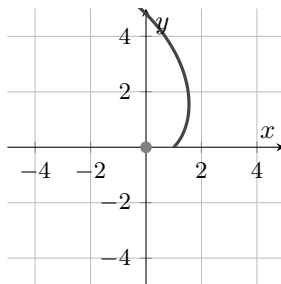
Question 7

Problem

For $r = e^\theta$, show that the angle between the radius vector and tangent is constant.



Question 7 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

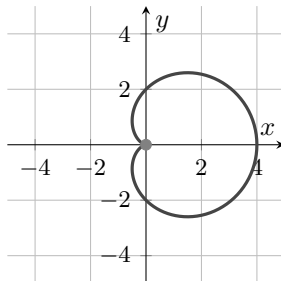
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. In polar components, tangent has radial component $r' = r$ and transverse component r .
4. Thus it makes angle $\arctan(r/r) = \boxed{\pi/4}$ with the radius vector.

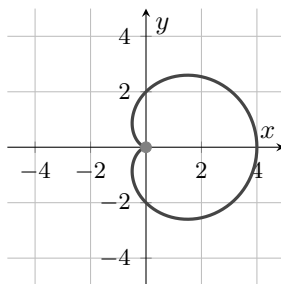
Question 8

Problem

Find a value of a so that $r = a + 2 \cos \theta$ has tangent slope 1 at $\theta = \pi/2$.



Question 8 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

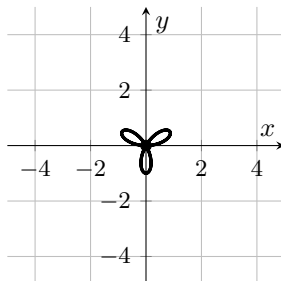
3. At $\pi/2$, the slope is $2/a$.

4. Setting $2/a = 1$ gives $\boxed{a = 2}$.

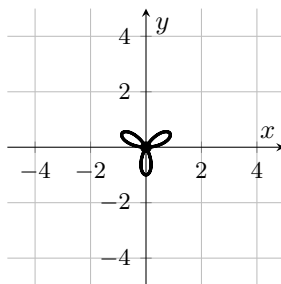
Question 9

Problem

A student cancels r before using the polar slope formula at a pole. Explain the danger.



Question 9 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

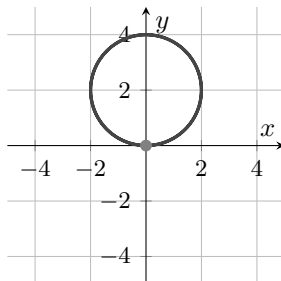
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $r = 0$, direct division by r is undefined. Valid cancellation where $r \neq 0$ can help compute a limit, but does not by itself justify substitution at the pole.
4. One must use a limit or inspect the parameter values producing the pole.

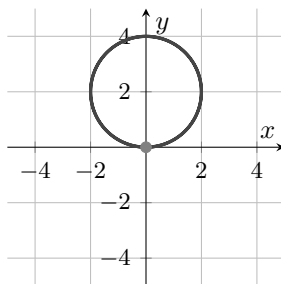
Question 10

Problem

For the circle $r = 4 \sin \theta$, find the tangent at its highest point using polar calculus.



Question 10 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

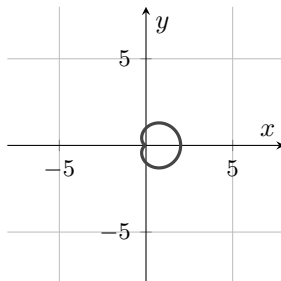
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

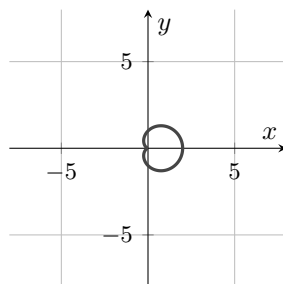
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The highest point occurs at $\theta = \pi/2$, giving $(0, 4)$.
4. The slope numerator is 0 and denominator -4 , so the tangent is horizontal: $y = 4$.

Exercise 1**Problem**

Find the area enclosed by the cardioid $r = 1 + \cos \theta$.



Exercise 1: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

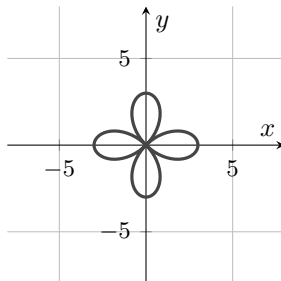
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

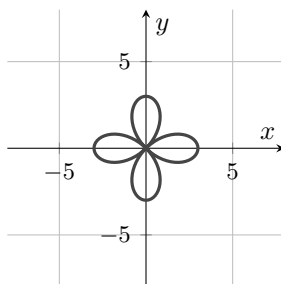
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. $A = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \boxed{3\pi/2}.$

Exercise 2**Problem**

Find the area of one petal of $r = 3 \cos 2\theta$.



Exercise 2: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.

2. For a polar boundary $r = f(\theta)$, use

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

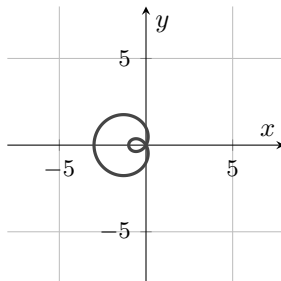
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

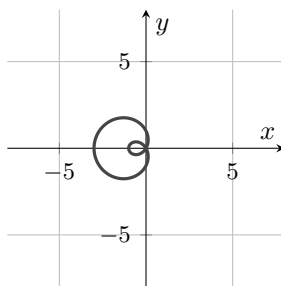
3. One petal is traced on $[-\pi/4, \pi/4]$.

4. Thus $A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} 9 \cos^2 2\theta d\theta = \boxed{9\pi/8}$.

Exercise 3**Problem**

Find the area of the inner loop of $r = 1 - 2 \cos \theta$.



Exercise 3: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

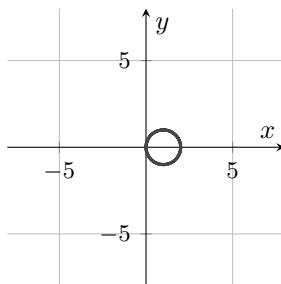
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

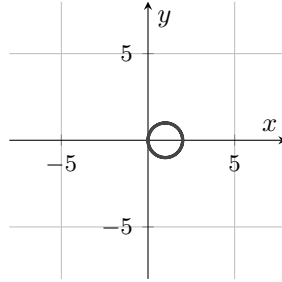
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. The loop is traced for $-\pi/3 \leq \theta \leq \pi/3$.
4. Hence $A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (1 - 2 \cos \theta)^2 d\theta = \boxed{\pi - 3\sqrt{3}/2}$.

Exercise 4**Problem**

Find the common area of the circles $r = 2 \cos \theta$ and $r = 2 \sin \theta$.



Exercise 4: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

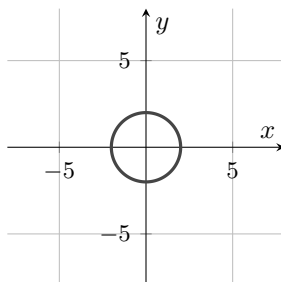
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

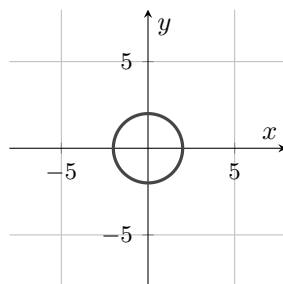
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. In the first quadrant the smaller radius switches at $\pi/4$.
4. Symmetry gives $A = \int_0^{\pi/4} (2 \sin \theta)^2 d\theta = \boxed{\pi/2 - 1}$.

Exercise 5**Problem**

Find the area inside $r = 2$ but outside $r = 2 \cos \theta$.



Exercise 5: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.

2. For a polar boundary $r = f(\theta)$, use

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

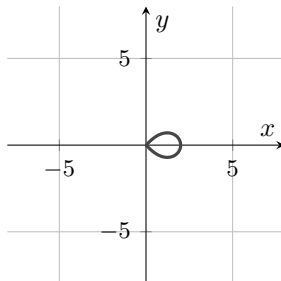
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

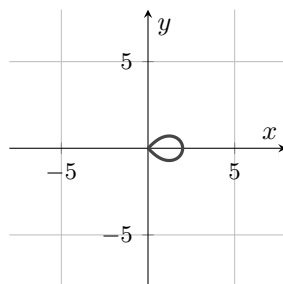
3. The circle $r = 2 \cos \theta$ lies entirely inside $r = 2$.

4. Subtract areas: $4\pi - \pi = \boxed{3\pi}$.

Exercise 6**Problem**

The curve $r^2 = 4 \cos 2\theta$ has two loops. Find its total area.



Exercise 6: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.

2. For a polar boundary $r = f(\theta)$, use

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

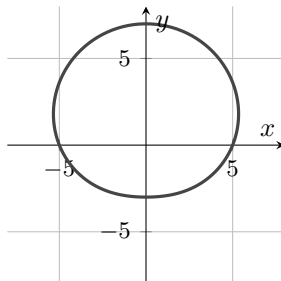
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

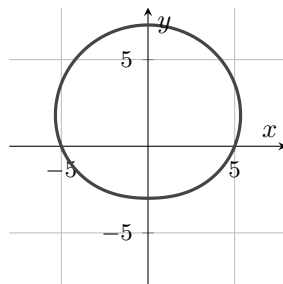
3. One loop uses $-\pi/4 \leq \theta \leq \pi/4$: $\text{area} = \frac{1}{2} \int 4 \cos 2\theta d\theta = 2$.

4. Doubling gives $\boxed{4}$.

Exercise 7**Problem**

A radar beam reaches distance $r = 5 + 2 \sin \theta$. Find the area swept while $0 \leq \theta \leq \pi$.



Exercise 7: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

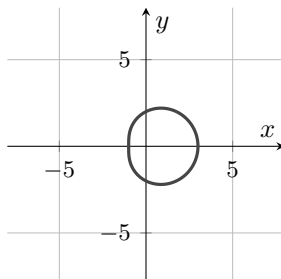
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

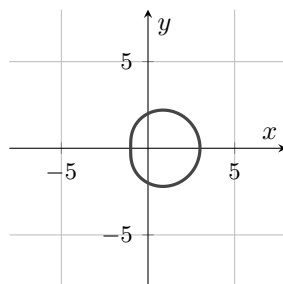
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. $A = \frac{1}{2} \int_0^\pi (5 + 2 \sin \theta)^2 d\theta = \boxed{27\pi/2 + 20}.$

Exercise 8**Problem**

Two polar curves satisfy $0 \leq r_1(\theta) \leq r_2(\theta)$. Explain why the area between them involves squares, not merely $r_2 - r_1$.



Exercise 8: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

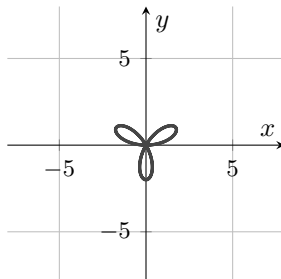
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

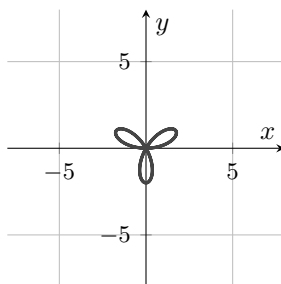
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. A thin polar sector has area $\frac{1}{2}r^2 d\theta$.
4. Subtracting sector areas yields $\frac{1}{2}(r_2^2 - r_1^2)d\theta$, reflecting increasing width with radius.

Exercise 9**Problem**

Find the area enclosed by one loop of $r = 2 \sin 3\theta$ and explain why multiplying by six is wrong for total area.



Exercise 9: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.

2. For a polar boundary $r = f(\theta)$, use

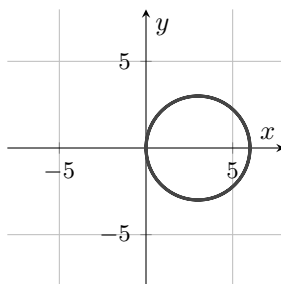
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

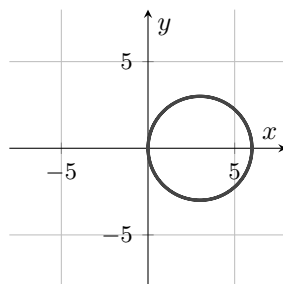
For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

3. One loop area is $\frac{1}{2} \int_0^{\pi/3} 4 \sin^2 3\theta d\theta = \pi/3$.
4. The rose has three petals, not six, because negative radii retrace opposite petals; total area is $\boxed{\pi}$.

Exercise 10**Problem**

Choose $a > 0$ so that the circle $r = a \cos \theta$ has area 9π .



Exercise 10: Solution**Solution**

1. Determine an interval of angles that sweeps out the requested region exactly once, using symmetry when it simplifies the work.
2. For a polar boundary $r = f(\theta)$, use

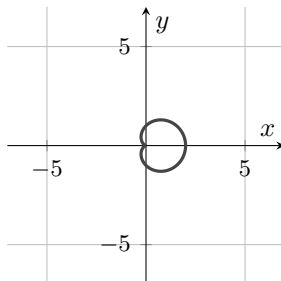
$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

For a region between two curves, subtract the inner squared radius from the outer squared radius before integrating.

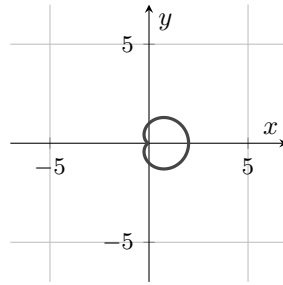
3. It is a circle of radius $a/2$, so area is $\pi a^2/4$.
4. Setting this to 9π gives $\boxed{a = 6}$.

Question 1**Problem**

Find the length of the cardioid $r = 1 + \cos \theta$.



Question 1 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

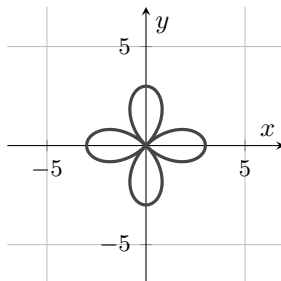
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

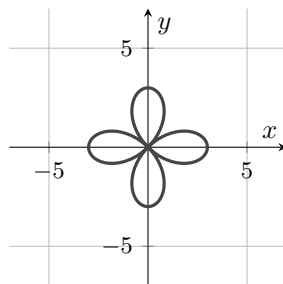
3. The speed simplifies to $2|\cos(\theta/2)|$.
4. Integrating from 0 to 2π gives $\boxed{8}$.

Question 2**Problem**

Find the length of one petal of $r = 3 \cos 2\theta$.



Question 2 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

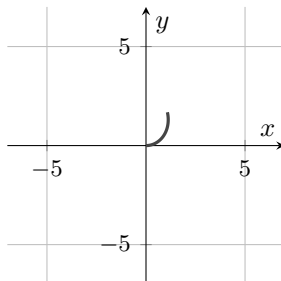
Simplify the expression under the square root before evaluating or reporting the integral.

3. On $[-\pi/4, \pi/4]$, speed is $3\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}$.

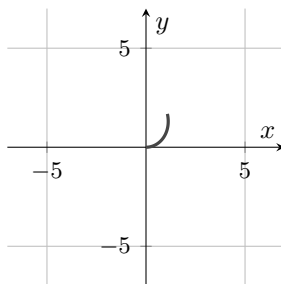
4. Thus $L = 3 \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta} d\theta$; it is an elliptic integral.

Question 3**Problem**

Find the length of the spiral $r = 2\theta$, $0 \leq \theta \leq 1$.



Question 3 – Solution



Solution

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

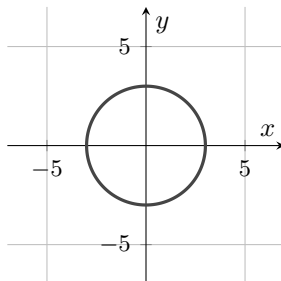
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

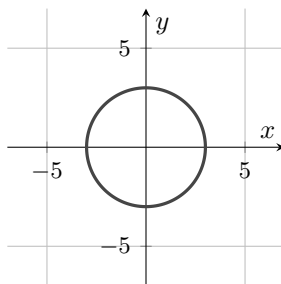
3. $L = 2 \int_0^1 \sqrt{1 + \theta^2} d\theta = \boxed{\sqrt{2} + \operatorname{arsinh} 1}.$

Question 4**Problem**

Use the polar arc-length formula to recover the circumference of $r = R$.



Question 4 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

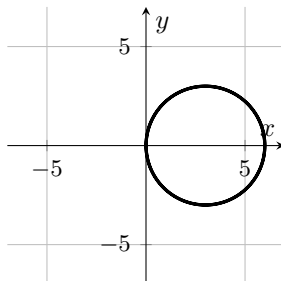
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

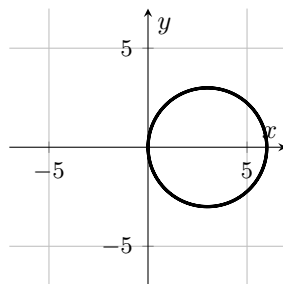
3. Since $r' = 0$, speed is R .
4. Hence $L = \int_0^{2\pi} R d\theta = \boxed{2\pi R}$.

Question 5**Problem**

A circle of the same radius is $r = 2R \cos \theta$. Which interval traces it once, and what length results?



Question 5 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

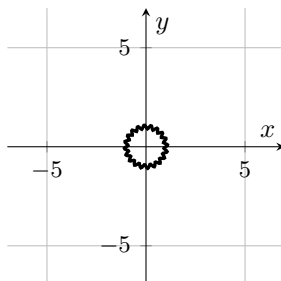
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

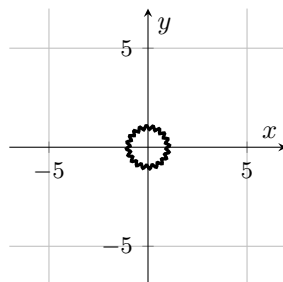
3. $-\pi/2 \leq \theta \leq \pi/2$ traces it once.
4. The speed simplifies to $2R$, so $L = 2R\pi = \boxed{2\pi R}$.

Question 6**Problem**

Without evaluating, compare the lengths of $r = 1$ and $r = 1 + 0.1 \sin(20\theta)$ over $0 \leq \theta \leq 2\pi$.



Question 6 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

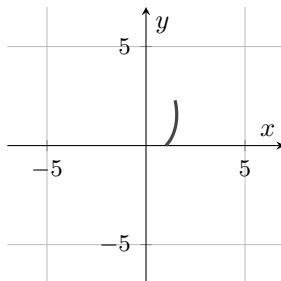
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

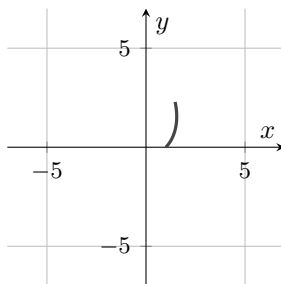
3. Here $r > 0$ and $r' = 2 \cos 20\theta$. Thus $\sqrt{r^2 + (r')^2} \geq r$, with strict inequality except at finitely many angles. Consequently $L > \int_0^{2\pi} (1 + 0.1 \sin 20\theta) d\theta = 2\pi$, the unit circle's length.

Question 7**Problem**

Find the length of $r = e^\theta$, $0 \leq \theta \leq \ln 2$.



Question 7 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

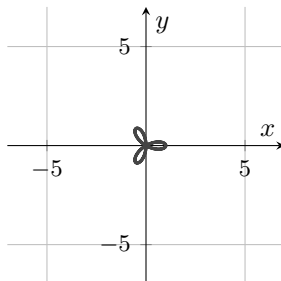
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

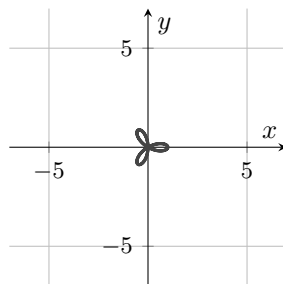
3. Since $r' = r$, speed is $\sqrt{2}e^\theta$.
4. Thus $L = \sqrt{2}[e^\theta]_0^{\ln 2} = \boxed{\sqrt{2}}$.

Question 8**Problem**

Explain why integrating the rose $r = \cos 3\theta$ from 0 to 2π overcounts its geometric length.



Question 8 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

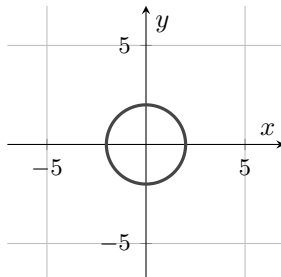
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Simplify the expression under the square root before evaluating or reporting the integral.

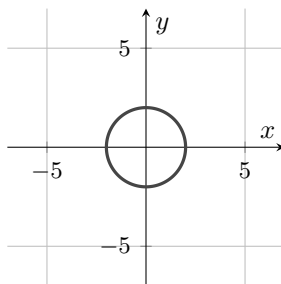
3. For odd 3, the full curve is completed on $0 \leq \theta \leq \pi$; the next π retraces it because negative radius duplicates opposite points.
4. The 0 to 2π integral counts every petal twice.

Question 9**Problem**

A polar path has constant arc-length speed 5 and runs for an angular interval of width $2\pi/3$. What is its length?



Question 9 – Solution

**Solution**

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

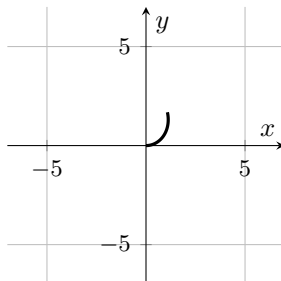
Simplify the expression under the square root before evaluating or reporting the integral.

3. Arc length is the integral of speed: $L = 5(2\pi/3) = \boxed{10\pi/3}$.

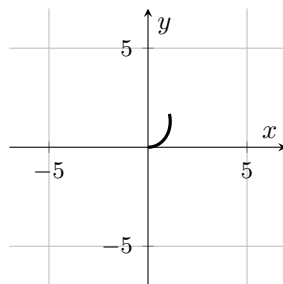
Question 10

Problem

Find a so that $r = a\theta$, $0 \leq \theta \leq 1$, has length $\sqrt{2} + \operatorname{arsinh} 1$.



Question 10 – Solution



Solution

1. Differentiate the polar radius to obtain $r' = dr/d\theta$ and choose an interval that traces the requested arc exactly once.
2. Use the polar arc-length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

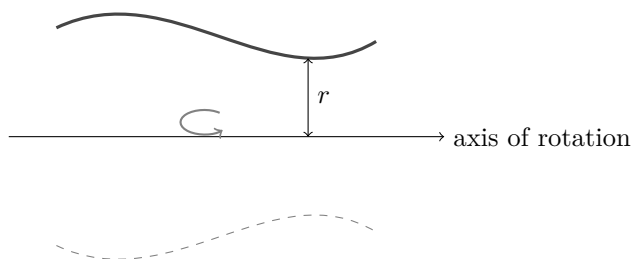
Simplify the expression under the square root before evaluating or reporting the integral.

3. Its length is $|a| \int_0^1 \sqrt{1 + \theta^2} d\theta = \frac{|a|}{2}(\sqrt{2} + \operatorname{arsinh} 1)$.
4. Thus $\boxed{a = \pm 2}$.

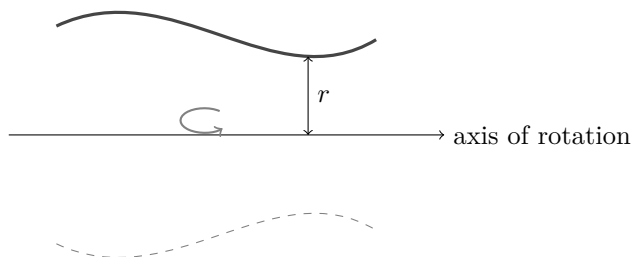
Question 1

Problem

The upper semicircle $r = 2R \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the polar axis. Find the generated surface area.



Question 1 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

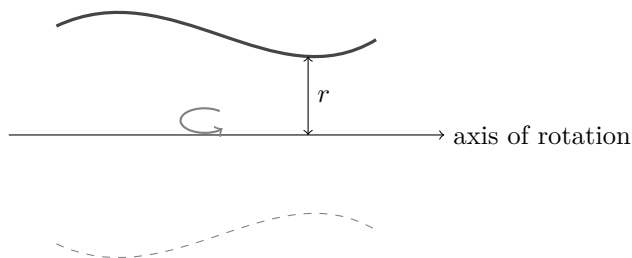
3. Here $y = 2R \sin \theta \cos \theta$ and $ds = 2R d\theta$.

4. Thus $S = 8\pi R^2 \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \boxed{4\pi R^2}$.

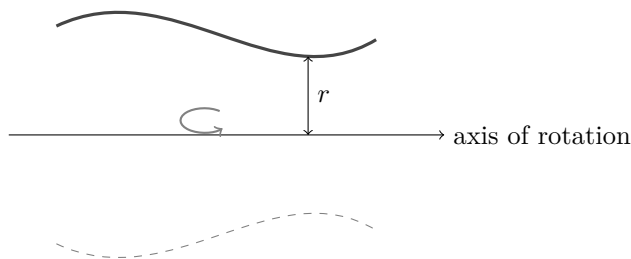
Question 2

Problem

Set up the surface area formed by rotating the cardioid $r = 1 + \cos \theta$, $0 \leq \theta \leq \pi$, about the polar axis.



Question 2 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

3. Use $y = r \sin \theta$ and $ds = 2 \cos(\theta/2) d\theta$ on this interval.

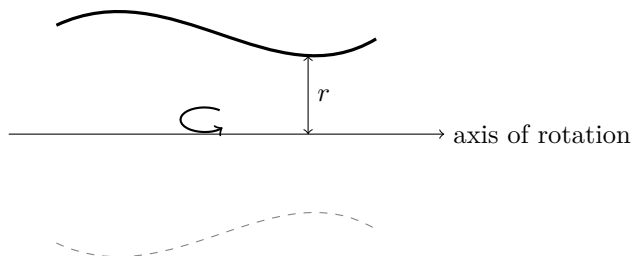
4. Therefore

$$S = 4\pi \int_0^\pi (1 + \cos \theta) \sin \theta \cos(\theta/2) d\theta.$$

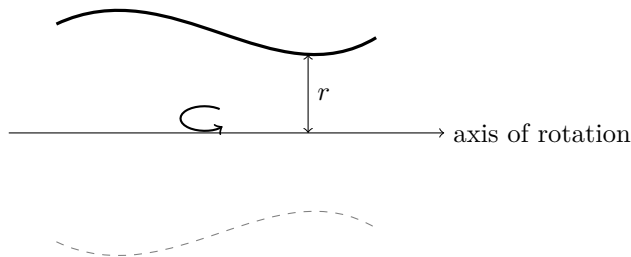
Question 3

Problem

Rotate the spiral arc $r = \theta$, $0 \leq \theta \leq \pi/2$, about the y -axis. Write the exact surface-area integral.



Question 3 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

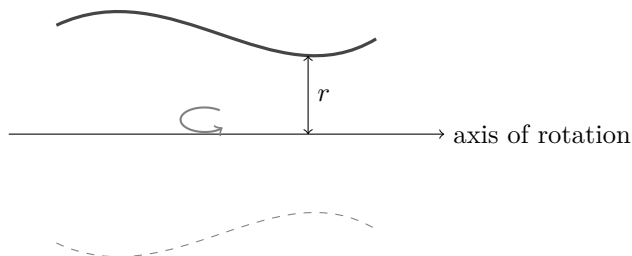
3. Distance to the y -axis is $x = \theta \cos \theta$, and $ds = \sqrt{1 + \theta^2} d\theta$.

4. Thus
$$S = 2\pi \int_0^{\pi/2} \theta \cos \theta \sqrt{1 + \theta^2} d\theta.$$

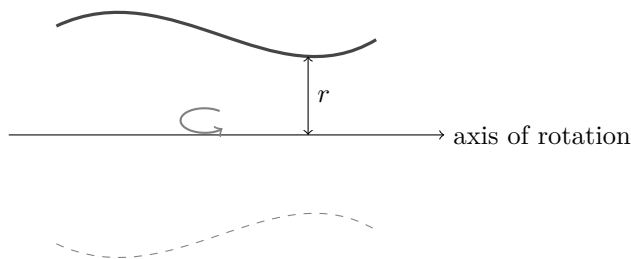
Question 4

Problem

Why must $|r \sin \theta|$ replace $r \sin \theta$ when a polar curve crosses the x -axis during rotation about that axis?



Question 4 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

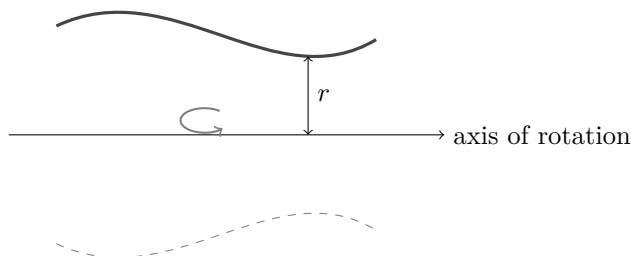
over an interval that generates the surface exactly once.

3. Radius to the axis is the nonnegative distance $|y| = |r \sin \theta|$.
4. Without the absolute value, areas on opposite sides would cancel.

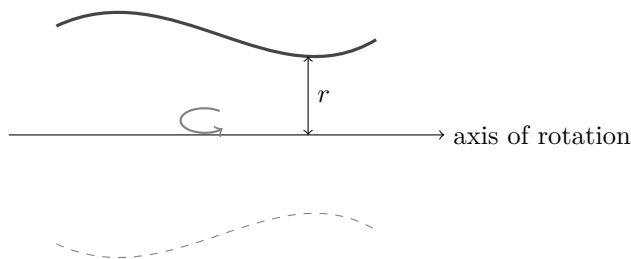
Question 5

Problem

A student proposes using only $0 \leq \theta \leq \pi/2$ when the circle $r = 4 \sin \theta$ rotates about the x -axis. Explain why that omits part of the surface and give a correct interval.



Question 5 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

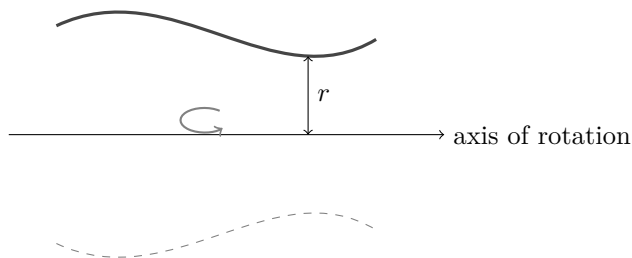
over an interval that generates the surface exactly once.

3. The interval $0 \leq \theta \leq \pi/2$ traces only the right semicircle; rotation preserves the x -coordinate, so the left half is not duplicated.
4. Use $\boxed{0 \leq \theta \leq \pi}$ to trace the full generating circle once.

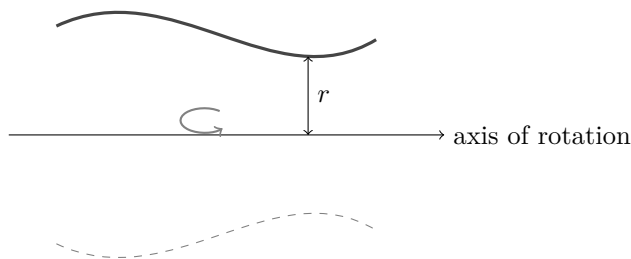
Question 6

Problem

For $r = R$, $0 \leq \theta \leq \pi$, rotate the semicircle about the x -axis and recover the sphere area.



Question 6 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

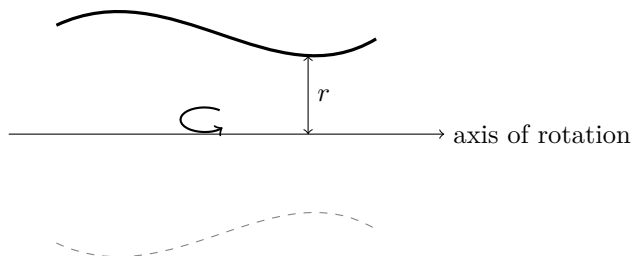
3. $y = R \sin \theta$ and $ds = R d\theta$.

4. Thus $S = 2\pi R^2 \int_0^\pi \sin \theta d\theta = \boxed{4\pi R^2}$.

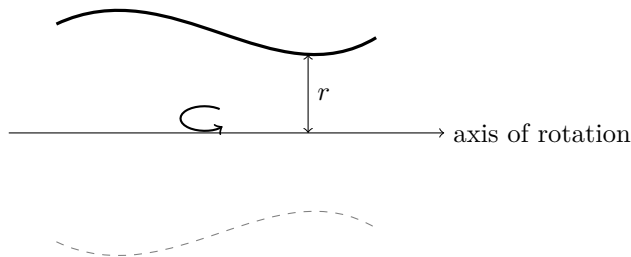
Question 7

Problem

Set up the surface area obtained by rotating one petal of $r = \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$, about the y -axis.



Question 7 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

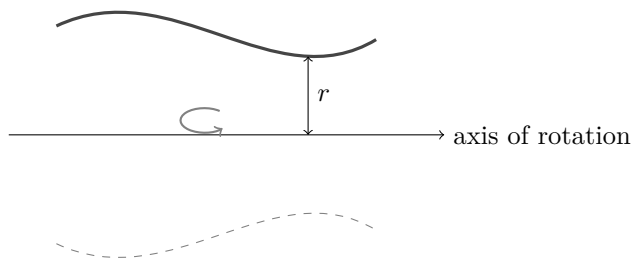
3. Use radius $|x| = |r \cos \theta|$ and $ds = \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta$.
4. On this interval $r \cos \theta \geq 0$, so the requested setup is

$$S = 2\pi \int_{-\pi/4}^{\pi/4} \cos 2\theta \cos \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta.$$

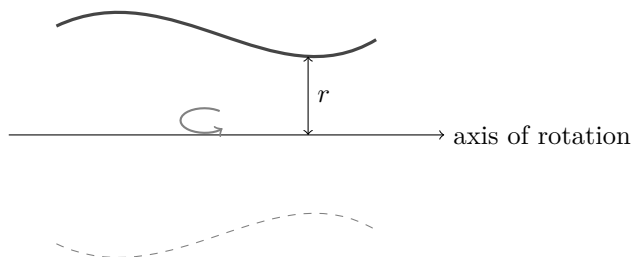
Question 8

Problem

A curve is entirely in the first quadrant. Compare its areas of revolution about the x - and y -axes using only weighted average coordinates.



Question 8 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

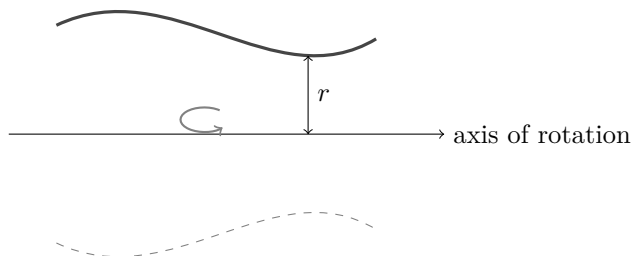
over an interval that generates the surface exactly once.

3. $S_x = 2\pi L \bar{y}_s$ and $S_y = 2\pi L \bar{x}_s$, where the bars are arc-length-weighted means.
4. Therefore $S_x/S_y = \bar{y}_s/\bar{x}_s$.

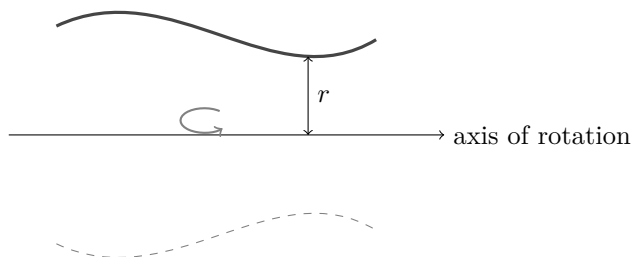
Question 9

Problem

Find the area generated when $r = 2 \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the line $y = -1$.



Question 9 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

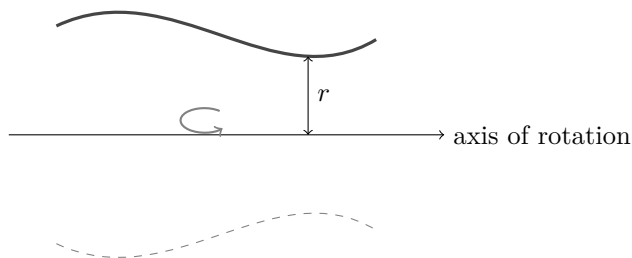
3. Here $ds = 2d\theta$ and radius is $y + 1 = 2 \sin \theta \cos \theta + 1$.

4. Hence $S = 4\pi \int_0^{\pi/2} (1 + 2 \sin \theta \cos \theta) d\theta = \boxed{2\pi^2 + 4\pi}$.

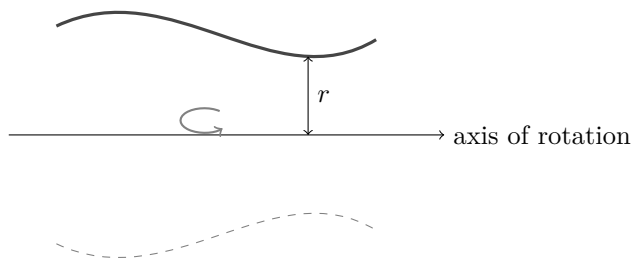
Question 10

Problem

A student integrates a closed polar curve over $0 \leq \theta \leq 2\pi$ and gets twice the expected surface area. Name two likely causes.



Question 10 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

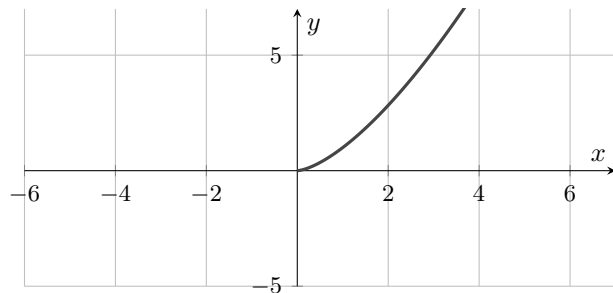
over an interval that generates the surface exactly once.

3. The curve may itself be traced twice on that interval, or two different portions of the generating curve may sweep the same surface after rotation.
4. Both are geometric double-counting.

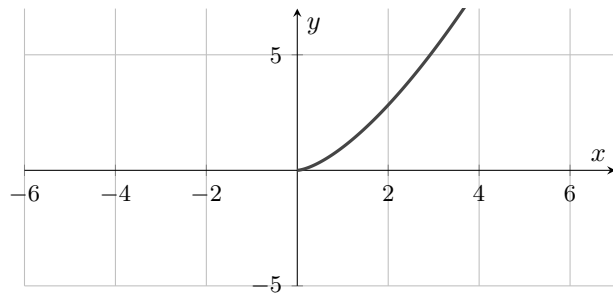
Question 1

Problem

For $y = x^{3/2}$, $0 \leq x \leq 4$, compare the Cartesian and parametric arc-length setups using $x = t^2$, $y = t^3$.



Question 1 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

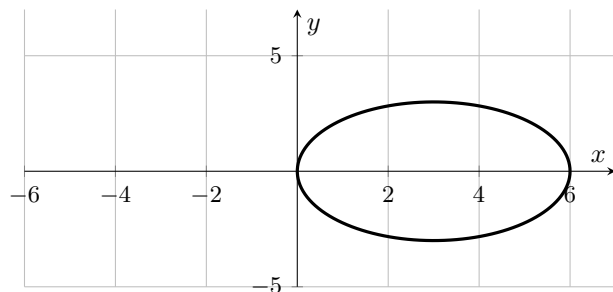
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian: $\int_0^4 \sqrt{1 + 9x/4} dx$.
4. Parametric with $0 \leq t \leq 2$: $\int_0^2 t\sqrt{4 + 9t^2} dt$.
5. Substitution $x = t^2$ makes them identical.

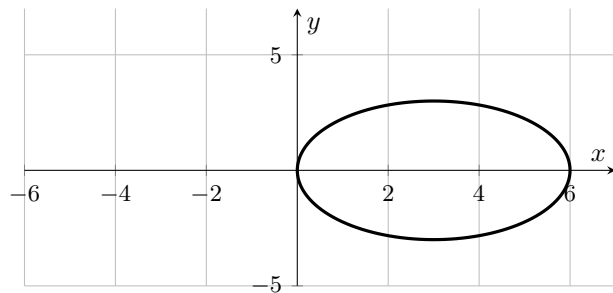
Question 2

Problem

A circle is given parametrically and polarly. Which representation makes its circumference easiest: $x = 3 + 3 \cos t$, $y = 3 \sin t$ or $r = 6 \cos \theta$? Explain.



Question 2 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

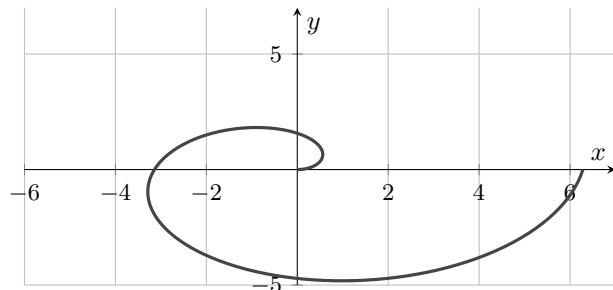
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametric speed is 3 on $0 \leq t \leq 2\pi$. The polar arc-length factor is 6 on $-\pi/2 \leq \theta \leq \pi/2$. Each interval traces the circle once, so both forms give $L = 6\pi$.
4. The best choice is the one whose single-tracing interval is clearest.

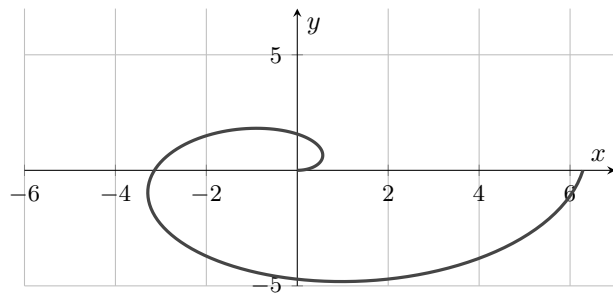
Question 3

Problem

The spiral $r = \theta$ can be parametrized by $x = \theta \cos \theta, y = \theta \sin \theta$. Verify that both arc-length formulas give the same integrand.



Question 3 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

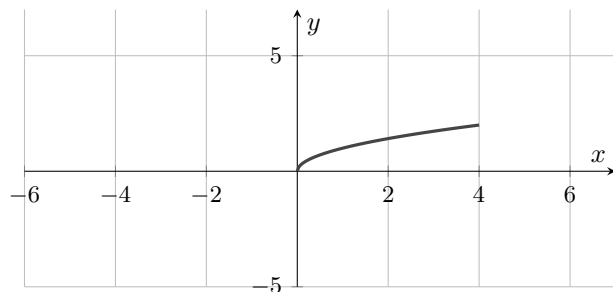
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Polar gives $\sqrt{r^2 + (r')^2} = \sqrt{\theta^2 + 1}$.
4. Squaring the Cartesian derivatives cancels cross terms and gives the same $\sqrt{\theta^2 + 1}$.

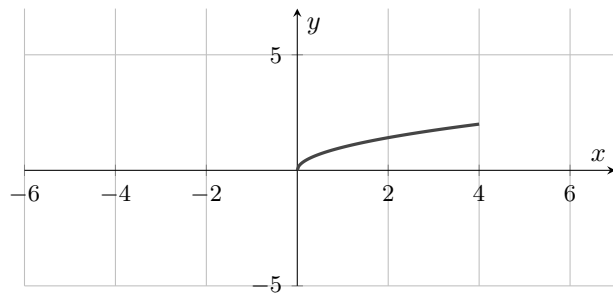
Question 4

Problem

A surface is generated from $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x -axis. Decide whether Cartesian or the parametrization $x = t^2, y = t$ is cleaner.



Question 4 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametrization removes the singular derivative at $x = 0$:

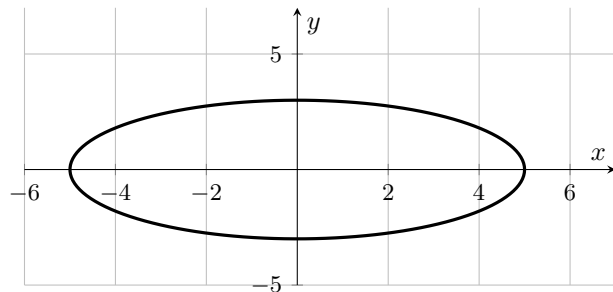
$$ds = \sqrt{4t^2 + 1} dt, \quad S = 2\pi \int_0^2 t\sqrt{4t^2 + 1} dt.$$

Therefore the parametric form is cleaner.

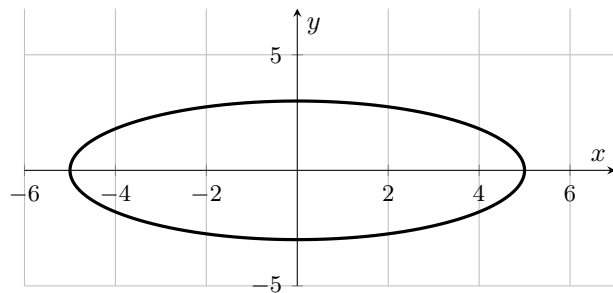
Question 5

Problem

A noncircular ellipse lacks an elementary perimeter formula. Explain why switching among Cartesian, parametric, and polar forms does not remove the essential difficulty.



Question 5 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

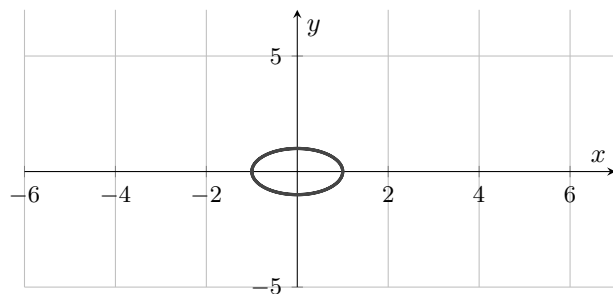
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Each representation reduces to an elliptic integral.
4. Reparametrization changes appearance, not the intrinsic arc-length integral or its non-elementary nature.

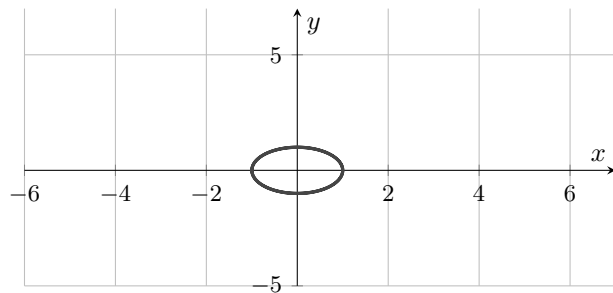
Question 6

Problem

A curve is traced twice by a parametrization but once by a polar interval. Which arc length is the geometric length, and which is the traveled distance?



Question 6 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

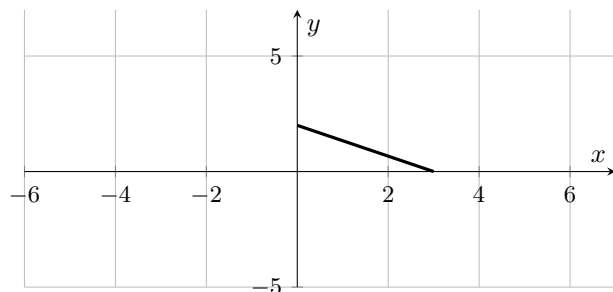
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The once-traced polar integral gives geometric curve length.
4. The twice-traced parametric integral gives total traveled distance, twice the geometric length.

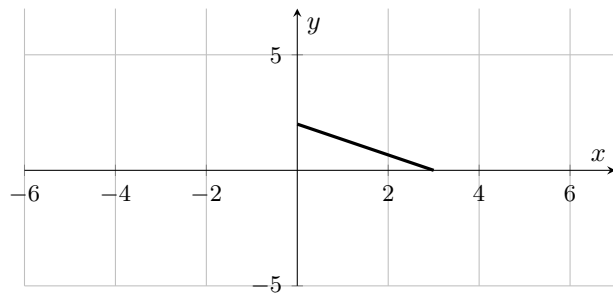
Question 7

Problem

Recover the cone lateral-area formula by parametrizing its slanted edge, then compare it with the slant-height mnemonic.



Question 7 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

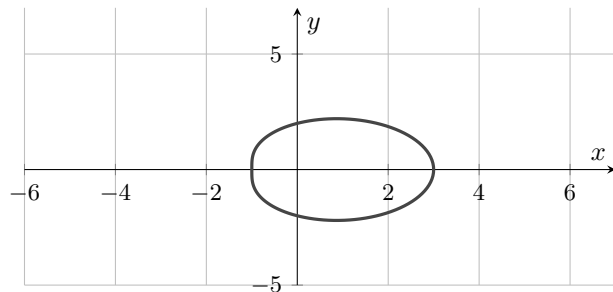
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. For radius $R > 0$ and height $h > 0$, rotate $(x, y) = (Rt/h, h - t)$, $0 \leq t \leq h$, about the y -axis.
4. Then $ds = (\sqrt{R^2 + h^2}/h)dt$ and $S = \pi R\sqrt{R^2 + h^2} = \pi R\ell$, matching the mnemonic.

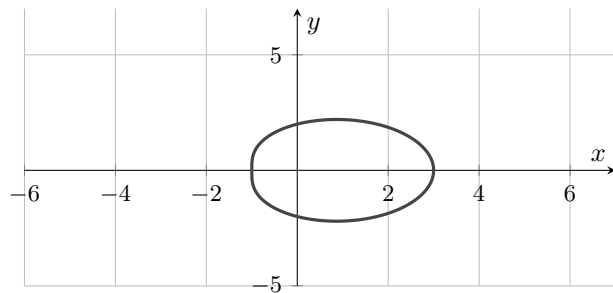
Question 8

Problem

A curve has polar form $r = f(\theta)$ but is rotated about the y -axis. State when converting to Cartesian form might be safer.



Question 8 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

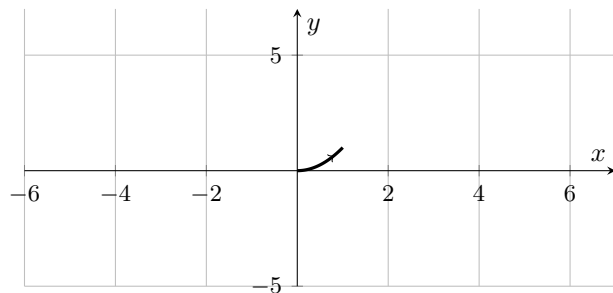
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Conversion is safer when $x = r \cos \theta$ changes sign or the polar interval retraces points, because the radius to the y -axis is $|x|$ and double-covering is easier to see geometrically.

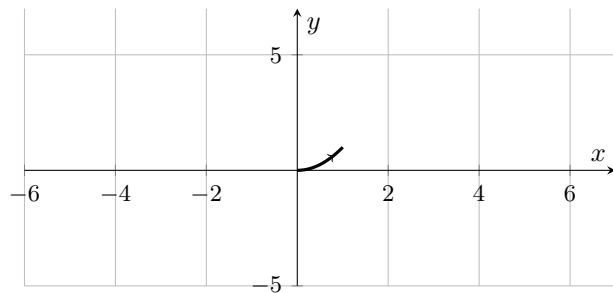
Question 9

Problem

Design a consistency check for any computed surface area of revolution using dimensions and scaling.



Question 9 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

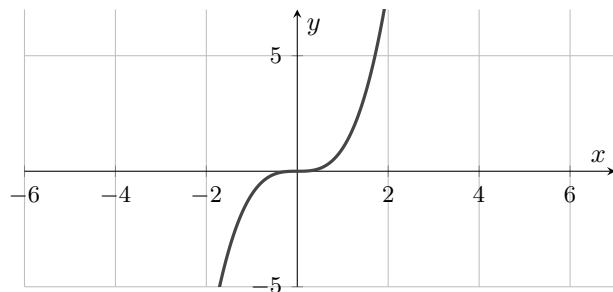
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Area must have length-squared units.
4. If every coordinate is scaled by $k > 0$, both radius and arc length scale by k , so the answer must scale by k^2 .

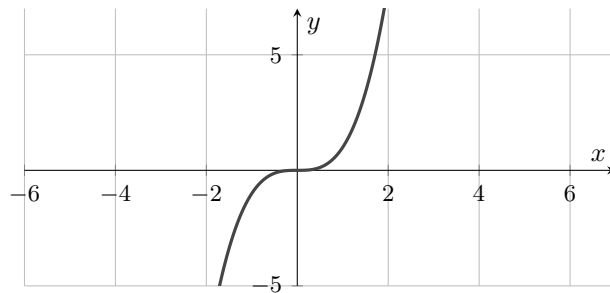
Question 10

Problem

A numerical system can evaluate either a Cartesian or parametric arc-length integral. Give one reason to prefer each form.



Question 10 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian is convenient for a single-valued smooth graph with simple derivative.
4. Parametric form is preferable for loops, vertical tangents, motion data, or when it avoids an infinite derivative.

Question 1

Let $a_n = \frac{3n-1}{n+2}$ for $n \geq 1$.

- (a) Find $L = \lim_{n \rightarrow \infty} a_n$.
- (b) Solve $|a_n - L| < 0.01$ exactly and determine the smallest integer N such that the inequality holds for every $n \geq N$.

Question 1 – Solution

Step 1: Find the candidate limit.

The numerator and denominator have the same degree. Divide both by n :

$$a_n = \frac{3 - 1/n}{1 + 2/n} \rightarrow \frac{3}{1} = 3.$$

Since $1/n \rightarrow 0$ and $2/n \rightarrow 0$, the quotient law gives $L = 3$.

Step 2: Express the error exactly.

Rather than approximate numerically, subtract the limit and simplify:

$$|a_n - 3| = \left| \frac{3n - 1}{n + 2} - 3 \right| = \frac{7}{n + 2}.$$

Step 3: Solve the strict accuracy inequality.

Therefore

$$\frac{7}{n + 2} < 0.01 = \frac{1}{100} \iff n > 698.$$

Thus every integer $n \geq 699$ satisfies the requirement.

Step 4: Verify minimality.

The smallest possible integer is $N = 699$. At the preceding index, $n = 698$, the error is $7/700 = 0.01$, which equals the tolerance and is not strictly less than it. Hence no smaller N works.

Question 2

Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ for $n \geq 1$.

- (a) Show that (a_n) is bounded.
- (b) Find the limits of the even and odd subsequences (a_{2k}) and (a_{2k-1}) .
- (c) Use those limits to decide whether (a_n) converges and to evaluate the claim, “Every bounded sequence converges.”

Question 2 – Solution**Step 1: Prove boundedness.**

For $n \geq 1$, $1 < 1 + 1/n \leq 2$. Since $|(-1)^n| = 1$,

$$|a_n| = 1 + \frac{1}{n} \leq 2.$$

Thus $-2 \leq a_n \leq 2$, and the sequence is bounded.

Step 2: Analyze the even subsequence.

For even indices,

$$a_{2k} = 1 + \frac{1}{2k} \rightarrow 1.$$

Step 3: Analyze the odd subsequence.

For odd indices,

$$a_{2k-1} = -1 - \frac{1}{2k-1} \rightarrow -1.$$

Step 4: Apply the subsequence criterion.

If (a_n) converged to L , every subsequence would also converge to L . The even and odd subsequences instead approach 1 and -1 , respectively. Since these limits differ, (a_n) diverges.

Step 5: Evaluate the student's claim.

This bounded divergent sequence is a counterexample to the claim that every bounded sequence converges. Boundedness together with monotonicity, however, would guarantee convergence by the Monotone Convergence Theorem.

Question 3

Let $a_n = \frac{n}{n+1}$ for $n \geq 1$.

- (a) Prove algebraically (without derivatives) that (a_n) is strictly increasing.
- (b) Find an upper bound and use the Monotone Convergence Theorem to explain why a limit exists.
- (c) Find the limit and express the error $|a_n - L|$ exactly.

Question 3 – Solution

Step 1: Prove monotonicity algebraically.

Compare consecutive terms by subtraction:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

The denominator is positive for $n \geq 1$, so $a_{n+1} - a_n > 0$ and (a_n) is strictly increasing.

Step 2: Find an upper bound.

Rewrite the term as

$$a_n = 1 - \frac{1}{n+1} < 1,$$

Thus $a_n < 1$ for every n , so the sequence is bounded above by 1.

Step 3: Establish convergence and calculate the limit.

An increasing sequence bounded above converges by the Monotone Convergence Theorem. Using the rewritten formula,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Step 4: Express the error exactly.

Since $a_n < 1$, the exact error is $|a_n - 1| = 1 - a_n = 1/(n+1)$, which tends to zero.

Question 4

Let $a_n = \left(1 + \frac{2}{n}\right)^n$.

- (a) Interpret a_n as the balance produced by compounding a principal of 1 at a total rate of 2 over one time unit, using n periods.
- (b) Evaluate $\lim_{n \rightarrow \infty} a_n$ using the standard exponential limit.
- (c) Give a decimal approximation of the limiting balance.

Question 4 – Solution

Step 1: Interpret the compounding model.

The principal is $P = 1$. Dividing the total rate 2 into n equal periods gives rate $2/n$ per period; applying that growth factor n times produces $1(1 + 2/n)^n$.

Step 2: Rewrite in the standard exponential form.

Put $m = n/2$. Then $m \rightarrow \infty$ and

$$a_n = \left[\left(1 + \frac{1}{m} \right)^m \right]^2.$$

Step 3: Apply the standard limit.

Using $\lim_{m \rightarrow \infty} (1 + 1/m)^m = e$ and continuity of the squaring function gives

$$\lim_{n \rightarrow \infty} a_n = e^2 \approx 7.38906.$$

Step 4: Interpret and verify.

This agrees with the continuous-compounding formula $Pe^{rt} = e^2$ for $P = 1$, $r = 2$, and $t = 1$. The standard exponential limit supplies the proof.

Question 5

Let $a_n = \sin(n\pi/2)$ for $n \geq 1$.

- (a) Make a table giving a_n for each residue class of n modulo 4, and list the first eight terms.
- (b) Determine whether the sequence converges. Justify your answer using subsequences.
- (c) Explain whether changing or rearranging finitely many initial terms can change convergence.

Question 5 – Solution**Step 1: Use periodicity.**

Since adding 2π does not change sine and increasing n by 4 adds 2π to $n\pi/2$, the values repeat every four indices:

$n \pmod{4}$	1	2	3	0
a_n	1	0	-1	0

Thus the first eight terms are $1, 0, -1, 0, 1, 0, -1, 0$.

Step 2: Select decisive subsequences.

In particular,

$$a_{4k+1} = 1 \quad \text{and} \quad a_{4k+3} = -1$$

for all applicable k . The first subsequence converges to 1, while the second converges to -1 .

Step 3: Conclude divergence.

If (a_n) converged, all its subsequences would share the same limit. These constant subsequences have different limits, so (a_n) diverges.

Step 4: Analyze finite modifications.

Changing, deleting, or rearranging only finitely many initial terms cannot change convergence because the definition concerns all terms beyond some index. No finite modification removes the infinitely recurring values 1 and -1 . Even an arbitrary permutation of all these terms still contains infinitely many occurrences of both values, so it cannot converge.

Question 6

Let $a_n = n^{1/n}$ for $n \geq 1$.

- (a) Use logarithms to find $\lim_{n \rightarrow \infty} a_n$.
- (b) For which integers n is $a_n > 1$?
- (c) Use the continuous extension $f(x) = x^{1/x}$ to explain why the terms first rise and then decrease, even though their limit is 1.

Question 6 – Solution

Step 1: Transform the variable exponent.

Because $a_n = n^{1/n} > 0$, logarithms are valid. Let $b_n = \ln a_n$. Then

$$b_n = \ln(n^{1/n}) = \frac{\ln n}{n}.$$

Step 2: Evaluate the logarithmic limit.

Extend the quotient to $g(x) = \ln x/x$. It has the indeterminate form ∞/∞ , and L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Restricting this continuous limit to integer values gives $b_n \rightarrow 0$. By continuity of the exponential function, $a_n = e^{b_n} \rightarrow e^0 = 1$.

Step 3: Determine when the terms exceed the limit.

For every integer $n > 1$, the base $n > 1$ and exponent $1/n > 0$, so $a_n > 1$; also $a_1 = 1$. A sequence may approach a limit entirely from one side, so convergence to 1 does not require later terms to equal or cross 1.

Step 4: Explain the rise and fall.

For the continuous extension $f(x) = x^{1/x}$, logarithmic differentiation gives

$$\frac{f'(x)}{f(x)} = \frac{1 - \ln x}{x^2}.$$

Because $f(x) > 0$, the sign of $f'(x)$ is the sign of $1 - \ln x$. Thus f increases for $x < e$ and decreases for $x > e$. Among integer indices, the maximum occurs at $n = 3$, after which the sequence decreases toward 1 while remaining above it.

Question 7

Let $a_n = \frac{\ln n}{\sqrt{n}}$ for $n \geq 1$.

- (a) Use a continuous extension and L'Hopital's Rule to find the limit.
- (b) Explain why taking logarithms of a_n is not the most direct method.
- (c) Use the derivative of $f(x) = \ln x / \sqrt{x}$ to explain the initial increase of the sequence and identify where the eventual decrease begins.

Question 7 – Solution

Step 1: Choose a continuous extension.

Let $f(x) = \ln x / \sqrt{x}$ for $x \geq 1$. Both numerator and denominator tend to infinity, so the quotient has the indeterminate form ∞/∞ and L'Hopital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Restricting the continuous limit to integer inputs gives $a_n \rightarrow 0$.

Step 2: Compare possible methods.

Taking $\ln(a_n) = \ln(\ln n) - \frac{1}{2} \ln n$ would introduce an $\infty - \infty$ expression and still require comparing logarithmic growth. Applying L'Hopital's Rule directly to the original quotient is cleaner.

Step 3: Determine where the sequence increases and decreases.

Differentiate the extension:

$$f'(x) = x^{-3/2} \left(1 - \frac{1}{2} \ln x \right).$$

Since $x^{-3/2} > 0$, $f'(x) > 0$ for $x < e^2$, equals 0 at $e^2 \approx 7.39$, and is negative afterward. The integer terms rise through $n = 7$ and then decrease very slowly. This initial increase is consistent with the eventual limit of zero.

Question 8

Define $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

- (a) Prove by induction that $0 \leq a_n < 2$ for every n .
- (b) Prove that (a_n) is increasing, and conclude that it converges.
- (c) Find its limit by solving the appropriate fixed-point equation, and explain why the other algebraic root is inadmissible.

Question 8 – Solution**Step 1: Prove an invariant bound.**

We prove $0 \leq a_n < 2$ by induction. The base case holds because $a_1 = 0$. Assume $0 \leq a_n < 2$. Then

$$0 < 2 + a_n < 4 \implies 0 < a_{n+1} = \sqrt{2 + a_n} < 2.$$

Thus the bound holds for a_{n+1} , completing the induction.

Step 2: Prove monotonicity.

Use induction again. Since $a_2 = \sqrt{2} > a_1$, the base case holds. If $a_n > a_{n-1}$, the square-root function is strictly increasing, so

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{2 + a_{n-1}} = a_n.$$

Thus (a_n) is increasing. Together with the upper bound 2, the Monotone Convergence Theorem guarantees $a_n \rightarrow L$ for some $0 \leq L \leq 2$.

Step 3: Solve the fixed-point equation.

Only after proving convergence may we pass to the limit in the recurrence. Continuity of the square root gives

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0.$$

The algebraic candidates are $L = 2$ and $L = -1$. Because every term and hence its limit are non-negative, $L = -1$ is inadmissible. Therefore $L = 2$. Solving this equation alone would identify only possible limits; boundedness and monotonicity are what prove a limit actually exists.

Question 9

Let $a_n = \frac{(n!)^{1/n}}{n}$ for $n \geq 1$.

- (a) Apply Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ to rewrite a_n as a dominant constant times a factor tending to 1.
- (b) Find $\lim_{n \rightarrow \infty} a_n$ and give its decimal value.
- (c) Explain why the first dozen terms alone do not reveal the limit accurately.

Question 9 – Solution**Step 1: Insert Stirling's approximation.**

Stirling's formula states

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Step 2: Apply the n th root and normalize.All quantities are positive, so taking n th roots and dividing by n gives

$$a_n = \frac{1}{n} (2\pi n)^{1/(2n)} \frac{n}{e} (1 + o(1))^{1/n} = \frac{1}{e} (2\pi n)^{1/(2n)} (1 + o(1))^{1/n}.$$

Step 3: Evaluate the correction factors.

The extra factors tend to 1. For the first factor, take logarithms:

$$\ln \left((2\pi n)^{1/(2n)} \right) = \frac{\ln(2\pi n)}{2n} \rightarrow 0,$$

so exponentiating gives $(2\pi n)^{1/(2n)} \rightarrow 1$. Also, if $u_n = o(1)$, then $(1 + u_n)^{1/n} \rightarrow 1$ because its logarithm is $\ln(1 + u_n)/n \rightarrow 0$. Consequently,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{e} \approx 0.367879.$$

Step 4: Interpret the numerical evidence.

At $n = 12$, $a_{12} \approx 0.4407$, still about 0.0729 above $1/e$. The slowly decaying correction factor $(2\pi n)^{1/(2n)}$ explains why a short numerical table is suggestive but cannot identify the hidden constant reliably.

Question 10

Let $a_n = \cos(1/n)$, where angles are measured in radians.

- (a) Find $\lim_{n \rightarrow \infty} a_n$ using continuity.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain an explicit upper bound for the error $|a_n - L|$.
- (c) Find a simple integer N that guarantees $|a_n - L| < 10^{-4}$ for every $n \geq N$.

Question 10 – Solution

Step 1: Find the limit by continuity.

Since $1/n \rightarrow 0$ and cosine is continuous at 0,

$$L = \lim_{n \rightarrow \infty} \cos(1/n) = \cos 0 = 1.$$

Step 2: Express and bound the error.

Because $\cos(1/n) \leq 1$, the absolute error simplifies to

$$|a_n - L| = 1 - \cos(1/n).$$

Apply $0 \leq 1 - \cos x \leq x^2/2$ with $x = 1/n$:

$$0 \leq |a_n - 1| \leq \frac{1}{2n^2}.$$

Step 3: Solve for a sufficient index.

It is enough to make the upper bound smaller than the desired tolerance:

$$\frac{1}{2n^2} < 10^{-4} \iff n^2 > 5000 \iff n > \sqrt{5000} \approx 70.71.$$

Step 4: State the guarantee precisely.

Thus $N = 71$ guarantees $|a_n - 1| < 10^{-4}$ for every $n \geq 71$, because $1/(2n^2)$ decreases with n . This is a sufficient value obtained from an upper bound; it need not be the smallest possible N for the exact cosine error.

Question 1

Define $a_1 = 2$ and $a_{n+1} = \frac{1}{2} \left(a_n + \frac{3}{a_n} \right)$.

- (a) Explain why this is Newton's method applied to $x^2 - 3 = 0$.
- (b) Prove that $a_n \geq \sqrt{3}$ for every n , and prove that (a_n) is decreasing.
- (c) Conclude that the sequence converges and determine its limit.

Question 1 – Solution**Step 1: Identify the algorithm.**

For $f(x) = x^2 - 3$, Newton's method uses $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Since $f'(x) = 2x$, the update becomes

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{1}{2} \left(x + \frac{3}{x} \right),$$

which is exactly the given recurrence.

Step 2: Establish positivity and a lower bound.

All terms are positive because $a_1 = 2 > 0$ and, whenever $a_n > 0$, both a_n and $3/a_n$ are positive. More precisely,

$$a_{n+1} - \sqrt{3} = \frac{(a_n - \sqrt{3})^2}{2a_n} \geq 0,$$

The right side is nonnegative, so $a_{n+1} \geq \sqrt{3}$. Since $a_1 \geq \sqrt{3}$, induction proves $a_n \geq \sqrt{3}$ for every n .

Step 3: Prove monotonicity.

Using the lower bound,

$$a_{n+1} - a_n = \frac{3 - a_n^2}{2a_n} \leq 0,$$

Thus $a_{n+1} \leq a_n$, so (a_n) is decreasing.

Step 4: Prove convergence before solving for the limit.

The sequence is decreasing and bounded below by $\sqrt{3}$. The Monotone Convergence Theorem therefore guarantees $a_n \rightarrow L$ for some $L \geq \sqrt{3}$. Continuity and $L > 0$ allow passage to the limit:

$$L = \frac{1}{2} \left(L + \frac{3}{L} \right), \quad L^2 = 3.$$

The equation gives $L = \pm\sqrt{3}$, and the established positive lower bound selects $L = \sqrt{3}$.

Question 2

Let (b_n) be a bounded real sequence and define $a_n = \sup\{b_k : k \geq n\}$.

- (a) Explain why every a_n exists.
- (b) Prove that (a_n) is decreasing and bounded, and hence convergent.
- (c) State what its limit represents, and compute it when $b_n = (-1)^n + 1/n$.

Question 2 – Solution

Step 1: Verify that each supremum exists.

Let $T_n = \{b_k : k \geq n\}$. Each tail is nonempty, and because (b_n) is bounded, T_n is bounded above. The least-upper-bound property of $\mathbb{R}$ therefore guarantees that $a_n = \sup T_n$ exists.

Step 2: Prove monotonicity.

Removing the first element of a tail cannot increase its set of possible values. Formally, $T_{n+1} \subseteq T_n$, so

$$a_{n+1} = \sup T_{n+1} \leq \sup T_n = a_n.$$

Thus (a_n) is decreasing.

Step 3: Establish a lower bound and convergence.

If m is any lower bound for (b_n) , then $m \leq b_k \leq a_n$ for every $k \geq n$, so $m \leq a_n$. Hence (a_n) is bounded below. The Monotone Convergence Theorem now gives convergence, and its limit is called the limit superior:

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n.$$

Step 4: Apply the construction to the example.

For $b_n = (-1)^n + 1/n$, even terms equal $1 + 1/(2j)$ and odd terms are negative. Among the even terms in a tail, the earliest one is largest. If e_n is the first even integer satisfying $e_n \geq n$, then

$$a_n = 1 + \frac{1}{e_n} \longrightarrow 1.$$

Thus $\limsup b_n = 1$. The tail-supremum process has turned oscillating bounded data into a decreasing upper envelope.

Question 3

Let $a_n = \{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$.

- (a) Prove that $0 \leq a_n < 1$, so the sequence is bounded.
- (b) Using the fact that the fractional parts of multiples of an irrational number are dense in $[0, 1]$, find two different subsequential limits.
- (c) Decide whether (a_n) converges and explain why boundedness is insufficient.

Question 3 – Solution**Step 1: Establish boundedness.**

For every real x , its fractional part satisfies $0 \leq \{x\} < 1$. Therefore $0 \leq a_n < 1$ for all n , and the sequence is bounded.

Step 2: Use irrationality and density.

Because $\sqrt{2}$ is irrational, the density theorem for irrational rotations says that the set $\{n\sqrt{2}\} : n \geq 1\}$ is dense in $[0, 1]$. In particular, every open subinterval contains infinitely many terms, allowing the selection of increasing subsequence indices.

Step 3: Construct two subsequences.

For each positive integer j , choose increasing indices n_j and m_j such that

$$0 < a_{n_j} < 1/j, \quad 1/2 - 1/j < a_{m_j} < 1/2 + 1/j.$$

The first inequality and Squeeze Theorem give $a_{n_j} \rightarrow 0$; the second gives $a_{m_j} \rightarrow 1/2$.

Step 4: Apply the subsequence criterion.

Every subsequence of a convergent sequence must approach the same limit as the original sequence. Since these two subsequences have different limits, (a_n) diverges. Boundedness alone guarantees neither monotonicity nor convergence.

Question 4

Let $a_n = \sum_{k=1}^n 2^{-k}$.

- (a) Derive a closed formula for a_n from the finite geometric-sum formula.
- (b) Prove that (a_n) converges and find its limit.
- (c) Find the exact remaining gap $L - a_n$ and the smallest n for which this gap is below 10^{-3} .

Question 4 – Solution

Step 1: Identify and sum the finite geometric series.

The terms form a geometric sequence with first term $a = 1/2$ and common ratio $r = 1/2$. Therefore

$$a_n = \frac{(1/2)(1 - (1/2)^n)}{1 - 1/2} = 1 - 2^{-n}.$$

Step 2: Determine convergence and the limit.

Since $2^{-n} \rightarrow 0$, the closed formula gives $a_n \rightarrow L = 1$. Also $a_{n+1} - a_n = 2^{-(n+1)} > 0$ and $a_n < 1$, so the sequence is increasing and bounded above, independently confirming convergence by the Monotone Convergence Theorem.

Step 3: Compute the exact remaining gap.

The exact error after n terms is

$$L - a_n = 2^{-n}.$$

Step 4: Solve the accuracy inequality.

We need $2^{-n} < 10^{-3}$, equivalently $2^n > 1000$. Since $2^9 = 512 \leq 1000$ but $2^{10} = 1024 > 1000$, the smallest qualifying index is $n = 10$.

Question 5

Let $a_n = \frac{n^2}{2^n}$ for $n \geq 1$.

- (a) Use a_{n+1}/a_n to determine exactly when $a_{n+1} < a_n$.
- (b) Identify the largest term and the first index from which the sequence is strictly decreasing.
- (c) Prove that $a_n \rightarrow 0$.

Question 5 – Solution

Step 1: Compare consecutive positive terms.

Because $a_n > 0$, the inequality $a_{n+1} < a_n$ is equivalent to $a_{n+1}/a_n < 1$. Compute

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2}.$$

Step 2: Solve the ratio inequality exactly.

Thus $a_{n+1} < a_n$ exactly when

$$(n+1)^2 < 2n^2 \iff n^2 - 2n - 1 > 0 \iff n > 1 + \sqrt{2}.$$

For integer $n \geq 1$, this is equivalent to $n \geq 3$. Since $a_1 = 1/2 < a_2 = 1 < a_3 = 9/8$ and $a_{n+1} < a_n$ for every $n \geq 3$, a_3 is the unique largest term and the sequence is strictly decreasing from index 3 onward.

Step 3: Prove the decreasing tail approaches zero.

A decreasing positive sequence has a limit, but the ratio bound identifies it directly. For $n \geq 7$,

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \leq \frac{1}{2} \left(\frac{8}{7}\right)^2 = \frac{32}{49} < 1.$$

Iterating this inequality gives $0 < a_n \leq a_7(32/49)^{n-7}$ for $n \geq 7$. The right side is geometric with ratio less than 1, so it tends to zero. The Squeeze Theorem yields $a_n \rightarrow 0$.

Question 6

Let $a_n = \frac{(-1)^n}{n}$.

- (a) Prove that $a_n \rightarrow 0$ using an absolute-value estimate.
- (b) Show that the full sequence is not monotone and is not eventually monotone.
- (c) Describe the monotonic behavior of its even and odd subsequences.

Question 6 – Solution**Step 1: Prove convergence.**

Use absolute values to remove the alternating sign:

$$|a_n - 0| = \frac{1}{n} \longrightarrow 0,$$

Since $1/n \rightarrow 0$, the definition of convergence gives $a_n \rightarrow 0$. Equivalently, $-1/n \leq a_n \leq 1/n$ and the Squeeze Theorem applies.

Step 2: Test eventual monotonicity.

For every positive integer k ,

$$a_{2k} = \frac{1}{2k} > 0, \quad a_{2k+1} = -\frac{1}{2k+1} < 0.$$

Thus $a_{2k} > a_{2k+1}$ but $a_{2k+1} < a_{2k+2}$, producing infinitely many decreases and increases. Because this occurs for arbitrarily large k , no tail can be monotone.

Step 3: Analyze the two subsequences.

The even subsequence $a_{2k} = 1/(2k)$ decreases to 0, while the odd subsequence $a_{2k-1} = -1/(2k-1)$ increases to 0. Both close in on the same limit from opposite sides. This demonstrates that convergence does not require eventual monotonicity.

Question 7

Let $a_n = \sqrt{n+1} - \sqrt{n}$.

- (a) Rationalize the expression and use the result to find the limit.
- (b) Prove that (a_n) is positive and decreasing.
- (c) Show that $a_n \sim 1/(2\sqrt{n})$ and interpret this as the spacing between neighboring square roots.

Question 7 – Solution

Step 1: Rationalize the difference.

Multiplying numerator and denominator by the conjugate avoids the indeterminate form $\infty - \infty$:

$$a_n = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Step 2: Find the limit and monotonicity.

The denominator tends to infinity, so $a_n \rightarrow 0$. It is also positive. Furthermore, $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$; taking reciprocals of positive quantities reverses the inequality and gives $a_{n+1} < a_n$. Thus the sequence decreases to zero.

Step 3: Determine the asymptotic gap size.

Compare a_n with $1/(2\sqrt{n})$ by taking their ratio:

$$\frac{a_n}{1/(2\sqrt{n})} = \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{1+1/n} + 1} \rightarrow 1.$$

Because the ratio tends to 1, $a_n \sim 1/(2\sqrt{n})$. Therefore neighboring square roots get closer, and their gap is approximately $1/(2\sqrt{n})$ for large n .

Question 8

Let $a_n = \int_0^1 x^n dx$ for $n \geq 1$.

- (a) Evaluate the integral and find $\lim_{n \rightarrow \infty} a_n$.
- (b) Prove directly from the integrands that (a_n) is decreasing.
- (c) Give a geometric explanation for why the areas approach zero, despite $x^n = 1$ at $x = 1$.

Question 8 – Solution

Step 1: Evaluate the area exactly.

By the power rule,

$$a_n = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1} \rightarrow 0.$$

Step 2: Prove monotonicity directly from the integrands.

For $0 \leq x \leq 1$, $0 \leq x^{n+1} \leq x^n$, with strict inequality on $(0, 1)$. Integrating over $[0, 1]$ gives $0 < a_{n+1} < a_n$, so the areas decrease.

Step 3: Explain the geometry rigorously.

For every fixed $x \in [0, 1)$, $x^n \rightarrow 0$; only the single endpoint $x = 1$ remains at height 1, and one point has zero width and contributes no area. For a quantitative version, fix $0 < c < 1$ and split the interval:

$$\int_0^1 x^n dx = \int_0^c x^n dx + \int_c^1 x^n dx \leq c^n + (1 - c).$$

Given $\varepsilon > 0$, first choose c so that $1 - c < \varepsilon/2$. Then choose n so large that $c^n < \varepsilon/2$. The displayed bound gives $a_n < \varepsilon$, proving geometrically that the areas vanish.

Question 9

For $n \geq 2$, let $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$.

- (a) Take logarithms and use $\ln(1 - x) \leq -x$ for $0 < x < 1$ to obtain a useful bound.
- (b) Prove that $a_n \rightarrow 0$.
- (c) Use $\ln(1 - x) = -x - x^2/2 + O(x^3)$ to describe the rate of decay more precisely.

Question 9 – Solution**Step 1: Take logarithms.**

For $n \geq 2$, $0 < 1 - 1/n < 1$, so $a_n > 0$ and logarithms are valid:

$$\ln a_n = n^2 \ln \left(1 - \frac{1}{n} \right).$$

Step 2: Obtain a decisive upper bound.

For $0 < x < 1$, $\ln(1-x) \leq -x$. With $x = 1/n$, this gives $\ln a_n \leq -n$. Because the exponential function is increasing,

$$0 < a_n \leq e^{-n} \longrightarrow 0.$$

The Squeeze Theorem proves $a_n \rightarrow 0$.

Step 3: Refine the decay rate.

For a sharper description, substitute $x = 1/n$ into $\ln(1-x) = -x - x^2/2 + O(x^3)$:

$$\ln a_n = n^2 \left(-\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n - \frac{1}{2} + O(n^{-1}).$$

Exponentiating both sides gives

$$a_n = e^{-n-1/2+O(1/n)} \sim e^{-n-1/2}.$$

The notation $u_n \sim v_n$ means $u_n/v_n \rightarrow 1$. Thus the quadratic exponent repeats a base close to 1 often enough to produce decay essentially like e^{-n} , with the additional constant factor $e^{-1/2}$.

Question 10

Define $a_1 = 0$ and $a_{n+1} = \frac{1}{2 + a_n}$.

- (a) Show that $0 \leq a_n \leq 1/2$ for all n .
- (b) Find the unique fixed point L in this interval and prove that consecutive terms alternate around it.
- (c) Establish an error contraction and use it to prove $a_n \rightarrow L$.

Question 10 – Solution**Step 1: Establish an invariant interval.**

The bound follows by induction. The base case $a_1 = 0$ lies in $[0, 1/2]$. If $0 \leq a_n \leq 1/2$, then $2 + a_n \geq 2$, so $0 < a_{n+1} = 1/(2 + a_n) \leq 1/2$. Hence every term remains in $[0, 1/2]$.

Step 2: Find the admissible fixed point.

A fixed point satisfies

$$L = \frac{1}{2 + L} \iff L^2 + 2L - 1 = 0,$$

so $L = -1 \pm \sqrt{2}$. Only $L = \sqrt{2} - 1$ lies in the invariant interval $[0, 1/2]$.

Step 3: Explain the alternating behavior.

Let $f(x) = 1/(2 + x)$. Because $f'(x) = -1/(2 + x)^2 < 0$, f is strictly decreasing. Since $a_1 < L$, $a_2 = f(a_1) > f(L) = L$. Applying the decreasing map reverses the side of L at every iteration, so successive terms alternate across the fixed point.

Step 4: Prove convergence with an error contraction.

$$|a_{n+1} - L| = \left| \frac{1}{2 + a_n} - \frac{1}{2 + L} \right| = \frac{|a_n - L|}{(2 + a_n)(2 + L)} \leq \frac{1}{4} |a_n - L|.$$

Both denominator factors are at least 2, which justifies the factor $1/4$. Iterating gives

$$|a_n - L| \leq \left(\frac{1}{4}\right)^{n-1} |a_1 - L| \longrightarrow 0.$$

Therefore $a_n \rightarrow \sqrt{2} - 1$; the even and odd subsequences approach the same value from opposite sides.

Question 1

Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

- (a) Identify the first term and common ratio, then evaluate the series using the infinite geometric-series formula.
- (b) Derive a formula for the N th partial sum s_N and evaluate $\lim_{N \rightarrow \infty} s_N$.
- (c) Find the exact remainder $R_N = S - s_N$ and determine the smallest N for which $R_N < 10^{-4}$.

Question 1 – Solution

Step 1: Identify the geometric structure.

Writing the first terms as $3/4 + 3/16 + 3/64 + \cdots$ shows that the first term is $a = 3/4$ and the common ratio is $r = 1/4$. Since $|r| < 1$, the infinite geometric-series formula applies:

$$S = \frac{a}{1-r} = \frac{3/4}{1-1/4} = 1.$$

Step 2: Confirm the result from partial sums.

For a geometric series, the finite sum is $s_N = a(1 - r^N)/(1 - r)$. Substitution gives

$$s_N = \frac{3}{4} \frac{1 - (1/4)^N}{1 - 1/4} = 1 - 4^{-N},$$

Since $4^{-N} \rightarrow 0$, $s_N \rightarrow 1$, confirming the result from the definition of an infinite series.

Step 3: Find and use the exact remainder.

The exact remainder after N terms is

$$R_N = S - s_N = 4^{-N}.$$

We require $4^{-N} < 10^{-4}$, equivalently $4^N > 10^4$. Since $4^6 = 4096 \leq 10^4$ but $4^7 = 16384 > 10^4$, the smallest choice is $N = 7$. The strict inequality is important.

Question 2

Consider $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

- (a) Write out the first four terms and display the cancellation.
- (b) Find a formula for the N th partial sum, identifying the surviving boundary terms.
- (c) Decide whether the series converges and, if so, find its sum and exact remainder.

Question 2 – Solution**Step 1: Form a finite partial sum.**

Never telescope directly to infinity. First write the N th partial sum:

$$\begin{aligned}s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}.\end{aligned}$$

Step 2: Identify the surviving boundary terms.

Every interior fraction appears once positively and once negatively, so it cancels. Only the initial 1 and final $-1/(N+1)$ survive.

Step 3: Take the partial-sum limit.

By definition, the series converges exactly when (s_N) has a finite limit. Here

$$S = \lim_{N \rightarrow \infty} s_N = 1.$$

Step 4: Compute and verify the remainder.

The exact tail after N terms is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As an algebraic check, $s_N - s_{N-1} = 1/N - 1/(N+1)$, which reproduces the original N th summand exactly.

Question 3

For a real parameter x , consider $\sum_{n=0}^{\infty} x^n$.

- (a) For $x \neq 1$, derive the finite identity $s_N = (1 - x^{N+1})/(1 - x)$.
- (b) Determine every x for which the infinite series converges and find its sum there.
- (c) Analyze $x = 1$, $x = -1$, and $|x| > 1$ separately.

Question 3 – Solution

Step 1: Derive the finite identity.

Let $s_N = 1 + x + \cdots + x^N$. Then $xs_N = x + x^2 + \cdots + x^{N+1}$. Subtracting the second equation from the first gives

$$(1 - x)s_N = 1 - x^{N+1},$$

so for $x \neq 1$ division by $1 - x$ yields

$$s_N = \frac{1 - x^{N+1}}{1 - x}.$$

Step 2: Determine the convergent parameter region.

If $|x| < 1$, then $x^{N+1} \rightarrow 0$. Therefore the partial sums converge and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Step 3: Check the excluded cases.

At $x = 1$, $s_N = N + 1 \rightarrow \infty$. At $x = -1$, the partial sums alternate between 1 and 0 and have no limit. If $|x| > 1$, the terms x^n do not approach zero, so the nth-term test proves divergence. Thus convergence occurs exactly for $|x| < 1$; checking both endpoints separately is essential.

Question 4

Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

- (a) Rewrite the N th partial sum as the logarithm of a product.
- (b) Simplify that product and determine whether the series converges.
- (c) Explain how the term test and the growth of the partial sums illustrate different facts.

Question 4 – Solution**Step 1: Work with a finite partial sum.**

For positive factors, $\ln u + \ln v = \ln(uv)$. Therefore

$$s_N = \ln \prod_{n=1}^N \frac{n+1}{n} = \ln \left(\frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{N+1}{N} \right) = \ln(N+1).$$

Step 2: Simplify the product.

Every interior factor cancels: the numerator $2 \cdot 3 \cdots N$ cancels the same factors in the denominator, leaving $N+1$. This is multiplicative telescoping.

Step 3: Classify the series.

Since $s_N = \ln(N+1) \rightarrow \infty$, the sequence of partial sums is unbounded and the series diverges to $+\infty$.

Step 4: Interpret the nth-term condition.

The individual terms do satisfy

$$\ln \left(\frac{n+1}{n} \right) = \ln(1 + 1/n) \rightarrow 0,$$

which is necessary but not sufficient for convergence. The logarithm grows slowly, explaining why a short numerical table can look nearly stable even though the partial sums are unbounded.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find a partial-fraction decomposition that produces telescoping cancellation.
- (b) Derive the exact N th partial sum and verify it for $N = 1, 2, 3$.
- (c) Find the sum and the exact remainder after N terms.

Question 5 – Solution**Step 1: Find the partial fractions.**

Write $1/[n(n+2)] = A/n + B/(n+2)$. Multiplying by $n(n+2)$ gives $1 = A(n+2) + Bn$. Matching coefficients yields $A = 1/2$ and $B = -1/2$, so

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite partial sum.

Hence

$$s_N = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{3}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}.$$

The surviving terms are 1, $1/2$, $-1/(N+1)$, and $-1/(N+2)$. For $N = 1, 2, 3$, the formula gives $1/3$, $11/24$, and $21/40$, matching direct addition.

Step 3: Take the limit.

Since both remaining reciprocal terms approach zero,

$$S = \frac{3}{4}.$$

Step 4: State the exact error.

Subtracting the partial sum from S gives the exact positive tail

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

Question 6

Consider $\sum_{n=1}^{\infty} (-1)^n$.

- (a) List the first six terms and partial sums.
- (b) Prove that the sequence of partial sums is bounded but divergent.
- (c) Apply the nth-term test and explain why bounded partial sums alone do not imply convergence.

Question 6 – Solution**Step 1: Distinguish terms from partial sums.**

The series terms are $a_n = (-1)^n$, so the first six are $-1, 1, -1, 1, -1, 1$. Adding successively gives the partial sums

$$-1, 0, -1, 0, -1, 0, \dots$$

More precisely,

$$s_{2k} = 0, \quad s_{2k-1} = -1.$$

Step 2: Test convergence of the partial sums.

The sequence (s_N) is bounded between -1 and 0 , but $s_{2k} = 0 \rightarrow 0$ while $s_{2k-1} = -1 \rightarrow -1$. Since these subsequences have different limits, (s_N) does not converge. By definition, the series therefore diverges.

Step 3: Apply the nth-term test as a check.

The terms $(-1)^n$ do not approach zero, so the nth-term test also proves divergence immediately. Indeed, if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. This example shows that bounded partial sums are not enough; the partial sums must approach one number.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric-series identity to derive a formula for $\sum_{n=1}^{\infty} nx^{n-1}$ when $|x| < 1$.
- (b) Use that formula at $x = 1/2$ to evaluate the series.
- (c) Derive the finite partial sum and exact remainder after N terms.

Question 7 – Solution

Step 1: Start from the geometric series.

For $|x| < 1$, the geometric identity is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Step 2: Differentiate within the interval of convergence.

Power series may be differentiated term by term at every interior point of their interval of convergence. Thus

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x aligns the power with the desired form:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Step 3: Substitute the required value.

Since $|1/2| < 1$, substitution is valid and gives $(1/2)/(1/2)^2 = 2$.

Step 4: Derive the finite sum and remainder.

Differentiating the finite identity $1 + x + \cdots + x^N = (1 - x^{N+1})/(1 - x)$ and multiplying by x yields

$$\sum_{n=1}^N nx^n = \frac{x - (N+1)x^{N+1} + Nx^{N+2}}{(1-x)^2}.$$

At $x = 1/2$, this simplifies to

$$s_N = 2 - \frac{N+2}{2^N}, \quad R_N = 2 - s_N = \frac{N+2}{2^N}.$$

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$.

- (a) Reindex the series using $k = n + 1$.
- (b) Compare it with the Maclaurin series for e and identify every omitted term.
- (c) Find the sum and give a simple upper bound for the remainder after N terms.

Question 8 – Solution**Step 1: Reindex carefully.**

Let $k = n + 1$. When $n = 1$, the new index is $k = 2$, and as $n \rightarrow \infty$, so does k . Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{k=2}^{\infty} \frac{1}{k!}.$$

Step 2: Compare with the exponential series.

The Maclaurin series for $e = e^1$ is

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{\infty} \frac{1}{k!}.$$

The desired tail begins at $k = 2$, so the omitted terms are exactly $1/0! = 1$ and $1/1! = 1$. Therefore

$$S = e - 2.$$

Step 3: Bound the remainder.

After the original terms through $n = N$, the tail starts at $k = N + 2$. For $j \geq 0$, each new factorial factor is at least $N + 3$, so

$$(N + 2 + j)! \geq (N + 2)!(N + 3)^j.$$

Taking reciprocals and summing gives

$$0 < R_N \leq \frac{1}{(N + 2)!} \sum_{j=0}^{\infty} \frac{1}{(N + 3)^j} = \frac{N + 3}{(N + 2)(N + 2)!}.$$

This bound is explicit, positive, and tends to zero, confirming the convergence of the partial sums to $e - 2$.

Question 9

Consider $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

- (a) Write the first four terms and expose the telescoping cancellation.
- (b) Derive the N th partial sum and determine whether the series converges.
- (c) Reconcile the divergence with the fact that the individual terms approach zero.

Question 9 – Solution**Step 1: Form the finite partial sum.**

Write the N th partial sum before passing to infinity:

$$\begin{aligned}s_N &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{N+1} - \sqrt{N}) \\ &= \sqrt{N+1} - 1.\end{aligned}$$

Step 2: Analyze the partial-sum limit.

All interior square-root terms cancel, leaving only the first negative boundary term and final positive boundary term. Because $\sqrt{N+1} - 1 \rightarrow \infty$, the partial sums are unbounded and the series diverges to $+\infty$.

Step 3: Check the n th-term condition.

The summand does approach zero. Rationalization gives

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Step 4: Reconcile the two facts.

There is no contradiction: the n th-term condition is necessary, not sufficient. Here every increment is positive and shrinks to zero, but the accumulated total still grows without bound. Since the denominator above is asymptotic to $2\sqrt{n}$, each increment is asymptotic to $1/(2\sqrt{n})$, consistent with a divergent p -series scale.

Question 10

For a real parameter r , consider $\sum_{n=1}^{\infty} r^{2n}$.

- (a) Identify the first term and common ratio as functions of r .
- (b) Determine all r for which the series converges and find its sum.
- (c) Analyze the boundary values $r = \pm 1$ and the region $|r| > 1$ using the n th-term test.

Question 10 – Solution**Step 1: Identify the geometric data.**

Expanding the first terms gives

$$r^2 + r^4 + r^6 + \cdots ,$$

Thus the first term is $a = r^2$ and the common ratio is $q = r^2$. A geometric series converges precisely when $|q| < 1$, so

$$|r^2| < 1 \iff |r| < 1.$$

Step 2: Find the sum on the convergence interval.For $|r| < 1$, the geometric formula gives

$$S(r) = \frac{r^2}{1 - r^2}.$$

The finite partial sum provides a direct verification:

$$s_N = \frac{r^2(1 - r^{2N})}{1 - r^2} \quad (r^2 \neq 1),$$

When $|r| < 1$, $r^{2N} \rightarrow 0$, so $s_N \rightarrow r^2/(1 - r^2)$.**Step 3: Check the endpoints and exterior region.**

At both $r = 1$ and $r = -1$, every term r^{2n} equals 1, so $s_N = N \rightarrow \infty$. If $|r| > 1$, then r^{2n} grows rather than approaches zero; the nth-term test proves divergence. Thus the convergence set is exactly $(-1, 1)$.

Question 1

Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

- (a) Apply the nth-term test and explain why its result is inconclusive.
- (b) Group terms between consecutive perfect squares and find a lower bound for each block.
- (c) Use the block estimates to classify the series.

Question 1 – Solution

Step 1: Check the necessary condition.

Let $a_n = 1/\sqrt{n}$. Since $\sqrt{n} \rightarrow \infty$, $a_n \rightarrow 0$. This necessary condition is satisfied, so the n th-term test is inconclusive; it can prove divergence when the limit is nonzero, but it can never prove convergence.

Step 2: Choose a grouping strategy.

The terms decrease too slowly for a geometric comparison. Grouping between consecutive squares makes their sizes and the number of terms in each block easy to control. For $k \geq 1$, consider $k^2 \leq n < (k+1)^2$. There are

$$(k+1)^2 - k^2 = 2k + 1$$

terms, and every one satisfies $\sqrt{n} < k+1$, hence $1/\sqrt{n} > 1/(k+1)$. Therefore the k th block has sum greater than

$$\frac{2k+1}{k+1} > 1.$$

Step 3: Convert the block estimate into divergence.

The partial sum through $(m+1)^2 - 1$ contains m such blocks and consequently exceeds m . Since m can be arbitrarily large, the increasing sequence of partial sums is unbounded.

Step 4: State and check the conclusion.

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.}$$

This agrees with the p -series rule: $p = 1/2 \leq 1$.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

- (a) Verify that the terms approach zero and explain why that does not settle convergence.
- (b) Group the indices into dyadic blocks $2^k \leq n < 2^{k+1}$.
- (c) Find a lower bound for each block and use it to classify the series.

Question 2 – Solution**Step 1: Check the terms.**

For $n \geq 2$, $a_n = 1/(n \ln n) > 0$, and $n \ln n \rightarrow \infty$. Thus $a_n \rightarrow 0$. The n th-term test is inconclusive, so a tail estimate is required.

Step 2: Form dyadic blocks.

Use the block $2^k \leq n < 2^{k+1}$. It contains exactly 2^k integers. Within this block,

$$n < 2^{k+1}, \quad \ln n < (k+1) \ln 2,$$

and hence

$$\frac{1}{n \ln n} > \frac{1}{2^{k+1}(k+1) \ln 2}.$$

Step 3: Estimate one complete block.

Because the block contains 2^k terms, its sum is greater than

$$2^k \frac{1}{2^{k+1}(k+1) \ln 2} = \frac{1}{2(k+1) \ln 2}.$$

Step 4: Compare the block series.

Summing these lower bounds over k produces the constant multiple $(2 \ln 2)^{-1} \sum 1/(k+1)$ of the divergent harmonic series. Hence the original partial sums are unbounded and

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Step 5: Interpret the slow growth.

The hidden continuous substitution is $u = \ln x$, since $du = dx/x$ and $\int dx/(x \ln x) = \ln(\ln x) + C$. This explains the exceptionally slow divergence; a large numerical cutoff can misleadingly suggest stabilization.

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ converges or diverges (angles in radians).

- (a) Explain why $0 \leq \sin^2 n/n \leq 1/n$ does not determine convergence.
- (b) Use $\sin^2 n = (1 - \cos 2n)/2$ to split the partial sums.
- (c) Show that $\sum \cos(2n)/n$ converges by the Dirichlet Test, then classify the original series.

Question 3 – Solution

Step 1: Audit the obvious comparison.

The terms satisfy $0 \leq \sin^2 n/n \leq 1/n$ and approach zero. However, an upper comparison with a divergent series is logically inconclusive: a smaller positive series may converge or diverge.

Step 2: Separate the average and oscillatory parts.

Using $\sin^2 n = (1 - \cos 2n)/2$ gives

$$s_N = \frac{1}{2} \sum_{n=1}^N \frac{1}{n} - \frac{1}{2} \sum_{n=1}^N \frac{\cos(2n)}{n}.$$

Step 3: Control the oscillatory series.

To apply Dirichlet's Test to $\sum \cos(2n)/n$, verify both hypotheses. First, the partial sums of $\cos(2n)$ are bounded. Indeed,

$$\sum_{n=1}^N e^{2in} = e^{2i} \frac{1 - e^{2iN}}{1 - e^{2i}},$$

whose magnitude is at most $2/|1 - e^{2i}|$; taking real parts preserves boundedness. Second, $1/n$ is positive, decreasing, and tends to zero. Dirichlet's Test therefore shows that $\sum \cos(2n)/n$ converges.

Step 4: Combine the two parts.

The first part of s_N is one-half of the harmonic partial sum and tends to $+\infty$, while the oscillatory part approaches a finite limit. Subtracting a bounded quantity cannot stop the harmonic growth. Hence

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} \text{ diverges.}}$$

The needed extra structure is that $\sin^2 n$ has positive average $1/2$, not merely that it is bounded.

Question 4

Determine whether $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}$ converges or diverges.

- (a) Identify the reciprocal-power benchmark suggested by the leading terms.
- (b) Compute the limit-comparison ratio with that benchmark.
- (c) State the conclusion and give a direct upper comparison valid for $n \geq 1$.

Question 4 – Solution**Step 1: Identify a benchmark.**

The terms are positive. Comparing the highest powers gives $(2n)/(n^3) = 2/n^2$, so the natural benchmark is the convergent p -series $b_n = 1/n^2$.

Step 2: Compute the limit-comparison ratio.

$$\lim_{n \rightarrow \infty} \frac{(2n+1)/(n^3+1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{2n^3+n^2}{n^3+1} = 2.$$

The ratio limit is 2, which is finite and strictly positive. These are exactly the hypotheses of the Limit Comparison Test. Since $\sum 1/n^2$ converges ($p = 2 > 1$), the given series converges.

Step 3: Verify with a direct inequality.

For $n \geq 1$, $2n+1 \leq 3n$ and $n^3+1 \geq n^3$, so

$$0 < \frac{2n+1}{n^3+1} \leq \frac{3}{n^2}.$$

Because $\sum 3/n^2$ converges, the Direct Comparison Test independently confirms the conclusion. The lower-order terms affect numerical values but not tail behavior.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$ converges or diverges.

- (a) Simplify the factorial quotient completely before selecting a test.
- (b) Express the N th partial sum using harmonic numbers.
- (c) Classify the series and explain why its factorial notation is deceptive.

Question 5 – Solution

Step 1: Simplify before selecting a test.

Use $(n+1)! = (n+1)n!$ to cancel the common factorial:

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Step 2: Rewrite the partial sums.

Thus

$$s_N = \sum_{n=1}^N \frac{1}{n+1} = \sum_{k=2}^{N+1} \frac{1}{k} = H_{N+1} - 1,$$

where H_m is the m th harmonic number.

Step 3: Apply the known harmonic behavior.

Because $H_{N+1} \rightarrow \infty$, subtracting the fixed number 1 does not change the divergence. Therefore the partial sums are unbounded and

$$\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} \text{ diverges.}$$

Step 4: Interpret and verify.

The terms $1/(n+1)$ do approach zero, so the n th-term test alone is inconclusive. The factorial notation suggests rapid growth, but all of $n!$ cancels, leaving exactly a shifted harmonic series. Removing or shifting finitely many harmonic terms never changes convergence or divergence.

Question 6

Determine whether $\sum_{n=1}^{\infty} e^{-\sqrt{n}}$ converges or diverges.

- (a) Verify the necessary nth-term condition.
- (b) Group indices into blocks $k^2 \leq n < (k+1)^2$ and bound the sum of each block from above.
- (c) Compare the block bounds with a convergent series.

Question 6 – Solution

Step 1: Check the necessary condition.

Let $a_n = e^{-\sqrt{n}}$. Because $\sqrt{n} \rightarrow \infty$, the exponent tends to $-\infty$ and $a_n \rightarrow 0$. The nth-term condition holds but does not prove convergence.

Step 2: Organize the tail into square blocks.

In the block $k^2 \leq n < (k+1)^2$, there are exactly $2k+1$ indices. Also $\sqrt{n} \geq k$, so the decreasing exponential satisfies $e^{-\sqrt{n}} \leq e^{-k}$. Therefore

$$\sum_{n=k^2}^{(k+1)^2-1} e^{-\sqrt{n}} \leq (2k+1)e^{-k}.$$

Step 3: Classify the series of block bounds.

It remains to check $\sum_{k=1}^{\infty} (2k+1)e^{-k}$. Let $q = e^{-1} < 1$. Then

$$2 \sum_{k=1}^{\infty} kq^k + \sum_{k=1}^{\infty} q^k,$$

Both series converge: $\sum q^k$ is geometric, and $\sum kq^k = q/(1-q)^2$ follows by differentiating the geometric series.

Step 4: Conclude by comparison.

The original nonnegative series is bounded above block by block by a convergent series, so its increasing partial sums are bounded and

$$\sum_{n=1}^{\infty} e^{-\sqrt{n}} \text{ converges.}$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \arctan n$ converges or diverges.

- (a) Evaluate the limit of the summand.
- (b) Apply the nth-term test and describe the growth of the partial sums.
- (c) Explain why no comparison or ratio test is needed.

Question 7 – Solution

Step 1: Apply the nth-term test first.

The summand has limit

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0.$$

Because the limit is not zero, the series diverges immediately by the nth-term test.

Step 2: Describe how the partial sums diverge.

Since $\arctan n \rightarrow \pi/2 > 1$, the definition of a limit guarantees an N_0 such that $\arctan n > 1$ for all $n \geq N_0$. Then for $N \geq N_0$,

$$s_N \geq s_{N_0-1} + (N - N_0 + 1),$$

so $s_N \rightarrow +\infty$ at least linearly.

Step 3: Justify why the test is decisive.

Every convergent series must have terms tending to zero: if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. Since this necessary condition fails, comparison, ratio, or integral tests are unnecessary.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ converges or diverges.

- (a) Verify that the terms approach zero.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain a direct comparison.
- (c) Confirm the quadratic scale by computing a limit-comparison ratio.

Question 8 – Solution**Step 1: Check the terms.**

Since $1/n \rightarrow 0$ and cosine is continuous, $1 - \cos(1/n) \rightarrow 1 - \cos 0 = 0$. The necessary condition holds, so a convergence test is still needed.

Step 2: Build a direct comparison.

The standard small-angle inequality

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

holds for every real x . Substituting $x = 1/n$ yields

$$0 \leq 1 - \cos(1/n) \leq \frac{1}{2n^2}.$$

Both sides are nonnegative, and $\sum 1/(2n^2)$ is a convergent p -series with $p = 2 > 1$. The Direct Comparison Test therefore proves convergence.

Step 3: Confirm the asymptotic scale.

The standard Taylor limit confirms more precisely that

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Thus the summand is asymptotic to $1/(2n^2)$. The finite positive ratio also satisfies the hypotheses of the Limit Comparison Test, independently confirming the conclusion.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ converges or diverges.

- (a) Verify the nth-term condition.
- (b) Perform a limit comparison with the harmonic series.
- (c) Give a direct lower comparison that independently proves divergence.

Question 9 – Solution

Step 1: Check the terms and select a benchmark.

Dividing by n^2 shows $n/(n^2 + 1) = (1/n)/(1 + 1/n^2) \rightarrow 0$, so the n th-term test is inconclusive. The dominant-power behavior suggests the harmonic benchmark $b_n = 1/n$.

Step 2: Perform limit comparison.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

The limit equals 1, which is finite and strictly positive. Since both series have positive terms and $\sum 1/n$ diverges, the Limit Comparison Test proves that the given series diverges.

Step 3: Verify with direct comparison.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$, so taking positive reciprocals and multiplying by n gives

$$\frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The smaller series $\sum 1/(2n)$ diverges; therefore the larger given positive-term series also diverges. This direction of comparison is essential when proving divergence.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$ converges or diverges.

- (a) Show that the denominator is positive for every $n \geq 1$.
- (b) Use $-1 \leq \sin n \leq 1$ to produce a simple upper comparison for the tail.
- (c) Confirm the classification by limit comparison with $1/n^2$.

Question 10 – Solution

Step 1: Check positivity and the domain.

For $n = 1$, $1 + \sin 1 > 0$. For $n \geq 2$, use $\sin n \geq -1$ to obtain

$$n^2 + \sin n \geq n^2 - 1 > 0,$$

so every term is defined and positive.

Step 2: Create an upper bound for the tail.

For $n \geq 2$,

$$n^2 - 1 \geq \frac{3}{4}n^2,$$

because $n^2 \geq 4$. Hence

$$0 < \frac{1}{n^2 + \sin n} \leq \frac{1}{n^2 - 1} \leq \frac{4}{3n^2}.$$

The comparison series is a constant multiple of the convergent p -series $\sum 1/n^2$. The Direct Comparison Test therefore proves convergence of the tail, and adding the finite first term cannot change that conclusion.

Step 3: Confirm the dominant behavior.

Limit comparison gives the sharper asymptotic statement

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + \sin n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \sin n} = 1,$$

because dividing numerator and denominator by n^2 produces $1/(1 + \sin n/n^2)$, and $|\sin n|/n^2 \leq 1/n^2 \rightarrow 0$. The ratio limit 1 confirms that the oscillation in the denominator is asymptotically negligible.

Question 1

Consider the Basel series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- (a) Prove that the series converges using the p -series test.
- (b) State Euler's classical evaluation of its exact sum and give a decimal approximation.
- (c) Use an integral estimate to bound the remainder after N terms, and explain why convergence alone does not produce the exact constant.

Question 1 – Solution**Step 1: Classify the series.**

This is a p -series $\sum 1/n^p$ with $p = 2 > 1$. Therefore it converges. This test establishes existence of a finite sum but does not determine that sum.

Step 2: State the classical special value.

Euler's solution of the Basel problem gives the non-elementary identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934.$$

The equality with $\pi^2/6$ requires additional machinery, such as Euler's product for $\sin x$ or Fourier series; it does not follow from the p -series test alone.

Step 3: Quantify convergence.

Because $f(x) = 1/x^2$ is positive and decreasing,

$$\int_{N+1}^{\infty} \frac{dx}{x^2} \leq R_N \leq \int_N^{\infty} \frac{dx}{x^2},$$

so

$$\frac{1}{N+1} \leq R_N \leq \frac{1}{N}.$$

Thus the partial sums approach $\pi^2/6$ from below, with an error of order $1/N$.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- (a) Prove convergence and compare its speed with the Basel series.
- (b) State Euler's classical value $\zeta(4)$.
- (c) Using that value, compute the even-indexed and odd-indexed fourth-power subseries separately.

Question 2 – Solution**Step 1: Establish convergence.**

The series is a p -series with $p = 4 > 1$, so it converges. Moreover, $1/n^4 \leq 1/n^2$ for $n \geq 1$, so its tail decays substantially faster than the Basel-series tail.

Step 2: State the special value accurately.

Euler's classical evaluation is

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082323.$$

As with $\zeta(2)$, the p -series test proves convergence but not this exact constant.

Step 3: Extract the even contribution.

For even indices $n = 2k$,

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{1440}.$$

Step 4: Extract the odd contribution.

Subtracting the even terms from the full series gives

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{90} - \frac{\pi^4}{1440} = \frac{\pi^4}{96}.$$

The even and odd pieces recombine to $\pi^4/90$, providing an algebraic check.

Question 3

Consider $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

- (a) Identify the geometric data and derive the N th partial sum.
- (b) Find the sum and exact remainder.
- (c) Interpret the result as repeatedly adding half of the remaining distance to 2.

Question 3 – Solution**Step 1: Identify the series.**

The expansion is $1 + 1/2 + 1/4 + \cdots$, so the first term is $a = 1$ and the common ratio is $r = 1/2$. Since $|r| < 1$, the series converges.

Step 2: Derive the finite partial sum.

With indices 0 through N , there are $N + 1$ terms. The finite geometric formula gives

$$s_N = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2 - 2^{-N}.$$

Because $2^{-N} \rightarrow 0$,

$$S = \lim_{N \rightarrow \infty} s_N = 2.$$

Step 3: Compute and interpret the remainder.

The exact gap is $R_N = S - s_N = 2^{-N}$. Starting at $s_0 = 1$, each new term equals one-half of the current distance to 2; the gap is halved at every step. The process never exceeds 2, but it can get arbitrarily close.

Question 4

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the cancellation in the N th partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 4 – Solution**Step 1: Decompose one summand.**

Write $1/[n(n+1)] = A/n + B/(n+1)$. Multiplication by $n(n+1)$ gives $1 = A(n+1) + Bn$, so $A = 1$ and $B = -1$. Hence

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Step 2: Telescope a finite sum.

$$\begin{aligned} s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

All interior reciprocal terms cancel; only the two boundary terms remain.

Step 3: Pass to the limit and measure the error.

Since $1/(N+1) \rightarrow 0$, the series sums to $S = 1$. The exact remainder is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As a check, $s_N - s_{N-1} = 1/[N(N+1)]$, the original N th term.

Question 5

Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify the Alternating Series Test hypotheses.
- (b) Decide whether the convergence is absolute or conditional.
- (c) Derive the value $\ln 2$ from the power series for $\ln(1+x)$ and state an error bound.

Question 5 – Solution**Step 1: Prove convergence.**

Let $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Therefore the Alternating Series Test proves that $\sum (-1)^{n-1} b_n$ converges.

Step 2: Test absolute convergence.

Taking absolute values gives

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

the divergent harmonic series. Hence the original series is conditionally convergent, not absolutely convergent.

Step 3: Identify the sum.

For $-1 < x \leq 1$, the endpoint-valid power-series identity is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

Setting $x = 1$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2.$$

Step 4: Quantify the approximation.

The Alternating Series Estimation Theorem gives $|S - s_N| \leq b_{N+1} = 1/(N+1)$. Odd partial sums lie above $\ln 2$ and even partial sums lie below it.

Question 6

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

- (a) Explain why this selects exactly the odd-indexed terms of the Basel series.
- (b) Compute the even-indexed contribution by scaling $\zeta(2)$.
- (c) Subtract to obtain the odd-square sum and verify that the two pieces recombine correctly.

Question 6 – Solution**Step 1: Partition the Basel series.**

Every positive integer is either even or odd, and the terms are positive, so

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Step 2: Evaluate the even contribution.

Using Euler's value $\zeta(2) = \pi^2/6$,

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

Step 3: Subtract to isolate the odd terms.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$

Step 4: Verify the decomposition.

The even portion is one-fourth of the full sum, while the odd portion is three-fourths. Their sum is $(1/4 + 3/4)\zeta(2) = \zeta(2)$, as required.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$.

- (a) Factor the constant scale from the summand.
- (b) Use the Basel sum to evaluate the series.
- (c) Relate the answer to the odd-square contribution and the full series.

Question 7 – Solution

Step 1: Simplify the scaled denominator.

Since $(2n)^2 = 4n^2$,

$$\frac{1}{(2n)^2} = \frac{1}{4} \frac{1}{n^2}.$$

Step 2: Use linearity of convergent series.

The Basel series converges, so its constant factor may be pulled outside:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \boxed{\frac{\pi^2}{24}}.$$

Step 3: Interpret the result.

The substitution $m = 2n$ selects precisely the even-indexed Basel terms. Their total is one-fourth of the complete sum because squaring the index contributes the scale factor $2^2 = 4$.

Step 4: Check against the odd contribution.

Adding the odd-square value $\pi^2/8$ gives

$$\frac{\pi^2}{24} + \frac{\pi^2}{8} = \frac{\pi^2}{6},$$

recovering the full Basel sum.

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric power series inside its interval of convergence.
- (b) Align the powers and evaluate at $x = 1/2$.
- (c) Find the finite partial sum and exact remainder.

Question 8 – Solution

Step 1: Begin with a generating function.

For $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Power series may be differentiated term by term at interior points, giving

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Step 2: Match and evaluate the desired series.

Multiplying by x yields

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Since $|1/2| < 1$, substitution is valid:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \boxed{2}.$$

Step 3: Quantify the finite approximation.

Differentiating the finite geometric identity gives

$$s_N = \sum_{n=1}^N \frac{n}{2^n} = 2 - \frac{N+2}{2^N}, \quad R_N = \frac{N+2}{2^N}.$$

The remainder is positive and tends to zero, so the partial sums approach 2 from below.

Question 9

Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

- (a) Differentiate the geometric generating function twice.
- (b) Explain why $n^2 = n(n-1) + n$ requires restoring a first-derivative contribution.
- (c) Derive a closed generating function and evaluate it at $x = 1/2$.

Question 9 – Solution**Step 1: Differentiate twice.**

For $|x| < 1$, $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Two derivatives give

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiplying by x^2 produces

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Step 2: Restore the missing linear part.

Because $n^2 = n(n-1) + n$, add the first-derivative identity $\sum nx^n = x/(1-x)^2$:

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}.$$

Step 3: Evaluate at the requested point.

Since $x = 1/2$ lies inside the interval of convergence,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{(1/2)^3} = \boxed{6}.$$

Omitting the $\sum nx^n$ term would compute $n(n-1)$ rather than n^2 .

Question 10

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the two-step telescoping cancellation in a finite partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 10 – Solution

Step 1: Decompose the gap-two denominator.

Solving $1/[n(n+2)] = A/n + B/(n+2)$ gives $A = 1/2$ and $B = -1/2$. Thus

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite sum.

$$\begin{aligned} s_N &= \frac{1}{2} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right) - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N+2} \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right). \end{aligned}$$

The cancellation forms two interlaced chains because the indices differ by two.

Step 3: Take the limit.

Both ending reciprocals tend to zero, so

$$S = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \boxed{\frac{3}{4}}.$$

Step 4: Find the exact remainder.

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

As a check, subtracting consecutive partial-sum formulas reproduces $1/[N(N+2)]$.

Question 1

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

- (a) Define a continuous function f with $f(n) = 1/n^{3/2}$ and verify that it is positive, continuous, and decreasing on the required interval.
- (b) Use the Integral Test to determine whether the series converges or diverges.
- (c) If s_N is the N th partial sum and $R_N = S - s_N$, derive both an upper and a lower bound for R_N .

Show the improper-integral limit and all antiderivative work.

Question 1 – Solution

Step 1: Choose the continuous function.

Let $f(x) = x^{-3/2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{3}{2}x^{-5/2} < 0,$$

so it is decreasing. Also $f(n) = 1/n^{3/2}$.

Step 2: Evaluate the improper integral.

$$\int_1^{\infty} x^{-3/2} dx = \lim_{b \rightarrow \infty} \left[-2x^{-1/2} \right]_1^b = 2.$$

The integral converges, so the series converges by the Integral Test.

Step 3: Bound the remainder.

For a positive decreasing f ,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Therefore

$$\boxed{\frac{2}{\sqrt{N+1}} \leq R_N \leq \frac{2}{\sqrt{N}}}.$$

The test proves convergence; these inequalities additionally measure the truncation error.

Question 2

Consider the logarithmic series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

- (a) Explain why the series begins at $n = 2$, and verify all three Integral Test hypotheses for $f(x) = 1/(x \ln x)$.
- (b) Evaluate the associated improper integral using $u = \ln x$.
- (c) State the convergence or divergence conclusion and describe the growth rate of the truncated integral.

The n th-term test alone is not a complete solution.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \ln x)$ for $x \geq 2$. It is continuous and positive because $x > 1$. Moreover,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0,$$

so f is decreasing, and $f(n)$ equals the summand.

Step 2: Evaluate the improper integral.

With $u = \ln x$ and $du = dx/x$,

$$\int_2^b \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln b} \frac{du}{u} = \ln(\ln b) - \ln(\ln 2).$$

As $b \rightarrow \infty$, $\ln(\ln b) \rightarrow \infty$. Thus the improper integral diverges.

Step 3: Apply the test.

All hypotheses hold, so the Integral Test gives

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

The growth scale $\ln \ln b$ explains why divergence is extremely slow.

Question 3

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- (a) Verify that $f(x) = 1/[x(\ln x)^2]$ is positive, continuous, and decreasing for $x \geq 2$.
- (b) Apply the Integral Test, showing the substitution and evaluation of the improper integral.
- (c) Derive two-sided Integral Test bounds for the remainder R_N .
- (d) Briefly compare the result with $\sum 1/(n \ln n)$.

Question 3 – Solution**Step 1: Verify the hypotheses.**

Define $f(x) = 1/[x(\ln x)^2]$ for $x \geq 2$. It is continuous and positive, and

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

Let $u = \ln x$, $du = dx/x$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Hence the series converges. The extra logarithm power changes $\int du/u$ into the convergent p -integral $\int du/u^2$.

Step 3: Estimate the remainder.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

These bounds follow by integrating from $N+1$ and N , respectively.

Question 4

Consider

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}.$$

- (a) For $f(x) = x/(x^2 + 1)^2$, verify positivity, continuity, and decreasing behavior on $[1, \infty)$.
- (b) Evaluate the associated improper integral using an appropriate substitution.
- (c) Determine convergence and obtain upper and lower bounds for R_N .

Clearly distinguish the value of the integral from the value of the series.

Question 4 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = x/(x^2 + 1)^2$ for $x \geq 1$. It is continuous and positive. Differentiation gives

$$f'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3} < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the integral.

With $u = x^2 + 1$ and $du = 2x \, dx$,

$$\int_1^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int_2^\infty u^{-2} \, du = \frac{1}{4}.$$

The integral converges, so the series converges.

Step 3: Bound the tail.

Since

$$\int_A^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2(A^2 + 1)},$$

the remainder satisfies

$$\boxed{\frac{1}{2((N+1)^2 + 1)} \leq R_N \leq \frac{1}{2(N^2 + 1)}}.$$

Question 5

Consider the nested-logarithm series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}.$$

- (a) Explain why starting at $n = 3$ makes the terms positive and well defined.
- (b) Verify the Integral Test hypotheses for the corresponding function.
- (c) Evaluate the improper integral using two successive logarithmic substitutions.
- (d) State the verdict and identify the slow growth expression responsible for it.

Question 5 – Solution

Step 1: Check the domain and hypotheses.

Start at $n = 3$ so that $\ln n > 1$ and $\ln(\ln n) > 0$. Let

$$f(x) = \frac{1}{x \ln x \ln(\ln x)} \quad (x \geq 3).$$

The denominator is positive, continuous, and strictly increasing, so f is positive, continuous, and decreasing.

Step 2: Perform two substitutions.

First let $u = \ln x$, $du = dx/x$; then let $v = \ln u$, $dv = du/u$:

$$\int_3^b \frac{dx}{x \ln x \ln(\ln x)} = \int_{\ln 3}^{\ln b} \frac{du}{u \ln u} = [\ln(\ln u)]_{\ln 3}^{\ln b}.$$

Thus the antiderivative grows as $\ln(\ln(\ln b))$, which tends to infinity.

Step 3: Conclude.

By the Integral Test,

$$\boxed{\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)} \text{ diverges.}}$$

This is another borderline series: three nested logarithms still grow without bound.

Question 6

Consider the rapidly decaying series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

- (a) Verify that $f(x) = xe^{-x^2}$ satisfies the Integral Test hypotheses on $[1, \infty)$.
- (b) Evaluate the improper integral with a substitution and classify the series.
- (c) Derive upper and lower bounds for the remainder R_N .
- (d) Explain what the bound says about the speed of convergence.

Question 6 – Solution

Step 1: Verify the hypotheses.

Let $f(x) = xe^{-x^2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = e^{-x^2}(1 - 2x^2) < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the improper integral.

With $u = x^2$ and $du = 2x \, dx$,

$$\int_1^\infty xe^{-x^2} \, dx = \frac{1}{2} \int_1^\infty e^{-u} \, du = \frac{1}{2e}.$$

Thus the series converges by the Integral Test.

Step 3: Bound the remainder.

Since $\int_A^\infty xe^{-x^2} \, dx = \frac{1}{2}e^{-A^2}$,

$$\boxed{\frac{1}{2}e^{-(N+1)^2} \leq R_N \leq \frac{1}{2}e^{-N^2}}.$$

The Gaussian factor makes the tail decrease exceptionally rapidly.

Question 7

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}.$$

- (a) Verify positivity, continuity, and decreasing behavior for $f(x) = 1/(x^2 + 4)$.
- (b) Use the Integral Test to classify the series, showing the arctangent antiderivative and improper limit.
- (c) Write explicit upper and lower bounds for the remainder after N terms.

Your remainder bounds should contain N and inverse tangent functions.

Question 7 – Solution

Step 1: Verify the hypotheses.

Define $f(x) = 1/(x^2 + 4)$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{2x}{(x^2 + 4)^2} < 0,$$

so it decreases.

Step 2: Test the improper integral.

$$\int_1^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\arctan \left(\frac{x}{2} \right) \right]_1^\infty = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{1}{2} \right) < \infty.$$

Therefore the series converges.

Step 3: Estimate the remainder.

For $A > 0$,

$$\int_A^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{A}{2} \right).$$

Consequently,

$$\frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N+1}{2} \right) \leq R_N \leq \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N}{2} \right).$$

Question 8

Consider

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = \ln x/x^2$ on $[2, \infty)$.
- (b) Evaluate the improper integral using integration by parts.
- (c) Determine convergence and derive two-sided bounds for R_N .
- (d) Explain how the result illustrates polynomial growth dominating logarithmic growth.

Question 8 – Solution

Step 1: Verify the hypotheses on the tail.

Let $f(x) = \ln x/x^2$ for $x \geq 2$. It is continuous and positive. Also

$$f'(x) = \frac{1 - 2 \ln x}{x^3} < 0 \quad (x \geq 2),$$

so it decreases.

Step 2: Integrate by parts.

Take $u = \ln x$ and $dv = x^{-2}dx$. Then $du = dx/x$ and $v = -1/x$:

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Hence

$$\int_2^\infty \frac{\ln x}{x^2} dx = \frac{\ln 2 + 1}{2} < \infty,$$

and the series converges.

Step 3: Estimate the remainder.

Since the tail integral from A equals $(\ln A + 1)/A$,

$$\boxed{\frac{\ln(N+1) + 1}{N+1} \leq R_N \leq \frac{\ln N + 1}{N}}.$$

Quadratic denominator growth dominates the logarithmic numerator.

Question 9

Consider the shifted p -series

$$\sum_{n=1}^{\infty} \frac{1}{(n+3)^{5/4}}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = (x+3)^{-5/4}$.
- (b) Evaluate the associated improper integral and classify the series.
- (c) Obtain two-sided bounds for R_N .
- (d) Explain why a fixed horizontal shift does not alter the underlying p -series verdict.

Question 9 – Solution

Step 1: Verify the hypotheses.

Set $f(x) = (x + 3)^{-5/4}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{5}{4}(x + 3)^{-9/4} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

$$\int_1^{\infty} (x + 3)^{-5/4} dx = \left[-4(x + 3)^{-1/4} \right]_1^{\infty} = \frac{4}{\sqrt{2}} < \infty.$$

Thus the shifted p -series converges. The horizontal shift changes initial values but not the tail exponent $p = 5/4 > 1$.

Step 3: Bound the remainder.

$$\boxed{\frac{4}{(N + 4)^{1/4}} \leq R_N \leq \frac{4}{(N + 3)^{1/4}}}.$$

Although convergent, the exponent is close to 1, so the error decreases slowly.

Question 10

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}.$$

- (a) Verify the domain and all Integral Test hypotheses for $f(x) = 1/[x\sqrt{\ln x}]$.
- (b) Evaluate the improper integral using $u = \ln x$.
- (c) Determine convergence or divergence.
- (d) Place this series within the family $\sum 1/[n(\ln n)^p]$ and state the threshold value of p .

Question 10 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/[x\sqrt{\ln x}]$ for $x \geq 2$. It is continuous and positive. Its denominator $x\sqrt{\ln x}$ is increasing, so f is decreasing. Equivalently,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}} < 0.$$

Step 2: Evaluate the improper integral.

Put $u = \ln x$, so $du = dx/x$:

$$\int_2^b \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\ln b} u^{-1/2} du = 2\sqrt{\ln b} - 2\sqrt{\ln 2}.$$

This tends to infinity, so the series diverges by the Integral Test.

Step 3: Identify the threshold.

More generally,

$$\sum \frac{1}{n(\ln n)^p}$$

converges for $p > 1$ and diverges for $p \leq 1$. Here $p = 1/2$, which lies on the divergent side.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n}$.

- (a) Identify the dominant power and select a standard benchmark.
- (b) Prove a direct-comparison inequality with the correct direction.
- (c) Classify the series and confirm the result by limit comparison.

State why the benchmark converges or diverges.

Question 1 – Solution

Step 1: Select the benchmark.

The denominator behaves like n^2 , so use $b_n = 1/n^2$, a convergent p -series with $p = 2$.

Step 2: Prove direct comparison.

Since $n^2 + 3n \geq n^2 > 0$,

$$0 < \frac{1}{n^2 + 3n} \leq \frac{1}{n^2}.$$

The Direct Comparison Test proves convergence.

Step 3: Verify by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + 3n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 + 3/n} = 1.$$

The finite positive limit confirms that both series share the same verdict.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$.

- (a) Use dominant powers to select a harmonic benchmark.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and give an independent direct lower comparison.

Question 2 – Solution

Step 1: Select a benchmark.

The leading-term quotient is $2n/n^2 = 2/n$, so choose $b_n = 1/n$. All terms are positive.

Step 2: Compute the ratio limit.

$$L = \lim_{n \rightarrow \infty} \frac{(2n+1)/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{n^2+1} = 2.$$

Because $0 < L < \infty$ and the harmonic series diverges, the given series diverges.

Step 3: Check the direction directly.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$ and $2n + 1 \geq 2n$, so

$$\frac{2n+1}{n^2+1} \geq \frac{2n}{2n^2} = \frac{1}{n}.$$

A series larger than the divergent harmonic series must diverge.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ (radians).

- (a) Establish a pointwise trigonometric bound.
- (b) Compare with a standard p -series.
- (c) Classify the series and explain why no averaging argument is needed.

Question 3 – Solution

Step 1: Bound the numerator.

For every real number t , $0 \leq \sin^2 t \leq 1$. Hence for each positive integer n ,

$$0 \leq \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}.$$

Step 2: Classify the benchmark.

The series $\sum 1/n^2$ converges because it is a p -series with $p = 2 > 1$.

Step 3: Apply Direct Comparison.

The given nonnegative series is bounded term by term above by a convergent series, so it converges. No information about the average value or periodic distribution of $\sin^2 n$ is required.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$.

- (a) Select a benchmark using dominant powers.
- (b) Perform limit comparison.
- (c) Prove divergence again using a direct lower bound.

Question 4 – Solution

Step 1: Select the harmonic benchmark.

Since $\sqrt{n^2 + 1}$ behaves like n , choose $b_n = 1/n$.

Step 2: Compute the ratio.

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n^2}} = 1.$$

The finite positive ratio and divergence of $\sum 1/n$ imply divergence.

Step 3: Give a direct lower bound.

Since $n^2 + 1 \leq 2n^2$ for $n \geq 1$,

$$\frac{1}{\sqrt{n^2 + 1}} \geq \frac{1}{\sqrt{2}n}.$$

The smaller comparison series diverges, independently confirming the verdict.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4}$.

- (a) Choose a benchmark from the dominant powers.
- (b) Compute the limiting ratio and verify all Limit Comparison Test conditions.
- (c) Classify the series and give a direct upper bound.

Question 5 – Solution

Step 1: Select the benchmark.

The leading quotient $n^3/n^5 = 1/n^2$ suggests $b_n = 1/n^2$, a convergent p -series.

Step 2: Compute the ratio limit.

$$\lim_{n \rightarrow \infty} \frac{n^3/(n^5 + 4)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^5}{n^5 + 4} = 1.$$

All terms are positive and the limit lies in $(0, \infty)$, so Limit Comparison proves convergence.

Step 3: Verify directly.

Since $n^5 + 4 \geq n^5$,

$$0 < \frac{n^3}{n^5 + 4} \leq \frac{1}{n^2}.$$

This upper comparison with a convergent series gives the same conclusion.

Question 6

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$.

- (a) Prove useful upper and lower quadratic bounds for the denominator.
- (b) Convert them into bounds for the summand, noting reciprocal inequality directions.
- (c) Classify the series and confirm by limit comparison.

Question 6 – Solution

Step 1: Bound the denominator.

Clearly $n^2 - n + 1 \leq n^2$. Also

$$n^2 - n + 1 \geq \frac{n^2}{2} \iff n^2 - 2n + 2 = (n - 1)^2 + 1 \geq 0.$$

Thus $n^2/2 \leq n^2 - n + 1 \leq n^2$.

Step 2: Take reciprocals carefully.

All expressions are positive, so reciprocals reverse the inequalities:

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n + 1} \leq \frac{2}{n^2}.$$

The upper bound is the decisive one for convergence. Since $\sum 2/n^2$ converges, Direct Comparison proves convergence.

Step 3: Confirm asymptotically.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 - n + 1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n + 1/n^2} = 1.$$

Question 7

Consider $\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2}$.

- (a) Select a geometric benchmark from the dominant exponential terms.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and identify the benchmark's common ratio.

Question 7 – Solution

Step 1: Select a geometric benchmark.

The dominant quotient is $3^n/5^n = (3/5)^n$. Let $b_n = (3/5)^n$, whose common ratio $3/5$ has absolute value less than 1; hence $\sum b_n$ converges.

Step 2: Compute the limiting ratio.

$$\frac{(3^n + 1)/(5^n - 2)}{(3/5)^n} = \frac{1 + 3^{-n}}{1 - 2 \cdot 5^{-n}} \rightarrow 1.$$

The original terms and benchmark are positive for $n \geq 1$, and the ratio limit is finite and positive.

Step 3: Apply Limit Comparison.

The two series have the same convergence behavior. Since the benchmark is geometric and convergent,

$$\boxed{\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2} \text{ converges.}}$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$.

- (a) Prove a direct lower bound using $\sqrt{n} \leq n$.
- (b) Use the correct comparison direction to classify the series.
- (c) Confirm the conclusion by limit comparison with $1/n$.

Question 8 – Solution

Step 1: Bound the denominator.

For $n \geq 1$, $\sqrt{n} \leq n$, so

$$n + \sqrt{n} \leq 2n.$$

Because both denominators are positive, taking reciprocals reverses the inequality:

$$\frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}.$$

Step 2: Apply Direct Comparison.

The smaller series $\sum 1/(2n)$ diverges, so the larger given positive-term series diverges.

Step 3: Confirm by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n + \sqrt{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/\sqrt{n}} = 1.$$

This finite positive limit confirms harmonic tail behavior.

Question 9

Consider $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

- (a) Prove a usable power bound for $\ln n$.
- (b) Compare the summand with a convergent p -series.
- (c) Explain why limit comparison with $1/n^3$ is inconclusive.

Question 9 – Solution

Step 1: Bound the logarithm.

We claim $\ln x \leq \sqrt{x}$ for $x \geq 1$. Let $h(x) = \sqrt{x} - \ln x$. Then

$$h'(x) = \frac{\sqrt{x} - 2}{2x},$$

so the minimum occurs at $x = 4$, where $h(4) = 2 - \ln 4 > 0$. The claim follows.

Step 2: Compare directly.

For $n \geq 1$,

$$0 \leq \frac{\ln n}{n^3} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

The benchmark is a convergent p -series with $p = 5/2 > 1$, so Direct Comparison proves convergence.

Step 3: Explain benchmark choice.

Comparison with $1/n^3$ by ratios gives $(\ln n/n^3)/(1/n^3) = \ln n \rightarrow \infty$, outside the standard finite-positive Limit Comparison case. A slightly larger convergent power benchmark is more useful.

Question 10

Consider $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

- (a) Show why limit comparison with $1/n$ and $1/n^p$ does not immediately decide the series.
- (b) Select a natural logarithmic benchmark or use the Integral Test.
- (c) Classify the series and state an explicit tail bound.

Question 10 – Solution

Step 1: Audit ordinary power benchmarks.

Relative to $1/n$,

$$\frac{1/[n(\ln n)^2]}{1/n} = \frac{1}{(\ln n)^2} \rightarrow 0,$$

which does not transfer the harmonic series' divergence. For any $p > 1$, the ratio to $1/n^p$ is $n^{p-1}/(\ln n)^2 \rightarrow \infty$, which does not transfer that p -series' convergence.

Step 2: Use the natural logarithmic test.

Let $f(x) = 1/[x(\ln x)^2]$. It is positive, continuous, and decreasing for $x \geq 2$. With $u = \ln x$,

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Thus the series converges by the Integral Test.

Step 3: State the tail estimate.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

The correct benchmark retains the logarithm rather than discarding it.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify every Alternating Series Test hypothesis.
- (b) Test absolute convergence and classify the series.
- (c) State its familiar value and give the error bound after N terms.

Question 1 – Solution

Step 1: Verify the AST hypotheses.

Write $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Hence the series converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n},$$

which diverges. Therefore convergence is conditional.

Step 3: State the value and error.

The power series for $\ln(1+x)$ at $x=1$ gives $S = \ln 2$. The Alternating Series Estimation Theorem gives

$$|S - s_N| \leq b_{N+1} = \frac{1}{N+1}.$$

Odd and even partial sums approach $\ln 2$ from opposite sides.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

- (a) Verify that the Alternating Series Test applies.
- (b) Determine whether the series converges absolutely or conditionally.
- (c) Give an error bound for s_N and explain which conclusion is stronger.

Question 2 – Solution**Step 1: Apply the AST.**

For $b_n = 1/n^2$, we have $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2},$$

a convergent p -series with $p = 2$. Hence the original series converges absolutely, a stronger conclusion than the AST alone provides.

Step 3: Bound the alternating error.

$$|S - s_N| \leq \frac{1}{(N+1)^2}.$$

In fact $S = -\pi^2/12$, though this exact value is not needed for classification.

Question 3

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$.

- (a) Evaluate the limit of the term magnitude.
- (b) Apply the nth-term test before considering the AST.
- (c) Explain exactly which AST condition fails.

Question 3 – Solution

Step 1: Simplify the magnitude.

$$b_n = \frac{n+1}{n} = 1 + \frac{1}{n} \longrightarrow 1.$$

Thus the series terms $a_n = (-1)^n b_n$ alternate near ± 1 and do not approach zero.

Step 2: Apply the nth-term test.

Every convergent series must satisfy $a_n \rightarrow 0$. Here the even subsequence tends to 1 and the odd subsequence tends to -1 , so the term sequence has no limit and the series diverges.

Step 3: Audit the AST.

Although b_n is positive and decreasing, the required condition $b_n \rightarrow 0$ fails. Alternating signs cannot compensate for terms of persistent size.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify the AST hypotheses and classify absolute versus conditional convergence.
- (b) Find the smallest N for which the next-term bound b_{N+1} is strictly below 0.01, thereby guaranteeing $|S - s_N| < 0.01$.
- (c) Explain the role of the strict inequality.

Question 4 – Solution**Step 1: Verify convergence.**

With $b_n = 1/\sqrt{n}$, positivity is clear, b_n decreases, and $b_n \rightarrow 0$. The AST proves convergence. The absolute series $\sum 1/\sqrt{n}$ diverges ($p = 1/2$), so convergence is conditional.

Step 2: Apply the error estimate.

$$|S - s_N| \leq b_{N+1} = \frac{1}{\sqrt{N+1}}.$$

To guarantee an error strictly below 0.01 it is sufficient that

$$\frac{1}{\sqrt{N+1}} < \frac{1}{100} \iff N+1 > 10000.$$

Thus the smallest integer furnished by this bound is $\boxed{N = 10000}$.

Step 3: Verify strictness.

At $N = 9999$, the bound equals 0.01, not less than it. At $N = 10000$, it is $1/\sqrt{10001} < 0.01$.

Question 5

Consider $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$.

- (a) Use a continuous derivative to prove eventual decrease of the magnitude.
- (b) Verify the zero-limit condition and apply the AST.
- (c) Test absolute convergence and classify the series.

Question 5 – Solution**Step 1: Prove eventual decrease.**Let $f(x) = \ln x/x$. Then

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \quad (x > e).$$

Hence $b_n = \ln n/n$ decreases for integers $n \geq 3$. Eventual decrease is sufficient; finitely many initial terms do not affect convergence.

Step 2: Check the limit.

L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Therefore the AST proves convergence.

Step 3: Test absolute convergence.

For $n \geq 3$, $\ln n \geq 1$, so $\ln n/n \geq 1/n$. The absolute-value series diverges by comparison with the harmonic series. Thus convergence is conditional.

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n + (-1)^n}$.

- (a) Explain the corrected starting index and why the usual AST magnitude is not decreasing.
- (b) Pair consecutive even and odd terms to test convergence.
- (c) Test absolute convergence and classify the series.

Question 6 – Solution

Step 1: Check the domain and AST template.

At $n = 1$ the denominator is $1 - 1 = 0$, so the series must start at 2. Magnitudes satisfy $b_{2k} = 1/(2k+1)$ and $b_{2k+1} = 1/(2k)$; hence $b_{2k+1} > b_{2k}$ infinitely often, so the standard monotonic AST does not apply.

Step 2: Pair terms.

$$a_{2k} + a_{2k+1} = \frac{1}{2k+1} - \frac{1}{2k} = -\frac{1}{2k(2k+1)}.$$

The paired series converges absolutely by comparison with $1/(4k^2)$. Since the unpaired final term tends to zero, both even and odd partial-sum subsequences approach the same limit; the original series converges.

Step 3: Test absolute convergence.

$$|a_{2k}| + |a_{2k+1}| = \frac{1}{2k+1} + \frac{1}{2k} \geq \frac{1}{2k}.$$

Thus the absolute series diverges, so convergence is conditional.

Question 7

Consider $\sum_{n=1}^{\infty} (-1)^n e^{-n}$.

- (a) Verify the AST hypotheses.
- (b) Test absolute convergence using a geometric series.
- (c) Find the exact sum and classify convergence.

Question 7 – Solution**Step 1: Verify the AST.**

With $b_n = e^{-n} > 0$, we have $b_{n+1} = e^{-1}b_n < b_n$ and $b_n \rightarrow 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum |(-1)^n e^{-n}| = \sum (e^{-1})^n,$$

a geometric series with ratio $e^{-1} < 1$. Hence convergence is absolute.

Step 3: Find the sum.

The original series is geometric with first term $-e^{-1}$ and ratio $-e^{-1}$:

$$S = \frac{-e^{-1}}{1 + e^{-1}} = -\frac{1}{e + 1}.$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$.

- (a) Verify decrease without differentiating.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 8 – Solution**Step 1: Verify monotonicity.**

Since $n + 2 > n + 1$ and logarithm is strictly increasing,

$$\ln(n + 2) > \ln(n + 1) > 0.$$

Taking positive reciprocals reverses the inequality, so $b_{n+1} < b_n$ for $b_n = 1/\ln(n + 1)$.

Step 2: Check the limit and apply AST.

Because $\ln(n + 1) \rightarrow \infty$, $b_n \rightarrow 0$. Thus the alternating series converges.

Step 3: Test absolute convergence.

Since $\ln(n + 1) \leq n$ for $n \geq 1$,

$$\frac{1}{\ln(n + 1)} \geq \frac{1}{n}.$$

The absolute series diverges by comparison with the harmonic series. Therefore convergence is conditional.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$.

- (a) Verify the AST hypotheses using monotonicity of sine.
- (b) Test absolute convergence by limit comparison with $1/n$.
- (c) Classify the series and state the alternating error bound.

Question 9 – Solution**Step 1: Verify the AST.**

For $n \geq 1$, $1/n \in (0, 1]$ and $b_n = \sin(1/n) > 0$. Since $1/(n+1) < 1/n$ and sine is increasing on $[0, 1]$, $b_{n+1} < b_n$. Continuity gives $b_n \rightarrow \sin 0 = 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

The harmonic series diverges, so Limit Comparison shows $\sum \sin(1/n)$ diverges. Hence convergence is conditional.

Step 3: State the error bound.

$$|S - s_N| \leq \sin\left(\frac{1}{N+1}\right) < \frac{1}{N+1}.$$

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$.

- (a) Use a derivative to verify decreasing magnitudes.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 10 – Solution**Step 1: Verify monotonicity.**Let $f(x) = x/(x^2 + 1)$. Then

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0 \quad (x \geq 1).$$

Thus $b_n = n/(n^2 + 1)$ is nonincreasing and positive.**Step 2: Check the limit and apply AST.**

$$b_n = \frac{1/n}{1 + 1/n^2} \longrightarrow 0.$$

Therefore the alternating series converges.

Step 3: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

By comparison with the divergent harmonic series, the absolute series diverges. Hence convergence is conditional.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$.

- (a) Test the absolute-value series first.
- (b) Classify the original series and identify the strongest efficient theorem.
- (c) Give an upper bound for the absolute tail after N terms.

Question 1 – Solution

Step 1: Remove the signs.

$$\sum \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum \frac{1}{n^{3/2}}.$$

This is a convergent p -series because $p = 3/2 > 1$.

Step 2: Classify convergence.

Absolute convergence implies ordinary convergence, so the original series converges absolutely. Although the Alternating Series Test also proves ordinary convergence, it gives a weaker conclusion and is unnecessary here.

Step 3: Bound the absolute tail.

Since $x^{-3/2}$ is positive and decreasing,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{3/2}} \leq \int_N^{\infty} x^{-3/2} dx = \frac{2}{\sqrt{N}}.$$

This also bounds the signed remainder in magnitude.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

- (a) Prove ordinary convergence.
- (b) Test absolute convergence separately.
- (c) Classify the series, state its sum, and give an alternating error bound.

Question 2 – Solution

Step 1: Prove ordinary convergence.

For $b_n = 1/n$, we have $b_n > 0$, b_n decreases, and $b_n \rightarrow 0$. The Alternating Series Test therefore proves convergence.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n},$$

which diverges. Thus the original series is conditionally convergent.

Step 3: State the value and error.

This sign convention is the negative alternating harmonic series, so

$$S = -\ln 2, \quad |S - s_N| \leq \frac{1}{N+1}.$$

Absolute convergence fails because removing the cancellation exposes the harmonic series.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

- (a) Bound the absolute value of each term.
- (b) Apply Direct Comparison to classify absolute convergence.
- (c) Explain what this implies about the original oscillatory series.

Question 3 – Solution

Step 1: Remove oscillation by an inequality.

Since $|\cos n| \leq 1$,

$$0 \leq \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2}.$$

Step 2: Compare absolute values.

The benchmark $\sum 1/n^2$ converges. Direct Comparison gives $\sum |\cos n|/n^2 < \infty$.

Step 3: State the strongest conclusion.

The series converges absolutely, and therefore converges ordinarily. No cancellation or special information about the values of $\cos n$ is needed.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{\sin n}{n}$.

- (a) Prove ordinary convergence using Dirichlet's Test.
- (b) Prove the absolute-value series diverges.
- (c) Classify the series precisely.

Question 4 – Solution

Step 1: Bound the numerator partial sums.

From the finite geometric sum of e^{in} ,

$$\left| \sum_{n=1}^N e^{in} \right| \leq \frac{2}{|1 - e^i|}.$$

Taking imaginary parts shows that $\sum_{n=1}^N \sin n$ is uniformly bounded.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to zero. Hence $\sum (\sin n)/n$ converges.

Step 3: Rule out absolute convergence.

Since $0 \leq |\sin n| \leq 1$, we have $|\sin n| \geq \sin^2 n$. Section 4-4 established that $\sum \sin^2 n/n$ diverges; therefore

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \infty.$$

Thus the series is conditionally convergent.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify ordinary convergence.
- (b) Test the absolute-value series with the p -series rule.
- (c) Classify convergence and give an error bound.

Question 5 – Solution

Step 1: Apply the AST.

With $b_n = 1/\sqrt{n}$, positivity, decrease, and $b_n \rightarrow 0$ all hold. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{n^{1/2}},$$

which diverges because $p = 1/2 \leq 1$.

Step 3: Classify and estimate.

The convergence is conditional, and

$$|S - s_N| \leq \frac{1}{\sqrt{N+1}}.$$

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$.

- (a) Verify ordinary convergence with the AST.
- (b) Test absolute convergence using the logarithmic benchmark.
- (c) Classify the series and state an alternating error estimate.

Question 6 – Solution**Step 1: Verify the AST.**

Let $b_n = 1/(n \ln n)$. For $n \geq 2$ it is positive. Its denominator is increasing, so b_n decreases, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

The absolute series is $\sum 1/(n \ln n)$. For $f(x) = 1/(x \ln x)$,

$$\int_2^b f(x) dx = \ln(\ln b) - \ln(\ln 2) \rightarrow \infty.$$

The Integral Test proves divergence.

Step 3: Classify and estimate.

The original series is conditionally convergent, with

$$|S - s_N| \leq \frac{1}{(N+1) \ln(N+1)}.$$

Question 7

Consider $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$.

- (a) Apply the Ratio Test to the absolute-value series.
- (b) Classify convergence.
- (c) Identify the exact sum using the exponential series.

Question 7 – Solution**Step 1: Test absolute values.**Let $b_n = 2^n/n!$. Then

$$\frac{b_{n+1}}{b_n} = \frac{2}{n+1} \rightarrow 0 < 1.$$

The Ratio Test proves that $\sum b_n$ converges.**Step 2: Classify.**Therefore the signed series converges absolutely. More generally, replacing 2 by any fixed real base c gives ratio $|c|/(n+1) \rightarrow 0$.**Step 3: Find the sum.**From $e^x = \sum_{n=0}^{\infty} x^n/n!$, substitute $x = -2$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2}.$$

Question 8

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$.

- (a) Simplify the ratio of consecutive absolute terms.
- (b) Apply the Ratio Test and classify convergence.
- (c) Explain the effect of the doubled factorial.

Question 8 – Solution

Step 1: Form the absolute ratio.

For $b_n = n!/(2n)!$,

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} = \frac{1}{2(2n+1)}.$$

Step 2: Apply the Ratio Test.

The ratio tends to $0 < 1$, so $\sum b_n$ converges. Hence the original series converges absolutely.

Step 3: Interpret.

Each step in $(2n)!$ introduces two new denominator factors but only one new numerator factor. This drives the ratio rapidly to zero.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^{-n}$.

- (a) Find the limit of the magnitude.
- (b) Apply the nth-term test.
- (c) Explain why alternating signs cannot rescue the series.

Question 9 – Solution

Step 1: Find the magnitude limit.

The standard exponential limit gives

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e, \quad \left(1 + \frac{1}{n}\right)^{-n} \rightarrow e^{-1} \neq 0.$$

Step 2: Examine the signed terms.

Even-indexed terms approach e^{-1} and odd-indexed terms approach $-e^{-1}$. Thus the terms do not approach zero.

Step 3: Apply the nth-term test.

A necessary condition for series convergence fails, so the series diverges. The AST is unavailable because its magnitude condition $b_n \rightarrow 0$ fails.

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$.

- (a) Prove a useful power bound for $\ln n$.
- (b) Use it to test absolute convergence.
- (c) Classify the original series and explain why the AST is not needed.

Question 10 – Solution**Step 1: Dominate the logarithm.**

For $x \geq 1$, $\ln x \leq \sqrt{x}$. One proof minimizes $h(x) = \sqrt{x} - \ln x$; its minimum is $h(4) = 2 - \ln 4 > 0$.

Step 2: Test absolute convergence.

$$0 \leq \frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$

The upper benchmark is a convergent p -series, so Direct Comparison proves convergence of the absolute-value series.

Step 3: Classify.

The original series converges absolutely. The AST could prove ordinary convergence after checking eventual decrease, but absolute comparison is faster and stronger.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n!}{3^n}$.

- (a) Compute a_{n+1}/a_n .
- (b) Decide whether the series converges.
- (c) Explain what the calculation says about factorial growth compared with 3^n .

Question 1 – Solution**Step 1: Identify the term.**

Let $a_n = n!/3^n$. Its terms are positive, so absolute values do not change the ratio.

Step 2: Simplify before taking the limit.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} = \frac{n+1}{3}.$$

Step 3: Interpret the ratio.

Thus $L = \lim_{n \rightarrow \infty} (n+1)/3 = \infty > 1$. The terms eventually increase; indeed, $a_{n+1} > a_n$ for $n > 2$. Therefore a_n cannot approach 0.

Conclusion.

The series **diverges** by the Ratio Test (and by the nth-term test). The unbounded ratio shows that $n!$ eventually grows faster than 3^n .

Question 2

Consider $\sum_{n=0}^{\infty} \frac{3^n}{n!}$.

- (a) Apply the Ratio Test and classify the series.
- (b) Recognize the resulting series and find its exact sum.

Question 2 – Solution

Step 1: Form the ratio.

With $a_n = 3^n/n!$,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \frac{3}{n+1}.$$

Step 2: Take the limit.

$L = \lim_{n \rightarrow \infty} 3/(n+1) = 0 < 1$. The Ratio Test proves absolute convergence.

Step 3: Identify the sum.

Since $e^x = \sum_{n=0}^{\infty} x^n/n!$, substituting $x = 3$ gives

$$\boxed{\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3}.$$

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.

- (a) Compute and simplify a_{n+1}/a_n .
- (b) Apply the Ratio Test.
- (c) Explain how the limit compares polynomial and exponential growth.

Question 3 – Solution

Step 1: Compute the exact ratio.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2.$$

Step 2: Evaluate the limit.

$$L = \frac{1}{2}(1)^2 = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely**. Its consecutive terms eventually behave as though multiplied by $1/2$. The polynomial factor changes the ratio by a factor tending to 1, while 2^n supplies the decisive factor $1/2$.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 5^n}$. Compute the consecutive-term ratio carefully, apply the Ratio Test, and state the convergence classification.

Question 4 – Solution

Step 1: Name the term.

Let $a_n = (2n)! / [(n!)^2 5^n]$.

Step 2: Expand only the new factorial factors.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)!)^2 5^{n+1}} \frac{(n!)^2 5^n}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{5(n+1)^2} = \frac{2(2n+1)}{5(n+1)}.\end{aligned}$$

Step 3: Take the limit.

$$L = \lim_{n \rightarrow \infty} \frac{2(2n+1)}{5(n+1)} = \frac{4}{5} < 1.$$

Conclusion.

The positive-term series **converges**. The factorial expression is $\binom{2n}{n}$, but no approximation is needed.

Question 5

Use the Ratio Test, without Stirling's formula, to determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges. Show all algebra used to find the ratio limit.

Question 5 – Solution

Step 1: Form and simplify the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}.\end{aligned}$$

Step 2: Use the standard exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

Conclusion.

The series **converges**. This gives the decisive geometric-rate factor $1/e$ directly, without an asymptotic formula for $n!$.

Question 6

For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $x \in \mathbb{R}$:

- (a) Apply the Ratio Test to the absolute values.
- (b) Find every real x for which the series converges.
- (c) State the exact sum.

Question 6 – Solution

Step 1: Treat x as fixed.

For $a_n = x^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n+1}.$$

For $x = 0$ the series plainly converges; the same final limit covers every fixed nonzero x .

Step 2: Take the limit.

$$L = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1 \quad \text{for every fixed } x \in \mathbb{R}.$$

Conclusion.

The series converges absolutely for all real x , so its interval of convergence is $(-\infty, \infty)$. By the exponential-series identity,

$$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}.$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^{10}}$ converges or diverges.

- (a) Simplify a_{n+1}/a_n .
- (b) Use its limiting behavior to justify the conclusion.
- (c) Explain why the nth-term test also applies.

Question 7 – Solution

Step 1: Compute the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{10}} \frac{n^{10}}{n!} = (n+1) \left(\frac{n}{n+1} \right)^{10} \\ &= n \left(1 + \frac{1}{n} \right)^{-9}.\end{aligned}$$

Step 2: Evaluate its behavior.

The factor $(1 + 1/n)^{-9} \rightarrow 1$, while $n \rightarrow \infty$. Hence $L = \infty > 1$.

Conclusion.

The series **diverges**. Its terms eventually grow rather than approach zero, so the nth-term divergence test confirms the result. A factorial eventually dominates every fixed power n^{10} .

Question 8

Analyze $\sum_{n=1}^{\infty} \frac{2^{n^2}}{3^{n^2}}$. Rewrite its term using a single base, compute the consecutive-term ratio, and classify the series.

Question 8 – Solution

Step 1: Combine the powers.

$$a_n = (2/3)^{n^2}.$$

Step 2: Compute the ratio.

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{3}\right)^{(n+1)^2 - n^2} = \left(\frac{2}{3}\right)^{2n+1}.$$

Because $0 < 2/3 < 1$, this expression tends to 0.

Conclusion.

The ratio limit is $L = 0 < 1$, so the series **converges**. Its quadratic exponent makes the terms decay faster than those of $\sum (2/3)^n$.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ converges or diverges. Reduce the consecutive-term ratio completely before taking its limit.

Question 9 – Solution

Step 1: Form the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} = \frac{(n+1)^{n-1}}{n^n} \\ &= \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n-1}.\end{aligned}$$

Step 2: Separate the limiting factors.

We have $(1 + 1/n)^{n-1} \rightarrow e$ and $1/n \rightarrow 0$. Therefore $L = 0$.

Conclusion.

Since $L = 0 < 1$, the positive-term series **converges**. The simplified ratio resolves the competition between n^n and $(n!)^2$ without guesswork.

Question 10

Use the Ratio Test to analyze $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$. Use logarithms and a standard logarithmic expansion to evaluate the ratio limit.

Question 10 – Solution

Step 1: Describe the term logarithmically.

Let $a_n = (n/(n+1))^{n^2}$. Then

$$\log a_n = -n^2 \log \left(1 + \frac{1}{n} \right).$$

Using $\log(1+t) = t - t^2/2 + O(t^3)$,

$$\log a_n = -n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n + \frac{1}{2} + O(n^{-1}).$$

Step 2: Find the ratio limit.

$$\log \left(\frac{a_{n+1}}{a_n} \right) = \log a_{n+1} - \log a_n = -1 + O(n^{-1}) \rightarrow -1.$$

Exponentiating gives $L = e^{-1} < 1$.

Conclusion.

The series **converges**. More specifically, $a_n \sim e^{1/2}e^{-n}$, so its terms have geometric-scale decay.

Question 1

Consider

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{5^n n^n}.$$

- (a) Rewrite the summand as a single quantity raised to the n th power.
- (b) Apply the Root Test and classify the series.

Question 1 – Solution**Step 1: Rewrite the term.**

Let

$$a_n = \frac{(2n+1)^n}{5^n n^n} = \left(\frac{2n+1}{5n} \right)^n = \left(\frac{2}{5} + \frac{1}{5n} \right)^n.$$

Step 2: Take the n th root.Since $a_n > 0$,

$$\sqrt[n]{|a_n|} = \frac{2n+1}{5n} = \frac{2}{5} + \frac{1}{5n}.$$

Therefore $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 2/5 < 1$.**Conclusion.**

The series **converges absolutely** by the Root Test. The slowly varying $1/(5n)$ correction disappears in the limit, leaving geometric-scale decay with factor $2/5$.

Question 2

Analyze

$$\sum_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{2} \right)^n.$$

- (a) Examine the even and odd terms separately.
- (b) Apply the nth-term divergence test.
- (c) Confirm the conclusion using the limsup form of the Root Test.

Question 2 – Solution**Step 1: Split by parity.**If $n = 2k$, then

$$a_{2k} = \left(1 + \frac{1}{2}\right)^{2k} = \left(\frac{3}{2}\right)^{2k} \rightarrow \infty.$$

If $n = 2k + 1$, then $a_{2k+1} = (1/2)^{2k+1} \rightarrow 0$.**Step 2: Apply the necessary condition.**

A convergent series must have $a_n \rightarrow 0$. The even subsequence is unbounded, so the full term sequence does not approach zero. Hence the series **diverges**.

Step 3: Check with roots.

$$\sqrt[n]{|a_n|} = 1 + \frac{(-1)^n}{2}, \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{3}{2} > 1.$$

The limsup Root Test gives the same result; an ordinary root limit does not exist.

Question 3

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} (3^n + 2^n)^{1/n} x^n.$$

Use the Root Test for the interior and test both endpoints separately.

Question 3 – Solution

Step 1: Apply the root to the entire term.

For $a_n = (3^n + 2^n)^{1/n} x^n$,

$$\sqrt[n]{|a_n|} = (3^n + 2^n)^{1/n^2} |x|.$$

Write $3^n + 2^n = 3^n[1 + (2/3)^n]$. Then

$$(3^n + 2^n)^{1/n^2} = 3^{1/n} \left[1 + \left(\frac{2}{3} \right)^n \right]^{1/n^2} \longrightarrow 1.$$

Thus the Root Test limit is $L = |x|$: convergence holds for $|x| < 1$ and divergence for $|x| > 1$.

Step 2: Test the endpoints.

Since $(3^n + 2^n)^{1/n} \rightarrow 3$, the terms at $x = 1$ approach 3, and at $x = -1$ their magnitudes approach 3. Neither endpoint passes the nth-term test.

Conclusion.

The radius is $\boxed{R = 1}$ and the interval is $\boxed{(-1, 1)}$.

Question 4

Use the Root Test to determine whether $\sum_{n=1}^{\infty} \frac{n^5}{4^n}$ converges. Explain explicitly why the polynomial factor does not affect the final root limit.

Question 4 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\frac{n^5}{4^n}} = \frac{n^{5/n}}{4}.$$

Step 2: Evaluate the polynomial-root factor.

Take logarithms:

$$\log(n^{5/n}) = \frac{5 \log n}{n} \rightarrow 0.$$

Exponentiating gives $n^{5/n} \rightarrow e^0 = 1$.

Step 3: Apply the test.

$$L = \lim_{n \rightarrow \infty} \frac{n^{5/n}}{4} = \frac{1}{4} < 1.$$

Therefore the series **converges absolutely**. Under an n th root, any fixed power n^p becomes $n^{p/n} \rightarrow 1$, so the exponential denominator determines the result.

Question 5

Determine whether

$$\sum_{n=2}^{\infty} \left(\frac{\log n}{n} \right)^n$$

converges or diverges. Compute the Root Test limit and interpret what a limit of zero implies.

Question 5 – Solution

Step 1: Take the n th root.

Since $n \geq 2$,

$$\sqrt[n]{\left| \left(\frac{\log n}{n} \right)^n \right|} = \frac{\log n}{n}.$$

Step 2: Evaluate the limit.

By l'Hopital's Rule for the corresponding continuous quotient,

$$\lim_{x \rightarrow \infty} \frac{\log x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Hence $L = 0 < 1$.

Conclusion.

The series **converges absolutely**. A root limit of zero means that for every fixed $0 < r < 1$, the terms eventually satisfy $a_n \leq r^n$; thus the tail is eventually dominated by a geometric series with any chosen ratio r .

Question 6

Analyze $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n} \right)^n$ using the Root Test. State the root limit and the convergence classification.

Question 6 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\left|\left(\frac{n+1}{2n}\right)^n\right|} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

Step 2: Evaluate the limit.

$$L = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely** by the Root Test. Although the base varies with n , it approaches $1/2$, so the terms eventually have geometric-scale decay. For example, once $n \geq 5$, the root is at most $3/5$, so $a_n \leq (3/5)^n$.

Question 7

Use the Root Test to determine whether

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{\sqrt{n}}\right)^{n^2}$$

converges. Use logarithms to evaluate the limit of the n th root.

Question 7 – Solution

Step 1: Take the n th root.

$$r_n = \sqrt[n]{a_n} = \left(1 - \frac{1}{\sqrt{n}}\right)^n.$$

Step 2: Take logarithms.

Since $\log(1 - u) \leq -u$ for $0 < u < 1$,

$$\log r_n = n \log \left(1 - \frac{1}{\sqrt{n}}\right) \leq -\sqrt{n} \longrightarrow -\infty.$$

Thus $0 \leq r_n \leq e^{-\sqrt{n}} \rightarrow 0$. (The expansion $\log(1 - u) = -u - u^2/2 + O(u^3)$ gives the sharper relation $\log r_n = -\sqrt{n} - 1/2 + O(n^{-1/2})$.)

Conclusion.

The root limit is $L = 0 < 1$, so the series **converges absolutely** and does so faster than any fixed geometric rate.

Question 8

Analyze

$$\sum_{n=1}^{\infty} \left(\frac{2 + \sin n}{4} \right)^n.$$

Use the limsup form of the Root Test to handle the oscillating base, and explain why an ordinary root limit is unavailable.

Question 8 – Solution**Step 1: Take the root.**Since $1 \leq 2 + \sin n \leq 3$,

$$\sqrt[n]{|a_n|} = \frac{2 + \sin n}{4} \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

This sequence oscillates rather than approaching one value.

Step 2: Compute the limsup.Integer angles modulo 2π are dense because $1/(2\pi)$ is irrational. Consequently, $\sin n$ comes arbitrarily close to 1, so

$$L = \limsup_{n \rightarrow \infty} \frac{2 + \sin n}{4} = \frac{3}{4} < 1.$$

(For convergence alone, the uniform bound $\sqrt[n]{|a_n|} \leq 3/4$ already suffices.)**Conclusion.**The series **converges absolutely** by the limsup Root Test.

Question 9

Determine whether $\sum_{n=1}^{\infty} n^{-n}$ converges or diverges. Use the Root Test and explain why the exponent n makes this test especially efficient.

Question 9 – Solution

Step 1: Rewrite the term.

We have $a_n = n^{-n} = (1/n)^n$.

Step 2: Take the n th root.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{1}{n}\right)^n} = \frac{1}{n}.$$

Therefore

$$L = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1.$$

Conclusion.

The series **converges absolutely**. Rather than complicating the calculation, the exponent n is exactly canceled by the n th root. Moreover, for any $0 < r < 1$, eventually $1/n < r$, so $n^{-n} < r^n$ and the tail is dominated by a geometric series.

Question 10

Let $c \in \mathbb{R}$. Determine exactly which values of c make

$$\sum_{n=1}^{\infty} c^{n^2}$$

converge. Apply the Root Test where decisive and inspect the boundary cases separately.

Question 10 – Solution

Step 1: Compute the root expression.

$$\sqrt[n]{|c^{n^2}|} = |c|^n.$$

Step 2: Separate the parameter cases.

- If $|c| < 1$, then $|c|^n \rightarrow 0 < 1$, so the series converges absolutely.
- If $|c| > 1$, then $|c|^n \rightarrow \infty$, and the terms c^{n^2} do not approach zero.
- If $|c| = 1$, the Root Test gives the boundary value 1. Directly, $|c^{n^2}| = 1$, so the terms again fail to approach zero.

Conclusion.

The series converges exactly when $|c| < 1$, equivalently $-1 < c < 1$. Both endpoints diverge.

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

- (a) Choose an efficient convergence test and justify the choice.
- (b) Determine whether the series converges.
- (c) If possible, use a known power-series identity to find its exact sum.

Question 1 – Solution

Step 1: Prove convergence efficiently.

For $a_n = n/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3n} \longrightarrow \frac{1}{3} < 1.$$

The Ratio Test is quick and proves convergence.

Step 2: Choose a more informative method for the sum.

Begin with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiate and multiply by x :

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

At $x = 1/3$,

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \boxed{\frac{3}{4}}.$$

Strategy note.

The Ratio Test classifies the series; generating-function recognition gives both convergence and the exact value.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges or diverges. Explain why the Integral Test is especially well suited to this expression.

Question 2 – Solution

Step 1: Verify the hypotheses.

Let $f(x) = 1/(x \log x)$ for $x \geq 2$. It is positive and continuous. Also,

$$f'(x) = -\frac{\log x + 1}{x^2(\log x)^2} < 0,$$

so f is decreasing.

Step 2: Evaluate the improper integral.

The integrand contains the differential $dx/x = d(\log x)$, suggesting $u = \log x$:

$$\int_2^b \frac{dx}{x \log x} = \int_{\log 2}^{\log b} \frac{du}{u} = \log(\log b) - \log(\log 2).$$

As $b \rightarrow \infty$, this tends to ∞ .

Conclusion.

The series **diverges** by the Integral Test. The substitution exposes the antiderivative $\log(\log x)$ immediately, which is why this test is more natural than a ratio or root test here.

Question 3

Classify $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ as absolutely convergent, conditionally convergent, or divergent. Use separate tests for ordinary and absolute convergence.

Question 3 – Solution

Step 1: Test the alternating series.

Put $b_n = 1/\sqrt{n}$. Then $b_n > 0$, $b_{n+1} < b_n$, and

$$\lim_{n \rightarrow \infty} b_n = 0.$$

Therefore $\sum (-1)^n b_n$ converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$$

This is a divergent p -series because $p = 1/2 \leq 1$.

Conclusion.

The original series is **conditionally convergent**. One test establishes convergence; the second shows that convergence depends essentially on cancellation between alternating signs.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n}$.

- (a) Show why the Ratio Test is inconclusive.
- (b) Use a standard central-binomial-coefficient asymptotic to reach a conclusion.

Question 4 – Solution

Step 1: Try the natural factorial test.

If $a_n = (2n)!/[(n!)^2 4^n]$, then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = \frac{2n+1}{2n+2} \rightarrow 1.$$

The Ratio Test is therefore inconclusive.

Step 2: Replace it with asymptotic comparison.

Stirling's formula yields the standard estimate

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Consequently,

$$a_n = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

Equivalently, $a_n/(1/\sqrt{n}) \rightarrow 1/\sqrt{\pi} > 0$.

Conclusion.

Since $\sum n^{-1/2}$ diverges, the given series **diverges** by Limit Comparison. A ratio limit of 1 signaled that a finer growth estimate was needed.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \cos n}$ converges. For the tail $n \geq 2$, use only the bound $-1 \leq \cos n \leq 1$; no exact trigonometric values are needed.

Question 5 – Solution

Step 1: Ensure the terms are defined and positive.

For $n = 1$, $\cos 1 > 0$ because $0 < 1 < \pi/2$. For $n \geq 2$, $n^2 + \cos n \geq n^2 - 1 > 0$.

Step 2: Build a simple upper comparison.

For $n \geq 2$,

$$n^2 + \cos n \geq n^2 - 1 \geq \frac{1}{2}n^2.$$

Taking reciprocals reverses the denominator comparison:

$$0 < \frac{1}{n^2 + \cos n} \leq \frac{2}{n^2}.$$

Step 3: Apply the Comparison Test.

The series $\sum 2/n^2$ converges because it is a constant multiple of a p -series with $p = 2 > 1$.

Conclusion.

The given series **converges**. A direct comparison is more efficient than ratio or root tests because the bounded trigonometric perturbation does not change the dominant n^2 denominator.

Question 6

Analyze $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$.

- (a) Simplify the factorial quotient.
- (b) Decide whether telescoping or comparison is more informative.
- (c) Determine convergence and, if possible, the exact sum.

Question 6 – Solution

Step 1: Cancel factorials.

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}.$$

This already resembles a telescoping difference.

Step 2: Decompose and form partial sums.

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\begin{aligned} S_N &= \sum_{n=1}^N \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} - \frac{1}{N+2}. \end{aligned}$$

Step 3: Pass to the limit.

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \lim_{N \rightarrow \infty} S_N = \boxed{\frac{1}{2}}.$$

Strategy note.

Comparison with $1/n^2$ proves convergence, but telescoping is better because it also gives the exact sum.

Question 7

Classify $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ as absolutely convergent, conditionally convergent, or divergent. Use bounded partial sums for the sine factor.

Question 7 – Solution

Step 1: Verify bounded partial sums.

The finite identity

$$\sum_{n=1}^N \sin n = \frac{\sin(N/2) \sin((N+1)/2)}{\sin(1/2)}$$

shows that $\left| \sum_{n=1}^N \sin n \right| \leq 1/|\sin(1/2)|$.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to 0, and the partial sums of $\sin n$ are bounded. Hence $\sum \sin n/n$ converges.

Step 3: Check absolute convergence.

Since $|\sin n| \geq \sin^2 n$,

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \frac{1}{2} \sum \frac{1}{n} - \frac{1}{2} \sum \frac{\cos(2n)}{n}.$$

The cosine series converges by Dirichlet's Test while the harmonic series diverges, so the absolute-value series diverges.

Conclusion.

The series is **conditionally convergent**.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$ converges. Explain why the Root Test is the natural first choice.

Question 8 – Solution

Step 1: Recognize the growing exponent.

Let $a_n = (1 + 1/n)^{-n^2}$. Taking an n th root cancels one power of n :

$$\sqrt[n]{a_n} = \left(1 + \frac{1}{n}\right)^{-n}.$$

Step 2: Use the exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = e^{-1} < 1.$$

Conclusion.

The series **converges** by the Root Test. This method is especially efficient because the summand is already a varying base raised to a multiple of n ; the root exposes its limiting geometric rate directly.

Question 9

Determine whether $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges. Compare the Integral Test with Cauchy Condensation and identify the shorter calculation.

Question 9 – Solution**Step 1: Use the Integral Test.**

The function $f(x) = \log x/x^2$ is positive and continuous for $x \geq 2$. Also,

$$f'(x) = \frac{1 - 2 \log x}{x^3} < 0 \quad (x \geq 2).$$

Integration by parts gives

$$\int_2^\infty \frac{\log x}{x^2} dx = \left[-\frac{\log x + 1}{x} \right]_2^\infty = \frac{\log 2 + 1}{2} < \infty.$$

Thus the series converges.

Step 2: Compare with condensation.

Because the terms decrease,

$$2^k a_{2^k} = 2^k \frac{\log(2^k)}{(2^k)^2} = \frac{k \log 2}{2^k},$$

whose series converges by the Ratio Test. Condensation also works, but the integral is slightly shorter and matches the logarithm-over-power form directly.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^3 - 2}$ converges or diverges. Identify the dominant powers and use Limit Comparison with an appropriate benchmark series.

Question 10 – Solution**Step 1: Identify the benchmark.**

The dominant-power quotient is

$$\frac{3n^2}{n^3} = \frac{3}{n},$$

so compare with the harmonic term $b_n = 1/n$. A finite number of initial terms, including the negative $n = 1$ term, does not affect convergence.

Step 2: Compute the limit.

For $n \geq 2$ both sequences are positive, and

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{3n^2 + 1}{n^3 - 2} \cdot n \\ &= \lim_{n \rightarrow \infty} \frac{3 + 1/n^2}{1 - 2/n^3} = 3.\end{aligned}$$

The limit is finite and strictly positive.

Conclusion.

Since $\sum 1/n$ diverges, the given series **diverges** by the Limit Comparison Test. Dominant powers correctly predicted harmonic-size terms.

Question 1

Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ and $S_N = \sum_{n=1}^N (-1)^{n-1}/n$.

- (a) State the Alternating Series Estimation Theorem for $R_N = S - S_N$.
- (b) Find the least final index N for which the next-term bound $1/(N + 1)$ is strictly below 10^{-3} .

Question 1 – Solution**Step 1: Verify the theorem applies.**

The magnitudes $b_n = 1/n$ decrease to 0, so the alternating series converges and

$$|R_N| \leq b_{N+1} = \frac{1}{N+1}.$$

Step 2: Solve the strict inequality.

$$\frac{1}{N+1} < 10^{-3} \iff N+1 > 1000 \iff N > 999.$$

The least integer is $\boxed{N = 1000}$.

Step 3: Check minimality.

For $N = 999$, the displayed non-strict bound gives $|R_{999}| \leq 1/1000 = 10^{-3}$, so this bound is not strictly below the tolerance. (The actual remainder is strictly smaller than the next term.) For $N = 1000$, $|R_{1000}| \leq 1/1001 < 10^{-3}$.

Conclusion.

Use the first 1000 terms, ending at index 1000.

Question 2

For $S = \sum_{n=1}^{\infty} 1/n^2$, let $S_N = \sum_{n=1}^N 1/n^2$ and $R_N = S - S_N$. Use both sides of the Integral Test remainder estimate to bracket R_N , then bracket S in terms of S_N .

Question 2 – Solution

Step 1: State the two-sided estimate.

For a positive decreasing function f with $a_n = f(n)$,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Here $f(x) = x^{-2}$ is positive, continuous, and decreasing.

Step 2: Evaluate both integrals.

$$\int_{N+1}^{\infty} \frac{dx}{x^2} = \frac{1}{N+1}, \quad \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

Thus

$$\boxed{\frac{1}{N+1} \leq R_N \leq \frac{1}{N}}.$$

Step 3: Convert the tail bracket to a sum bracket.

Since $S = S_N + R_N$,

$$\boxed{S_N + \frac{1}{N+1} \leq S \leq S_N + \frac{1}{N}}.$$

The bracket width is $1/[N(N+1)]$.

Question 3

Let $S = \sum_{n=1}^{\infty} 1/n^3$. Using the Integral Test upper remainder bound, find the least final index N certified to give absolute error $|R_N| < \frac{1}{2}10^{-4}$. Explain whether this tolerance alone guarantees identical rounding to four decimal places.

Question 3 – Solution

Step 1: Obtain an upper tail bound.

With $f(x) = x^{-3}$,

$$0 < R_N \leq \int_N^{\infty} x^{-3} dx = \left[-\frac{1}{2x^2} \right]_N^{\infty} = \frac{1}{2N^2}.$$

Step 2: Impose the absolute-error tolerance.

The requested error is below $0.00005 = 5 \times 10^{-5}$:

$$\frac{1}{2N^2} < 5 \times 10^{-5} \iff N^2 > 10,000 \iff N > 100.$$

Thus the least integer certified by this bound is $\boxed{N = 101}$.

Step 3: Check the strict endpoint.

At $N = 100$, the bound equals 5×10^{-5} , so it does not guarantee a strict error smaller than half a unit in the fourth decimal place. At $N = 101$, it does.

This absolute-error tolerance alone does not guarantee identical rounding: two nearby numbers can lie on opposite sides of a rounding threshold.

Question 4

Approximate $e^{-1} = \sum_{n=0}^{\infty} (-1)^n/n!$ using $S_N = \sum_{n=0}^N (-1)^n/n!$. Find the least final index N that guarantees correct rounding through six decimal places.

Question 4 – Solution

Step 1: Use the alternating remainder bound.

The magnitudes $1/n!$ decrease to 0, so

$$|R_N| \leq \frac{1}{(N+1)!}.$$

The tolerance $|R_N| < 5 \times 10^{-7}$ suggests a candidate; correct rounding also requires a bracket within a single rounding interval.

Step 2: Find the first qualifying factorial.

$$9! = 362,880, \quad 10! = 3,628,800.$$

Hence

$$\frac{1}{9!} \approx 2.76 \times 10^{-6} > 5 \times 10^{-7}, \quad \frac{1}{10!} \approx 2.76 \times 10^{-7} < 5 \times 10^{-7}.$$

Therefore $N+1=10$, so $\boxed{N=9}$.

Step 3: Certify the rounding.

The alternating sign gives

$$0.3678791887 < S < S_9 + 1/10! < 0.3678794643.$$

Both endpoints lie in $[0.3678785, 0.3678795)$ and round to 0.367879. Also $S_8 \approx 0.3678819444$ and $S_7 \approx 0.3678571429$; the earlier even and odd partial sums lie still farther away. Hence no smaller final index gives this rounded value.

Conclusion.

Ten terms (indices 0 through 9) suffice. They give $S_9 \approx 0.3678791887$, which rounds to 0.367879, the same six-decimal value as e^{-1} .

Question 5

For $S = \sum_{n=0}^{\infty} (1/3)^n$, define $S_N = \sum_{n=0}^N (1/3)^n$.

- (a) Compute the tail $R_N = S - S_N$ exactly.
- (b) Compare it with the standard geometric-tail estimate based on the first omitted term.

Question 5 – Solution

Step 1: Identify the first omitted term.

Because S_N ends at index N , the tail begins at $n = N + 1$:

$$R_N = \sum_{n=N+1}^{\infty} \left(\frac{1}{3}\right)^n.$$

Step 2: Sum the geometric tail.

$$R_N = \frac{(1/3)^{N+1}}{1 - 1/3} = \frac{1}{2 \cdot 3^N}.$$

Thus the exact error is $\boxed{R_N = 3^{-N}/2}$.

Step 3: Compare with the general bound.

For a geometric tail with ratio magnitude $r < 1$,

$$|R_N| \leq \frac{|a_{N+1}|}{1 - r}.$$

Here $a_{N+1} = 3^{-(N+1)}$ and $r = 1/3$, producing $1/(2 \cdot 3^N)$. Because every term is positive and the ratio is exactly constant, the usual “bound” equals the exact tail.

Question 6

The Leibniz series is $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. If exactly M terms are used (indices 0 through $M-1$), find the least M certified by a next-term bound strictly below 10^{-4} .

Question 6 – Solution

Step 1: Identify the first omitted term.

With M terms, the final included index is $M - 1$, so the first omitted term has index M and magnitude $1/(2M + 1)$.

Step 2: Apply the alternating estimate.

$$|R| \leq \frac{1}{2M + 1}.$$

We need

$$\frac{1}{2M + 1} < 10^{-4} \iff 2M + 1 > 10,000 \iff M > 4999.5.$$

Thus the least integer is $\boxed{M = 5000}$.

Step 3: Verify minimality.

With 4999 terms the bound is $1/9999 > 10^{-4}$; with 5000 terms it is $1/10001 < 10^{-4}$. This large count illustrates the slow convergence of the Leibniz series.

Question 7

For $S = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, derive a two-sided Integral Test estimate for $R_N = S - S_N$. Then determine what the upper bound requires to guarantee $R_N < 0.01$ and interpret the result.

Question 7 – Solution**Step 1: Verify the setup.**

The function $f(x) = 1/[x(\log x)^2]$ is positive and decreasing for $x \geq 2$.

Step 2: Integrate the tail.

With $u = \log x$,

$$\int_A^\infty \frac{dx}{x(\log x)^2} = \int_{\log A}^\infty u^{-2} du = \frac{1}{\log A}.$$

Therefore

$$\boxed{\frac{1}{\log(N+1)} \leq R_N \leq \frac{1}{\log N}}.$$

Step 3: Solve the requested tolerance.

$$\frac{1}{\log N} < 0.01 \iff \log N > 100 \iff N > e^{100}.$$

Thus the least certified integer is $\lfloor e^{100} \rfloor + 1$, an astronomically large index (about 2.69×10^{43}).

Interpretation.

The lower and upper bounds are asymptotically equal, so this is not merely a loose estimate: the series itself converges extraordinarily slowly.

Question 8

Let $S = \sum_{n=1}^{\infty} ne^{-n}$ and $R_N = \sum_{n=N+1}^{\infty} ne^{-n}$. Use the Integral Test remainder theorem to give a two-sided bound for R_N .

Question 8 – Solution**Step 1: Check monotonicity.**For $f(x) = xe^{-x}$,

$$f'(x) = e^{-x}(1 - x) \leq 0 \quad (x \geq 1).$$

Thus f is positive, continuous, and decreasing on the needed interval.**Step 2: Find the antiderivative.**

Integration by parts gives

$$\int_A^\infty xe^{-x} dx = (A + 1)e^{-A}.$$

Step 3: Apply both remainder bounds.

$$\int_{N+1}^\infty xe^{-x} dx \leq R_N \leq \int_N^\infty xe^{-x} dx,$$

so

$$\boxed{(N + 2)e^{-(N+1)} \leq R_N \leq (N + 1)e^{-N}}.$$

The upper bound is a decreasing integral majorant for the full omitted tail; no extra first-term adjustment is needed under this standard formulation.

Question 9

Let $S = \sum_{n=1}^{\infty} (-1)^n/n^2$ and $S_N = \sum_{n=1}^N (-1)^n/n^2$.

- (a) Give the alternating-series error bound.
- (b) Compare it with an Integral Test bound for the tail of $\sum 1/n^2$.
- (c) Evaluate both bounds at $N = 100$.

Question 9 – Solution

Step 1: Use cancellation.

Since $1/n^2$ decreases to 0, the alternating estimate gives

$$|S - S_N| \leq \frac{1}{(N+1)^2}.$$

Step 2: Ignore signs and bound the absolute tail.

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

This also bounds the signed error, but it discards cancellation and is therefore much weaker.

Step 3: Compare numerically.

At $N = 100$,

$$\frac{1}{101^2} = \frac{1}{10201} \approx 9.80 \times 10^{-5}, \quad \frac{1}{100} = 10^{-2}.$$

The alternating bound is over 100 times smaller. In general, it is order N^{-2} rather than N^{-1} .

Question 10

For $S = \sum_{n=1}^{\infty} 2^{-n^2}$ and $S_N = \sum_{n=1}^N 2^{-n^2}$, bound $R_N = S - S_N$ by a geometric series. Clearly identify its first term and common ratio.

Question 10 – Solution**Step 1: Reindex the tail.**

Write $n = N + 1 + k$, where $k \geq 0$. Then

$$(N + 1 + k)^2 = (N + 1)^2 + k(2N + 2) + k^2 \geq (N + 1)^2 + k(2N + 3),$$

because $k^2 \geq k$ for integers $k \geq 0$.

Step 2: Compare term by term.

$$2^{-(N+1+k)^2} \leq 2^{-(N+1)^2} \left(2^{-(2N+3)} \right)^k.$$

The comparison series has first term $2^{-(N+1)^2}$ and ratio $q = 2^{-(2N+3)}$.

Step 3: Sum the geometric majorant.

$$0 < R_N \leq \frac{2^{-(N+1)^2}}{1 - 2^{-(2N+3)}}.$$

The bound is very close to the first omitted term because q becomes extremely small.

Question 1

Find the center, radius of convergence, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}.$$

Test $x = -3$ and $x = 3$ separately and identify each resulting endpoint series.

Question 1 – Solution

Step 1: Recognize the geometric series.

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n.$$

It converges exactly when $|x/3| < 1$, or $|x| < 3$. Thus the center is 0 and $R = 3$.

Step 2: Test the endpoints.

- At $x = 3$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$. Its terms do not approach 0, so it diverges.

Conclusion.

The interval is $\boxed{(-3, 3)}$. Neither endpoint is included.

Question 2

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{nx^n}{4^n}$. Explain why the polynomial factor n does not change the radius but does determine the endpoint behavior.

Question 2 – Solution**Step 1: Use the Ratio Test.**For $a_n = nx^n/4^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} \frac{|x|}{4} \longrightarrow \frac{|x|}{4}.$$

Convergence requires $|x|/4 < 1$, so $|x| < 4$ and $R = 4$.**Step 2: Test the endpoints.**

- At $x = 4$, the terms are n , which do not approach 0.
- At $x = -4$, the terms are $n(-1)^n$, whose magnitudes grow.

Both endpoint series diverge by the n th-term test.**Conclusion.**

The interval is $\boxed{(-4, 4)}$. The factor $(n+1)/n \rightarrow 1$, so n does not alter the radius; at the endpoints it prevents the terms from tending to zero.

Question 3

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n$. Identify the harmonic or alternating harmonic series arising at each endpoint.

Question 3 – Solution

Step 1: Find the interior interval.

$$\left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = |x| \frac{n}{n+1} \longrightarrow |x|.$$

The Ratio Test gives convergence for $|x| < 1$ and divergence for $|x| > 1$, so $R = 1$.

Step 2: Test $x = 1$.

The series becomes $\sum 1/n$, the divergent harmonic series.

Step 3: Test $x = -1$.

The series becomes $\sum (-1)^n/n$, which converges by the Alternating Series Test (but not absolutely).

Conclusion.

The interval is $\boxed{[-1, 1)}$. Convergence at the left endpoint is conditional.

Question 4

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n^2$. Explain why its radius matches Question 3 while both endpoints now converge.

Question 4 – Solution

Step 1: Find the radius.

$$\left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \left(\frac{n}{n+1} \right)^2 \rightarrow |x|.$$

Thus the interior condition is $|x| < 1$ and $R = 1$.

Step 2: Test the endpoints.

- At $x = 1$, the series is $\sum 1/n^2$, which converges.
- At $x = -1$, the absolute-value series is again $\sum 1/n^2$, so the series converges absolutely.

Conclusion.

The interval is $\boxed{[-1, 1]}$. Both this series and $\sum x^n/n$ have coefficient roots tending to 1, hence the same radius, but the stronger n^{-2} decay makes both boundary series absolutely convergent.

Question 5

Determine the radius and set of convergence of $\sum_{n=0}^{\infty} n!x^n$. Use the Ratio Test to show what happens for every nonzero x .

Question 5 – Solution

Step 1: Fix $x \neq 0$ and form the ratio.

With $a_n = n!x^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = (n+1)|x| \longrightarrow \infty.$$

The Ratio Test therefore gives divergence for every fixed nonzero x . Indeed, the terms eventually increase in magnitude and cannot tend to zero.

Step 2: Check $x = 0$.

At $x = 0$, the series has only its $n = 0$ term nonzero (using $0^0 = 1$ in the power-series convention), so it converges to 1.

Conclusion.

The radius is $\boxed{R = 0}$ and the set of convergence is the single point $\boxed{\{0\}}$. This is not a nondegenerate interval.

Question 6

Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} x^n/n!$. Explain why the factorial denominator allows every real x .

Question 6 – Solution

Step 1: Apply the Ratio Test for a fixed x .

$$\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \rightarrow 0.$$

For every fixed real x , the limit is $0 < 1$.

Step 2: State the convergence set.

The series converges absolutely for all $x \in \mathbb{R}$. Thus

$$\boxed{R = \infty}, \quad \boxed{(-\infty, \infty)}.$$

There are no finite endpoints to test.

Step 3: Interpret the factorial.

Each ratio contains the growing divisor $n+1$, which eventually dominates any fixed $|x|$. The series is the familiar expansion $e^x = \sum x^n/n!$.

Question 7

Find the center, radius, and interval of convergence of $\sum_{n=0}^{\infty} (x-2)^n/5^n$. Verify the behavior at both physical endpoints.

Question 7 – Solution

Step 1: Read the geometric structure.

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{5} \right)^n$$

has center $c = 2$ and converges when

$$\left| \frac{x-2}{5} \right| < 1 \iff |x-2| < 5 \iff -3 < x < 7.$$

Thus $R = 5$.

Step 2: Test the endpoints.

- At $x = 7$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$, which also diverges because its terms do not tend to zero.

Conclusion.

The interval is $\boxed{(-3, 7)}$, symmetric about the center 2.

Question 8

Find the center, radius, and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}.$$

Track the center carefully and determine which physical endpoint produces the alternating harmonic series.

Question 8 – Solution

Step 1: Find the interior condition.

The series is centered at $c = -1$. For $a_n = (-1)^n(x+1)^n/n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = |x+1| \frac{n}{n+1} \rightarrow |x+1|.$$

Thus $|x+1| < 1$, so $-2 < x < 0$ and $R = 1$.

Step 2: Test $x = -2$.

Here $x+1 = -1$, so

$$\frac{(-1)^n(-1)^n}{n} = \frac{1}{n}.$$

The harmonic series diverges; $x = -2$ is excluded.

Step 3: Test $x = 0$.

Here $x+1 = 1$, producing $\sum (-1)^n/n$, which converges conditionally.

Conclusion.

The interval is $\boxed{(-2, 0]}$.

Question 9

Find the center, radius, and interval of convergence of $\sum_{n=1}^{\infty} \frac{n^2(x-3)^n}{2^n}$. Explain the different roles of the exponential and polynomial factors.

Question 9 – Solution**Step 1: Apply the Ratio Test.**

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n} \right)^2 \frac{|x-3|}{2} \rightarrow \frac{|x-3|}{2}.$$

Thus $|x-3| < 2$, giving center $c = 3$, radius $R = 2$, and preliminary interval $1 < x < 5$.

Step 2: Test the endpoints.

- At $x = 5$, the terms reduce to n^2 , so the series diverges.
- At $x = 1$, the terms reduce to $n^2(-1)^n$, whose magnitudes do not approach zero.

Conclusion.

The interval is $\boxed{(1, 5)}$. The exponential coefficient 2^{-n} fixes the radius; the polynomial n^2 has root tending to 1 but makes both boundary terms fail the nth-term test.

Question 10

Find the interval of convergence and sum function of $\sum_{n=0}^{\infty} (x^2)^n$. Treat it as a geometric series in x^2 and test $x = \pm 1$.

Question 10 – Solution

Step 1: Apply the geometric criterion.

The common ratio is $r = x^2$. Therefore

$$|r| < 1 \iff |x^2| < 1 \iff x^2 < 1 \iff |x| < 1.$$

Step 2: Test the endpoints.

At both $x = 1$ and $x = -1$, we have $x^2 = 1$, so the series becomes $\sum 1$ and diverges.

Step 3: Find the sum.

For $|x| < 1$, the geometric formula gives

$$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1 - x^2}.$$

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$. As a power series in x , only even powers occur, but its center is still 0 and its radius is 1.

Question 1

Starting from the geometric-series formula, find a power-series representation centered at 0 for $f(x) = 1/(1 - x)$. State its radius and interval of convergence, including endpoint tests.

Question 1 – Solution

Step 1: Use the geometric prototype.

For a geometric series,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (|r| < 1).$$

Set $r = x$ to obtain

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Carry over the convergence condition.

The requirement $|r| < 1$ becomes $|x| < 1$, so the radius is $R = 1$.

Step 3: Test the endpoints.

At $x = 1$, the series is $\sum 1$; at $x = -1$, it is $\sum (-1)^n$. In both cases the terms fail to approach zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$. The identity holds only there, even though the rational function is defined at some points outside that interval.

Question 2

Obtain a power series centered at 0 for $f(x) = 1/(1+x)$ by substituting into the geometric series. State its radius and interval of convergence and test both endpoints directly.

Question 2 – Solution

Step 1: Make the sign substitution.

Replace x by $-x$ in $1/(1-x) = \sum x^n$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Step 2: Translate the restriction.

The geometric condition is $|-x| < 1$, equivalent to $|x| < 1$, so $R = 1$.

Step 3: Test the endpoints.

- At $x = 1$, the series is $\sum (-1)^n$, which diverges by the nth-term test.
- At $x = -1$, it is $\sum 1$, which diverges.

Conclusion.

The interval is $\boxed{(-1, 1)}$. Although $1/(1+x)$ is finite at $x = 1$, its Taylor series about 0 does not converge there.

Question 3

Differentiate the geometric series term by term to find a power series for $1/(1-x)^2$. Reindex it in powers x^n and state its full interval of convergence.

Question 3 – Solution**Step 1: Begin inside the radius.**For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

A power series may be differentiated term by term at every point strictly inside its radius.

Step 2: Differentiate.

$$\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Let $k = n - 1$:

$$\boxed{\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k.}$$

Step 3: Check the boundary.Differentiation preserves the radius $R = 1$, but endpoints must be retested. At $x = 1$ the terms are $k + 1$; at $x = -1$ their magnitudes are $k + 1$. Neither tends to zero.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 4

Derive a power series for $\log(1 + x)$ by integrating a geometric series. Determine the constant of integration and the exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .Term-by-term integration is valid when $|x| < 1$:

$$\begin{aligned} \log(1+x) - \log 1 &= \int_0^x \frac{dt}{1+t} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

Thus

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

The lower limit makes the constant 0.

Step 3: Test endpoints.At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series is $-\sum 1/n$ and diverges.**Conclusion.** $R = 1$ and the interval is $\boxed{(-1, 1]}$.

Question 5

Derive the Maclaurin series for $\arctan x$ by integrating the series for $1/(1+x^2)$. Explain the odd powers and determine the endpoint behavior.

Question 5 – Solution**Step 1: Expand the derivative.**With geometric ratio $-t^2$,

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad |t| < 1.$$

Step 2: Integrate from 0 to x .

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}, \quad |x| < 1.$$

Integrating the even powers t^{2n} produces only odd powers x^{2n+1} ; the value at 0 fixes the constant.**Step 3: Test endpoints.**At $x = 1$, the alternating series converges to $\pi/4$. At $x = -1$, it converges to $-\pi/4$. Both are conditional.**Conclusion.**The radius is $R = 1$ and the interval is $\boxed{[-1, 1]}$.

Question 6

Use the differentiated geometric identity to find a power series for $x/(1-x)^2$. State the index, radius, and exact interval of convergence.

Question 6 – Solution

Step 1: Differentiate the geometric series.

For $|x| < 1$,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Step 2: Multiply by x .

$$\boxed{\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n}.$$

Multiplication by x shifts every exponent upward by one but does not change the radius.

Step 3: Test the boundary.

The radius remains $R = 1$. At $x = 1$, the terms are n ; at $x = -1$, they are $n(-1)^n$. Neither term sequence approaches zero.

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$.

Question 7

Find a Maclaurin series for $1/(1 - x^2)$ by recognizing a geometric series. Explain why only even powers occur and determine the interval of convergence.

Question 7 – Solution**Step 1: Identify the ratio.**Apply $1/(1-r) = \sum r^n$ with $r = x^2$:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Only even powers occur because each power of the ratio x^2 adds two to the exponent.**Step 2: Translate the geometric restriction.**

$$|x^2| < 1 \iff |x| < 1.$$

Hence $R = 1$.**Step 3: Test the endpoints.**At both $x = 1$ and $x = -1$, every term x^{2n} equals 1. Thus the series diverges at both endpoints.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 8

Find a power series for $F(x) = \int_0^x \frac{dt}{1-t^4}$. Justify term-by-term integration, then determine the radius and exact interval of convergence of the resulting series.

Question 8 – Solution**Step 1: Expand the integrand.**For $|t| < 1$,

$$\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n}.$$

On every closed subinterval inside $(-1, 1)$, this power series converges uniformly, so it may be integrated term by term.

Step 2: Integrate.

$$F(x) = \sum_{n=0}^{\infty} \int_0^x t^{4n} dt = \boxed{\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}}.$$

Step 3: Test the endpoints.

Integration preserves radius $R = 1$. At $x = 1$, the series is $\sum 1/(4n+1)$, which diverges by Limit Comparison with $1/n$. At $x = -1$, it is the negative of the same divergent positive series.

Conclusion.The interval is $\boxed{(-1, 1)}$.

Question 9

Find a Maclaurin series for $1/(2 - x)$ by first factoring the denominator. State the radius and interval of convergence, including endpoint tests.

Question 9 – Solution

Step 1: Put the function in geometric form.

$$\frac{1}{2-x} = \frac{1}{2} \frac{1}{1-x/2}.$$

Using the geometric identity with $r = x/2$ gives

$$\frac{1}{2-x} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}.$$

Step 2: Translate the restriction.

The condition $|x/2| < 1$ becomes $|x| < 2$, so $R = 2$.

Step 3: Test endpoints.

At $x = 2$, each term equals $1/2$. At $x = -2$, the terms alternate between $\pm 1/2$. Neither tends to zero, so both series diverge.

Conclusion.

The interval is $\boxed{(-2, 2)}$.

Question 10

Find a Maclaurin series for $x^2/(1 + x^3)$. Identify the geometric ratio, explain the gaps between exponents, and determine the interval of convergence.

Question 10 – Solution**Step 1: Expand the reciprocal.**Use the geometric ratio $r = -x^3$:

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \quad |x^3| < 1.$$

Step 2: Multiply by x^2 .

$$\boxed{\frac{x^2}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}}.$$

The exponents are 2, 5, 8, ... because multiplying by each additional ratio contributes three powers of x .

Step 3: Determine convergence.

The interior condition is $|x| < 1$, so $R = 1$. At $x = 1$, terms are $(-1)^n$; at $x = -1$, every term is 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 1

Construct the Maclaurin series for e^x from its derivatives. Prove that it converges to e^x for every real x , and give a uniform Lagrange-remainder bound for the degree- N polynomial on $|x| \leq 1$.

Question 1 – Solution

Step 1: Compute the coefficients.

Every derivative of $f(x) = e^x$ is e^x , so $f^{(n)}(0) = 1$. Hence

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n!}, \quad \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

Step 2: Bound the remainder.

Taylor's theorem gives

$$R_N(x) = \frac{e^c x^{N+1}}{(N+1)!}$$

for some c between 0 and x . If $|x| \leq 1$, then $c \leq 1$, so

$$\boxed{|R_N(x)| \leq \frac{e|x|^{N+1}}{(N+1)!} \leq \frac{e}{(N+1)!}}.$$

Step 3: Justify equality with the function.

For any fixed real x , $e^c \leq e^{|x|}$ and $e^{|x|}|x|^{N+1}/(N+1)! \rightarrow 0$ by the Ratio Test. Thus $R_N(x) \rightarrow 0$ for every x , proving the series represents e^x on all of $\mathbb{R}$.

Question 2

Derive the Maclaurin series for $\sin x$ from its derivative cycle. Explain why only odd powers occur, give the degree-7 polynomial, and state the convergence domain.

Question 2 – Solution

Step 1: List the derivative cycle.

$$\sin x, \quad \cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

At $x = 0$, the values are $0, 1, 0, -1, 0, \dots$. All even-order derivatives vanish, while the odd derivatives alternate between 1 and -1 .

Step 2: Form the series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The requested polynomial is

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Step 3: Establish convergence.

The absolute-value ratio of consecutive displayed terms is

$$\frac{x^2}{(2n+2)(2n+3)} \longrightarrow 0$$

for every fixed x . Taylor's remainder is bounded by $|x|^{m+1}/(m+1)!$ because all derivatives of sine have magnitude at most 1, so it tends to 0. Thus the series equals $\sin x$ for every real x .

Question 3

Derive the Maclaurin series for $\cos x$ from its derivative cycle. Explain why every odd coefficient vanishes, give the degree-6 polynomial, and state the convergence domain.

Question 3 – Solution

Step 1: Evaluate the derivative cycle at zero.

$$\cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

gives values $1, 0, -1, 0, 1, \dots$ at 0. Therefore every odd-order coefficient is zero, and the even coefficients alternate in sign.

Step 2: Write the series and polynomial.

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}, \quad P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Step 3: Justify convergence to cosine.

Consecutive absolute terms have ratio

$$\frac{x^2}{(2n+1)(2n+2)} \rightarrow 0,$$

so the series converges for all x . Since every derivative of cosine has magnitude at most 1, the Lagrange remainder is at most $|x|^{m+1}/(m+1)! \rightarrow 0$. Hence the series equals $\cos x$ on $(-\infty, \infty)$.

Question 4

Derive the Maclaurin series for $\log(1 + x)$ by integrating a geometric series. Give its first five nonzero terms and determine its exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}}.$$

The first five nonzero terms are

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

Step 3: Determine endpoints.

The interior radius is 1. At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series becomes $-\sum 1/n$ and diverges.

Conclusion.The interval is $\boxed{(-1, 1]}$.

Question 5

Obtain the Maclaurin series and degree-4 Taylor polynomial for e^{2x} by substitution into the series for e^u . State the convergence domain and justify it.

Question 5 – Solution

Step 1: Start with the parent series.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (u \in \mathbb{R}).$$

Set $u = 2x$:

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

Step 2: Extract the requested polynomial.

$$P_4(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

Step 3: Check convergence.

For $a_n = (2x)^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2|x|}{n+1} \rightarrow 0$$

for every real x . Hence the radius is infinite and the series represents e^{2x} on all of $\mathbb{R}$.

Question 6

Use $\cosh x = (e^x + e^{-x})/2$ to derive its Maclaurin series. Explain why all odd powers cancel, give the degree-6 polynomial, and state the convergence domain.

Question 6 – Solution

Step 1: Expand both exponentials.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

When the series are added, odd-power coefficients cancel and even-power coefficients double.

Step 2: Divide by two.

$$\boxed{\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}, \quad P_6(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}.$$

Step 3: State convergence.

Both exponential series converge absolutely for every real x , so their sum and scalar multiple do also. Thus the Maclaurin series for $\cosh x$ has infinite radius and represents the function on $(-\infty, \infty)$.

Question 7

Derive the Maclaurin series for xe^x by multiplying a known series by x . Give two equivalent index forms, the first five nonzero terms, and the convergence domain.

Question 7 – Solution

Step 1: Multiply the exponential series.

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}.$$

Multiplication by x raises every exponent by one but leaves the coefficient denominator tied to the original index.

Step 2: Reindex if powers x^k are preferred.

Set $k = n + 1$:

$$xe^x = \boxed{\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}}.$$

The first five nonzero terms are

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}.$$

Step 3: State convergence.

Multiplying an everywhere-convergent series by x does not change its infinite radius. Therefore the identity holds for every real x .

Question 8

Derive the Maclaurin series for $f(x) = 1/(1 - x)$ from Taylor coefficients, then connect it with the finite geometric-sum identity. Determine the exact interval of convergence.

Question 8 – Solution**Step 1: Compute Taylor coefficients.**

Repeated differentiation gives

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad f^{(n)}(0) = n!.$$

Thus the Maclaurin coefficient is $f^{(n)}(0)/n! = 1$, and

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Verify using finite sums.

$$S_N = 1 + x + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then $x^{N+1} \rightarrow 0$, so $S_N \rightarrow 1/(1-x)$. This proves the Taylor series and geometric identity are the same construction.

Step 3: Check endpoints.

At $x = 1$ the terms are 1; at $x = -1$ they alternate between ± 1 . Both diverge by the nth-term test.

Conclusion.

$R = 1$ and the interval is $\boxed{(-1, 1)}$.

Question 9

Use the generalized binomial coefficients to find the first four nonzero terms of the Maclaurin series for $\sqrt{1+x}$. State the radius and determine convergence at both endpoints.

Question 9 – Solution

Step 1: Use the binomial pattern.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

With $\alpha = 1/2$,

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 2: State radius and endpoints.

The binomial series has $R = 1$. Its coefficients have magnitude asymptotic to a constant times $n^{-3/2}$, so it converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$, with sums $\sqrt{2}$ and 0 at the endpoints.

Question 10

Use complex exponentials to find the first four nonzero Maclaurin terms of $e^x \cos x$. Give a general coefficient formula and state the convergence domain.

Question 10 – Solution

Step 1: Encode the product as a real part.

$$e^x \cos x = \operatorname{Re} (e^x e^{ix}) = \operatorname{Re} (e^{(1+i)x}).$$

Step 2: Expand the complex exponential.

$$e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(1+i)^n x^n}{n!},$$

so the real coefficient of x^n is $\operatorname{Re}((1+i)^n)/n!$. Since $1+i = \sqrt{2}e^{i\pi/4}$,

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{2^{n/2} \cos(n\pi/4)}{n!} x^n.$$

Step 3: Extract nonzero terms.

The x^2 coefficient vanishes. The first four nonzero terms are

$$\boxed{1 + x - \frac{x^3}{3} - \frac{x^4}{6}} \quad \left(\text{next term } -\frac{x^5}{30} \right).$$

The factorial denominator gives infinite radius, so the series represents the function for every real x .

Question 1

Approximate $e^{0.1}$ using the cubic Maclaurin polynomial for e^x . Then use the Lagrange remainder to give a rigorous upper bound for the absolute error.

Question 1 – Solution

Step 1: Write the cubic polynomial.

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Step 2: Evaluate at $x = 0.1$.

$$\begin{aligned} P_3(0.1) &= 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} \\ &= 1 + 0.1 + 0.005 + 0.0001666667 \\ &= \boxed{1.1051666667}. \end{aligned}$$

Step 3: Bound the error.

Taylor's theorem gives

$$R_3(0.1) = \frac{e^c(0.1)^4}{4!} \quad \text{for some } 0 < c < 0.1.$$

Because $e^c \leq e^{0.1}$,

$$|R_3(0.1)| \leq \frac{e^{0.1}(0.1)^4}{24} < 4.61 \times 10^{-6}.$$

Thus $e^{0.1}$ lies within 0.00000461 of the stated approximation.

Question 2

Approximate $\sin(0.2)$ so that correct rounding through four decimal places is guaranteed. Use the lowest-degree odd Maclaurin polynomial whose error bound is below 5×10^{-5} .

Question 2 – Solution

Step 1: Check the linear approximation.

For $x = 0.2$, the sine series is alternating with decreasing term magnitudes. Using only x gives

$$|R| \leq \frac{0.2^3}{3!} = 0.001333\ldots > 5 \times 10^{-5},$$

so the degree-1 polynomial does not provide the requested guarantee.

Step 2: Use the cubic polynomial.

$$P_3(x) = x - \frac{x^3}{3!}, \quad P_3(0.2) = 0.2 - \frac{0.008}{6} = \boxed{0.1986666667}.$$

Step 3: Bound the first omitted term.

$$|R| \leq \frac{0.2^5}{5!} = \frac{0.00032}{120} \approx 2.67 \times 10^{-6} < 5 \times 10^{-5}.$$

The alternating sign gives $0.1986666666 < \sin(0.2) < 0.1986693334$; both endpoints round to 0.1987. Therefore the smallest qualifying odd polynomial is cubic, and $\boxed{\sin(0.2) \approx 0.1987}$ to four decimal places.

Question 3

Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ by using a Taylor expansion. Identify why the constant and linear terms disappear and which coefficient determines the limit.

Question 3 – Solution

Step 1: Expand far enough to survive the cancellation.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

Step 2: Subtract the specified terms.

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

The constant 1 and linear term x cancel exactly.

Step 3: Divide by x^2 .

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + O(x^2).$$

As $x \rightarrow 0$, every remaining positive-power term tends to zero.

Conclusion.

$$\boxed{\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}}.$$

The answer is the first Taylor coefficient not removed by the numerator's subtraction.

Question 4

Solve the initial-value problem $y' = y$, $y(0) = 1$ by assuming a power-series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the coefficient recursion and identify the resulting function.

Question 4 – Solution

Step 1: Differentiate and align powers.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The original series is $y = \sum_{n=0}^{\infty} a_n x^n$.

Step 2: Match coefficients in $y' = y$.

$$(n+1)a_{n+1} = a_n, \quad a_{n+1} = \frac{a_n}{n+1}.$$

The initial condition gives $a_0 = y(0) = 1$. Repeated use of the recursion yields

$$a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \quad \text{and in general} \quad a_n = \frac{1}{n!}.$$

Conclusion.

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{e^x}.$$

The series has infinite radius by the Ratio Test, so the solution is valid for every real x .

Question 5

Estimate $I = \int_0^{0.5} e^{-x^2} dx$ by integrating the first four terms of the Maclaurin series for e^{-x^2} . Give a rigorous error bound from the first omitted integrated term.

Question 5 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

For $0 \leq x \leq 0.5$, the alternating term magnitudes decrease.

Step 2: Integrate through the x^6 term.

$$\begin{aligned} I &\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right]_0^{0.5} \\ &= 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} \\ &= \boxed{0.4612723214}. \end{aligned}$$

Step 3: Bound the integrated remainder.

The omitted integrand has magnitude at most $x^8/4!$, and the integrated alternating remainder satisfies

$$|R| \leq \int_0^{0.5} \frac{x^8}{4!} dx = \frac{0.5^9}{9 \cdot 4!} < 9.05 \times 10^{-6}.$$

Thus the estimate is accurate to within 0.00000905.

Question 6

For $0 < x \leq 1$, compare the linear approximation $L(x) = x$ with $\log(1 + x)$. Decide whether L overestimates or underestimates, and use the alternating series to bound the absolute error.

Question 6 – Solution

Step 1: Expand the logarithm.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad 0 < x \leq 1.$$

The first neglected term after the linear term is negative.

Step 2: Determine the direction.

Subtract the function from the linear approximation:

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots.$$

This is positive by the Alternating Series Test. Therefore x is an **overestimate** of $\log(1+x)$.

Step 3: Bound the error.

The alternating remainder after the first term is no larger than the first omitted magnitude:

$$0 < x - \log(1+x) \leq \frac{x^2}{2}.$$

For example, at $x = 0.2$, the error is at most 0.02; the actual error is approximately 0.01768.

Question 7

Derive one forward Euler step for $y' = f(t, y)$ from Taylor's theorem. Identify the discarded term. Then apply one step with $h = 0.1$ to $y' = y$, $y(0) = 1$, and bound its local error.

Question 7 – Solution

Step 1: Expand the exact solution in time.

Taylor's theorem about t_n gives

$$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n)$$

for some ξ_n between t_n and $t_n + h$.

Step 2: Replace the derivative using the ODE.

Since $y'(t_n) = f(t_n, y(t_n))$, dropping the quadratic remainder produces Euler's update

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}.$$

The one-step local truncation error is $h^2y''(\xi_n)/2$, hence is order h^2 .

Step 3: Apply the method.

For $y' = y$, $y_0 = 1$, and $h = 0.1$,

$$y_1 = 1 + 0.1(1) = \boxed{1.1}.$$

The exact value is $e^{0.1}$. On $[0, 0.1]$, $y''(t) = e^t \leq e^{0.1}$, so

$$|e^{0.1} - 1.1| \leq \frac{e^{0.1}(0.1)^2}{2} < 0.00553.$$

Question 8

The small-angle model replaces $\sin \theta$ by θ when θ is measured in radians. At $\theta = 0.2$, compute the relative error $|\theta - \sin \theta|/|\sin \theta|$ and explain it using the sine series.

Question 8 – Solution

Step 1: Use the local series.

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots.$$

Thus the leading absolute error in replacing $\sin \theta$ by θ is approximately $\theta^3/6$, and the leading relative error is approximately $\theta^2/6$.

Step 2: Evaluate the exact relative error.

At $\theta = 0.2$ radians,

$$\sin(0.2) \approx 0.1986693308, \quad 0.2 - \sin(0.2) \approx 0.0013306692.$$

Therefore

$$\frac{0.2 - \sin(0.2)}{\sin(0.2)} \approx 0.0066979 = \boxed{0.6698\%}.$$

Step 3: Compare with the leading prediction.

$$\frac{\theta^2}{6} = \frac{0.04}{6} \approx 0.0066667 = 0.6667\%,$$

which is close to the exact relative error. This quantifies why the approximation is useful but not exact.

Question 9

A principal P earns annual rate r compounded n times per year, so $A_n = P(1 + r/n)^n$. Use the series for $\log(1 + u)$ to derive the continuous-compounding limit as $n \rightarrow \infty$.

Question 9 – Solution**Step 1: Take a logarithm.**For sufficiently large n , $1 + r/n > 0$, and

$$\log\left(\frac{A_n}{P}\right) = n \log\left(1 + \frac{r}{n}\right).$$

Step 2: Insert the logarithmic expansion.

$$\begin{aligned} n \log\left(1 + \frac{r}{n}\right) &= n \left(\frac{r}{n} - \frac{r^2}{2n^2} + \frac{r^3}{3n^3} + O(n^{-4}) \right) \\ &= r - \frac{r^2}{2n} + \frac{r^3}{3n^2} + O(n^{-3}) \longrightarrow r. \end{aligned}$$

Step 3: Exponentiate.

Continuity of the exponential function gives

$$\frac{A_n}{P} \longrightarrow e^r, \quad \boxed{A_n \longrightarrow Pe^r}.$$

The correction $-r^2/(2n)$ also explains the approach rate. For positive r , discrete compounding approaches the continuous value from below.

Question 10

Construct the quadratic Maclaurin model for the Gaussian e^{-x^2} . Determine where that polynomial is positive, and give an error bound for $|x| \leq 1$ using the first omitted term.

Question 10 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots.$$

Keeping terms through degree 2 gives

$$\boxed{Q_2(x) = 1 - x^2}.$$

Step 2: Analyze the model's sign.

$$Q_2(x) > 0 \iff 1 - x^2 > 0 \iff |x| < 1.$$

It vanishes at $x = \pm 1$ and becomes negative for $|x| > 1$, whereas the true Gaussian is positive for every real x . This shows the model is local.

Step 3: Bound the local error.

For $|x| \leq 1$, the series in x^2 alternates with decreasing magnitudes. Therefore

$$\boxed{|e^{-x^2} - (1 - x^2)| \leq \frac{x^4}{2}}.$$

The error is order x^4 near the origin.

Question 1

Use the generalized binomial theorem to find the first four nonzero terms of $(1 + x)^{1/2}$. State the radius and determine convergence at both endpoints.

Question 1 – Solution

Step 1: State the coefficient formula.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Step 2: Substitute $\alpha = 1/2$.

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 3: Determine the interval.

The radius is $R = 1$. The coefficient magnitudes are asymptotic to a constant times $n^{-3/2}$, so the series converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$.

Question 2

Treat $(1 + x)^{-1}$ as a generalized binomial series. Derive its coefficients, connect the result with a geometric series, and determine the exact interval of convergence.

Question 2 – Solution

Step 1: Compute the generalized coefficients.

For $n \geq 0$,

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = (-1)^n.$$

Thus

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots.$$

Step 2: Connect with geometry.

This is exactly

$$\frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n,$$

whose common ratio is $-x$. It converges when $|-x| < 1$, so $R = 1$.

Step 3: Test endpoints.

At $x = 1$, terms are $(-1)^n$; at $x = -1$, terms are 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 3

Find the first four nonzero terms of $(1 + x)^{-1/2}$ using generalized binomial coefficients. Explain the signs and determine the exact interval of convergence.

Question 3 – Solution**Step 1: Compute coefficients.**

$$\binom{-1/2}{1} = -\frac{1}{2}, \quad \binom{-1/2}{2} = \frac{3}{8}, \quad \binom{-1/2}{3} = -\frac{5}{16}.$$

Every factor in the numerator is negative, so the coefficient sign is $(-1)^n$. Hence

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots.$$

Step 2: Find the radius.

The generalized binomial series has $R = 1$.

Step 3: Test endpoints.

The coefficient magnitudes satisfy

$$\left| \binom{-1/2}{n} \right| = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

At $x = 1$ the signs alternate, so the series converges conditionally. At $x = -1$, all terms are positive and comparable with $1/\sqrt{n}$, so it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1, 1]}$.

Question 4

Expand $(1 - 2x)^{3/2}$ through the x^4 term. Track the substitution $u = -2x$ carefully, then find the radius and endpoint behavior.

Question 4 – Solution

Step 1: Expand in an intermediate variable.

$$(1 + u)^{3/2} = 1 + \frac{3}{2}u + \frac{3}{8}u^2 - \frac{1}{16}u^3 + \frac{3}{128}u^4 + \cdots.$$

Step 2: Substitute $u = -2x$.

$$\begin{aligned}(1 - 2x)^{3/2} &= 1 + \frac{3}{2}(-2x) + \frac{3}{8}(4x^2) - \frac{1}{16}(-8x^3) + \frac{3}{128}(16x^4) + \cdots \\ &= \boxed{1 - 3x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \cdots}.\end{aligned}$$

Each degree- n coefficient is multiplied by $(-2)^n$.

Step 3: Determine convergence.

The condition $|u| < 1$ gives $|x| < 1/2$, so $R = 1/2$. For exponent $3/2$, coefficient magnitudes are order $n^{-5/2}$, so both endpoint series converge absolutely.

Conclusion.

The interval is $\boxed{[-1/2, 1/2]}$.

Question 5

For $(1+x)^{1/3} = \sum_{n=0}^{\infty} c_n x^n$, derive a recurrence for c_{n+1} from c_n . Use it to compute through x^4 , then state the convergence interval.

Question 5 – Solution

Step 1: Derive the recurrence.

Since $c_n = \binom{1/3}{n}$,

$$\frac{c_{n+1}}{c_n} = \frac{(1/3) - n}{n + 1}, \quad \boxed{c_{n+1} = c_n \frac{1/3 - n}{n + 1}}, \quad c_0 = 1.$$

Step 2: Generate coefficients.

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{9}, \quad c_3 = \frac{5}{81}, \quad c_4 = -\frac{10}{243}.$$

Therefore

$$\boxed{(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \cdots}.$$

Step 3: State convergence.

The radius is 1. Since the coefficient magnitudes are order $n^{-4/3}$, both endpoint series converge absolutely.

Conclusion.

The interval is $\boxed{[-1, 1]}$.

Question 6

Derive the binomial series for $(1 - x)^{-2}$ and show that it agrees with differentiating the geometric series. Determine the exact interval of convergence.

Question 6 – Solution

Step 1: Use generalized coefficients.

For $n \geq 0$,

$$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-n-1)}{n!} = (-1)^n(n+1).$$

Substitute $u = -x$ into $(1+u)^{-2}$:

$$\binom{-2}{n}(-x)^n = (-1)^n(n+1)(-1)^n x^n = (n+1)x^n.$$

Thus

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 2: Verify by differentiation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Rightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 3: Check convergence.

The radius is 1. At $x = 1$ terms are $n+1$; at $x = -1$ their magnitudes are $n+1$. Both endpoints diverge.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 7

Approximate $\sqrt{1.04}$ using only the constant and linear terms of the binomial series for $\sqrt{1+x}$. Use the alternating-series structure to bound the omitted correction and report the approximation.

Question 7 – Solution

Step 1: Use the binomial expansion.

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots.$$

Set $x = 0.04$.

Step 2: Keep two terms.

$$\sqrt{1.04} \approx 1 + \frac{0.04}{2} = \boxed{1.02}.$$

Step 3: Bound the error.

Beginning with the quadratic term, the omitted terms alternate and decrease in magnitude at $x = 0.04$. Therefore

$$|R| \leq \frac{(0.04)^2}{8} = \boxed{0.0002}.$$

The first correction is negative, so the linear approximation is an overestimate. More precisely,

$$1.0198 \leq \sqrt{1.04} < 1.02.$$

The actual value, $1.0198039\dots$, is consistent with this bound.

Question 8

Let m be a nonnegative integer. Explain algebraically why the generalized binomial series for $(1+x)^m$ terminates, and recover the ordinary finite Binomial Theorem. Illustrate with $m = 4$.

Question 8 – Solution

Step 1: Examine the generalized coefficient.

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

If $n = m + 1$, the numerator includes the factor $m - m = 0$. For every $n > m$, it still includes that zero factor. Hence

$$\binom{m}{n} = 0 \quad (n > m).$$

Step 2: Recover the finite theorem.

The formally infinite series therefore reduces to

$$(1+x)^m = \sum_{n=0}^m \binom{m}{n} x^n.$$

Because this is a polynomial identity, it is valid for every real (and complex) x ; there is no finite radius restriction.

Step 3: Illustrate with $m = 4$.

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The next coefficient is $\binom{4}{5} = 0$, so no higher powers occur.

Question 9

Find the coefficient of x^4 in the Maclaurin series for $(1+x)^{1/2}$ directly from the generalized binomial formula. Do not compute the lower-degree coefficients.

Question 9 – Solution**Step 1: Identify the needed coefficient.**

In

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n,$$

the coefficient of x^4 is exactly $\binom{1/2}{4}$.**Step 2: Evaluate only that coefficient.**

$$\begin{aligned} \binom{1/2}{4} &= \frac{(1/2)(1/2-1)(1/2-2)(1/2-3)}{4!} \\ &= \frac{(1/2)(-1/2)(-3/2)(-5/2)}{24} \\ &= \frac{-15}{16 \cdot 24} = \boxed{-\frac{5}{128}}. \end{aligned}$$

There are three negative numerator factors, so the negative sign is expected.

Conclusion.The x^4 term is $\boxed{-5x^4/128}$. Computing preceding terms is unnecessary because generalized binomial coefficients directly target any desired power.

Question 10

Find the binomial series, radius, and exact interval of convergence for $(1 + 3x)^{-1/2}$. Test $x = -1/3$ and $x = 1/3$ separately.

Question 10 – Solution

Step 1: Substitute into the parent series.

$$(1 + u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} u^n.$$

With $u = 3x$,

$$(1 + 3x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 3^n x^n = 1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 + \cdots.$$

Step 2: Find the radius.

The condition $|u| < 1$ becomes $|3x| < 1$, so $R = 1/3$.

Step 3: Test endpoints.

At $x = 1/3$, $u = 1$, and the terms alternate with magnitude asymptotic to $1/\sqrt{\pi n}$; the series converges conditionally. At $x = -1/3$, $u = -1$ cancels the coefficient signs, leaving positive terms comparable with $1/\sqrt{n}$; it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

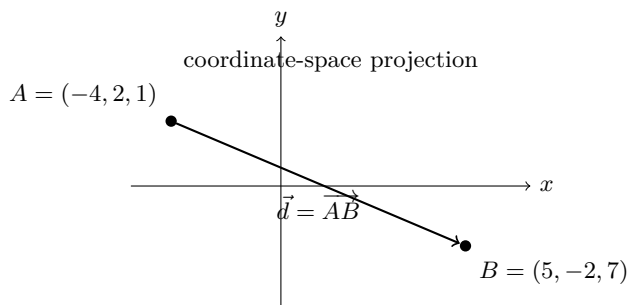
The interval is $\boxed{(-1/3, 1/3]}$.

Question 1

A rescue drone travels from $A = (-4, 2, 1)$ to $B = (5, -2, 7)$. Find the displacement vector, its magnitude, and the corresponding unit direction vector.

Question 1 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Subtract initial coordinates from terminal coordinates: $\vec{d} = B - A = \langle 9, -4, 6 \rangle$.

Step 2 Compute its length: $\|\vec{d}\| = \sqrt{9^2 + (-4)^2 + 6^2} = \sqrt{133}$.

Step 3 Normalize: $\hat{d} = \vec{d}/\|\vec{d}\| = \frac{1}{\sqrt{133}} \langle 9, -4, 6 \rangle$. The displacement magnitude is $\sqrt{133}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

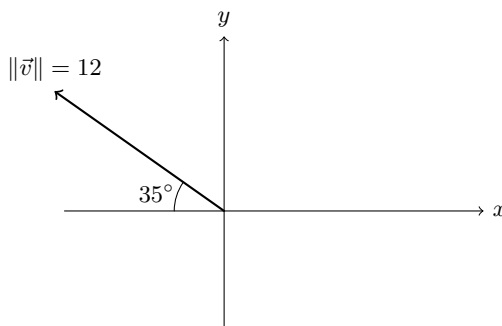
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 2

A vector in the plane has magnitude 12, points into Quadrant II, and makes an angle of 35° with the negative x -axis. Write it in component form.

Question 2 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Its standard-position angle is $180^\circ - 35^\circ = 145^\circ$.

Step 2 Use $\vec{v} = \|\vec{v}\|\langle \cos \theta, \sin \theta \rangle$.

Step 3 Thus $\vec{v} = 12\langle -\cos 35^\circ, \sin 35^\circ \rangle$. The signs agree with Quadrant II.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

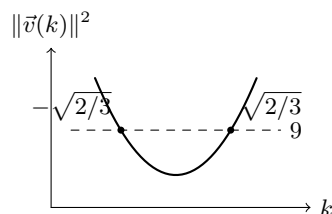
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 3

Find all real k for which $\vec{v} = \langle k, 2k - 1, k + 2 \rangle$ has magnitude 3.

Question 3 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Square the magnitude equation: $k^2 + (2k - 1)^2 + (k + 2)^2 = 9$.

Step 2 Expanding gives $6k^2 + 5 = 9$, so $3k^2 = 2$.

Step 3 Therefore $k = \pm\sqrt{2/3}$; substitution verifies both values.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

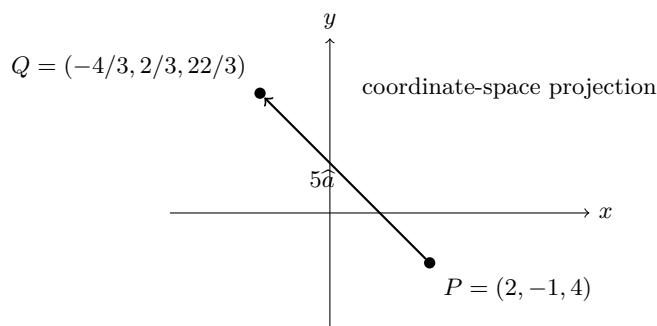
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 4

The point $P = (2, -1, 4)$ is translated five units in the direction of $\vec{a} = \langle -2, 1, 2 \rangle$. Find the image point Q .

Question 4 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 First normalize: $\|\vec{a}\| = 3$, so $\hat{a} = \frac{1}{3}\langle -2, 1, 2 \rangle$.

Step 2 A five-unit translation has vector $5\hat{a} = \langle -10/3, 5/3, 10/3 \rangle$.

Step 3 Add it to P : $Q = P + 5\hat{a} = \left(-\frac{4}{3}, \frac{2}{3}, \frac{22}{3} \right)$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

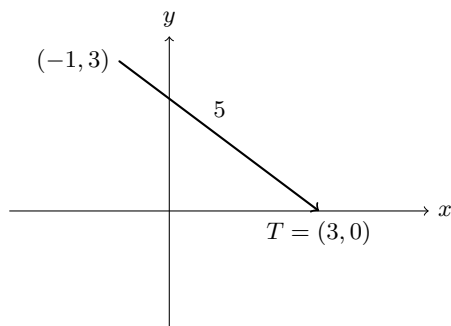
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 5

A vector $\vec{u}$ begins at $(-1, 3)$ and ends somewhere on the positive x -axis. If $\|\vec{u}\| = 5$, find every possible terminal point.

Question 5 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Write the terminal point as $T = (t, 0)$ with $t > 0$. Then $\vec{u} = \langle t + 1, -3 \rangle$.

Step 2 The length condition gives $(t + 1)^2 + 9 = 25$, hence $t + 1 = \pm 4$.

Step 3 The candidates are $t = 3, -5$; positivity leaves $\boxed{T = (3, 0)}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

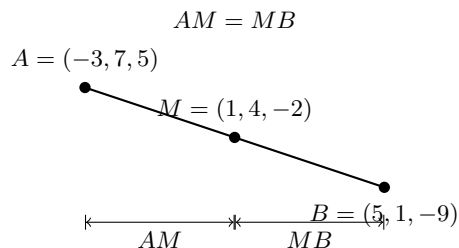
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 6

Show that $M = (1, 4, -2)$ is the midpoint of the segment joining $A = (-3, 7, 5)$ and an unknown point B , and determine B .

Question 6 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 The midpoint equation is $M = (A + B)/2$, so $B = 2M - A$.

Step 2 Compute $2M = \langle 2, 8, -4 \rangle$ and subtract A .

Step 3 Thus $B = \boxed{(5, 1, -9)}$. Averaging A and B returns $(1, 4, -2)$, which verifies the claim.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

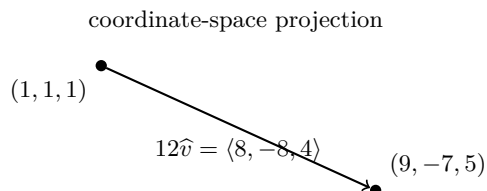
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 7

A particle moves from $(1, 1, 1)$ in the direction $\langle 2, -2, 1 \rangle$. At what point has it traveled exactly 12 units?

Question 7 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 The direction magnitude is $\sqrt{4 + 4 + 1} = 3$, so its unit vector is $\frac{1}{3}\langle 2, -2, 1 \rangle$.

Step 2 A 12-unit displacement is $12\hat{v} = \langle 8, -8, 4 \rangle$.

Step 3 Adding the starting point gives $\boxed{(9, -7, 5)}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

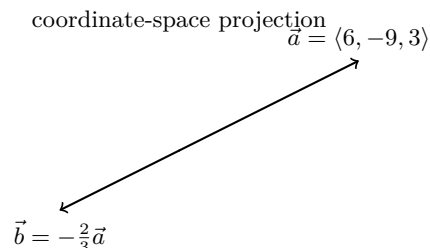
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 8

Determine whether $\vec{a} = \langle 6, -9, 3 \rangle$ and $\vec{b} = \langle -4, 6, -2 \rangle$ represent the same direction, opposite directions, or neither. Explain.

Question 8 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Compare corresponding nonzero components: $-4/6 = 6/(-9) = (-2)/3 = -2/3$.

Step 2 Thus $\vec{b} = (-2/3)\vec{a}$.

Step 3 Because the scalar is negative, the vectors are parallel but point in opposite directions.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

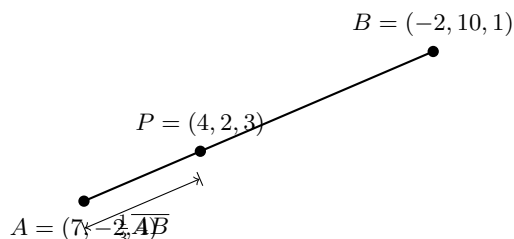
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 9

Find the point one-third of the way from $A = (7, -2, 4)$ to $B = (-2, 10, 1)$.

Question 9 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 The displacement from A to B is $B - A = \langle -9, 12, -3 \rangle$.

Step 2 One-third of this displacement is $\langle -3, 4, -1 \rangle$.

Step 3 Starting at A gives $A + \frac{1}{3}(B - A) = \boxed{(4, 2, 3)}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

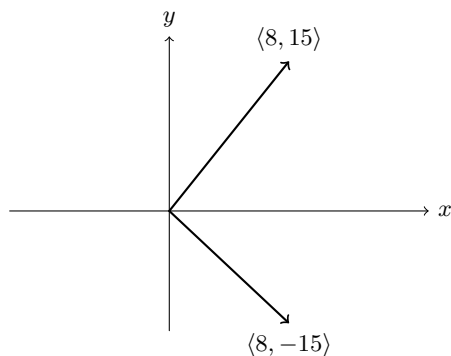
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Question 10

A velocity vector has horizontal component 8 and total speed 17. Find all possible vertical components and interpret the two answers.

Question 10 – Solution

Strategy Identify the initial and terminal information first. Build the vector by subtracting coordinates, then use the magnitude formula or normalize only after the components are correct.



Step 1 Let the vertical component be v . The speed condition is $\sqrt{8^2 + v^2} = 17$.

Step 2 Squaring yields $v^2 = 289 - 64 = 225$, so $v = \pm 15$.

Step 3 The velocities are $\boxed{\langle 8, 15 \rangle}$ and $\boxed{\langle 8, -15 \rangle}$, representing upward and downward motion with the same speed.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

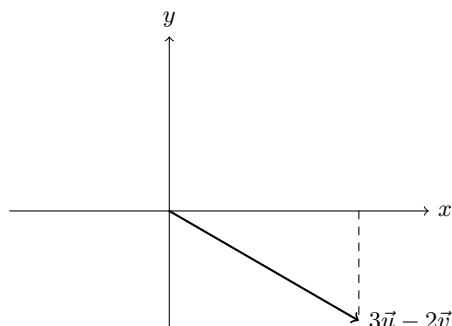
Key idea A vector separates two ideas: its components describe displacement, while its magnitude and unit vector describe size and direction. Always check signs and confirm that a unit vector has magnitude 1.

Question 1

Given $\vec{u} = \langle 2, -1, 3 \rangle$ and $\vec{v} = \langle -4, 5, 1 \rangle$, find $3\vec{u} - 2\vec{v}$ and its magnitude.

Question 1 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Scale componentwise: $3\vec{u} = \langle 6, -3, 9 \rangle$ and $2\vec{v} = \langle -8, 10, 2 \rangle$.

Step 2 Subtract to get $3\vec{u} - 2\vec{v} = \langle 14, -13, 7 \rangle$.

Step 3 Its magnitude is $\sqrt{196 + 169 + 49} = \boxed{\sqrt{414} = 3\sqrt{46}}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

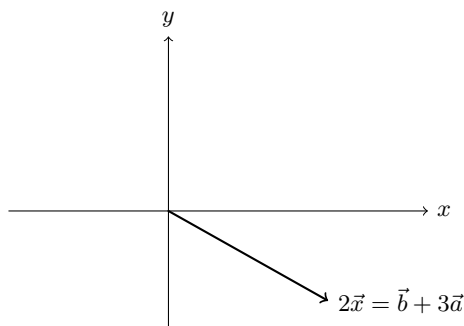
Key idea Vector equations behave like ordinary algebra provided that every operation is performed component by component. A final component check is often the fastest verification.

Question 2

Solve the vector equation $2\vec{x} - 3\vec{a} = \vec{b}$ when $\vec{a} = \langle 1, -2, 4 \rangle$ and $\vec{b} = \langle 5, 0, -6 \rangle$.

Question 2 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Isolate the unknown: $2\vec{x} = \vec{b} + 3\vec{a}$.

Step 2 Compute $\vec{b} + 3\vec{a} = \langle 5, 0, -6 \rangle + \langle 3, -6, 12 \rangle = \langle 8, -6, 6 \rangle$.

Step 3 Divide by 2: $\vec{x} = \langle 4, -3, 3 \rangle$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

Key idea Vector equations behave like ordinary algebra provided that every operation is performed component by component. A final component check is often the fastest verification.

Question 3

For what value of t are $\langle t + 1, 2t - 3, 5 \rangle$ and $\langle 4, 3, 5 \rangle$ equal? If none exists, identify the obstruction.

Question 3 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.

$$\begin{array}{c} \langle t+1, 2t-3, 5 \rangle \\ \diagup \\ = \langle 4, 3, 5 \rangle \end{array}$$

Step 1 Equal vectors require equality of every component.

Step 2 The first component gives $t + 1 = 4$, so $t = 3$; the second gives $2t - 3 = 3$, also $t = 3$.

Step 3 The third components already agree, so the unique value is $\boxed{t = 3}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

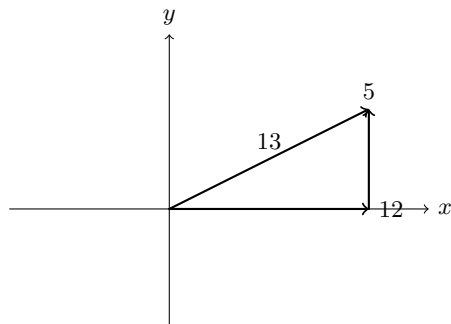
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Question 4

A boat moves at 12 km/h due east relative to the water while a current flows at 5 km/h due north. Find the ground velocity, speed, and direction north of east.

Question 4 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Add the velocity vectors: $\vec{v}_g = \langle 12, 0 \rangle + \langle 0, 5 \rangle = \langle 12, 5 \rangle$.

Step 2 The ground speed is $\sqrt{12^2 + 5^2} = 13$ km/h.

Step 3 The direction satisfies $\tan \theta = 5/12$, so $\theta = \tan^{-1}(5/12) \approx 22.6^\circ$ north of east.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

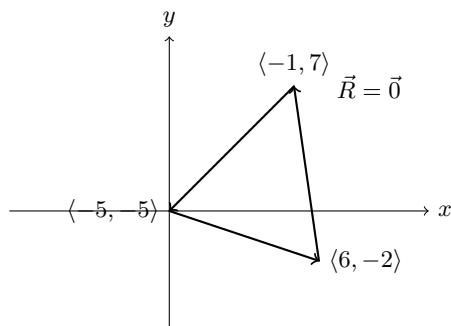
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Question 5

Three forces $\langle 6, -2 \rangle$, $\langle -1, 7 \rangle$, and $\langle p, -5 \rangle$ act on an object. Find p so the horizontal component of the resultant vanishes, then find the resultant. Is it a nonzero vertical force?

Question 5 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Add components: $\vec{R} = \langle 6 - 1 + p, -2 + 7 - 5 \rangle = \langle p + 5, 0 \rangle$.

Step 2 A vertical vector must have horizontal component zero, so $p + 5 = 0$ and $p = -5$.

Step 3 Here the vertical component is also zero, so the resultant is the zero vector: $\boxed{\vec{R} = \vec{0}}$. This is equilibrium. The zero vector has no direction, so no value of p produces a nonzero vertical resultant.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

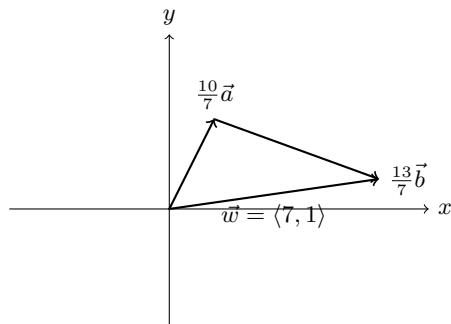
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Question 6

Write $\vec{w} = \langle 7, 1 \rangle$ as a linear combination of $\vec{a} = \langle 1, 2 \rangle$ and $\vec{b} = \langle 3, -1 \rangle$.

Question 6 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Let $\vec{w} = c\vec{a} + d\vec{b}$. Components give $c + 3d = 7$ and $2c - d = 1$.

Step 2 From the second equation $d = 2c - 1$. Substitute: $c + 3(2c - 1) = 7$, so $7c = 10$.

Step 3 Thus $c = 10/7$ and $d = 13/7$, giving $\vec{w} = \frac{10}{7}\vec{a} + \frac{13}{7}\vec{b}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

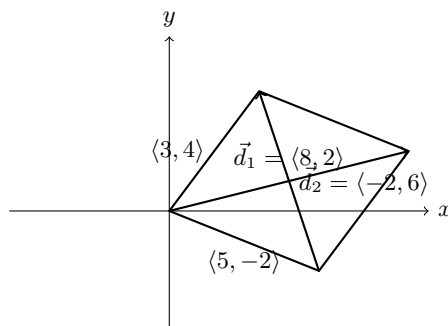
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Question 7

The diagonals of a parallelogram are $\vec{d}_1 = \langle 8, 2 \rangle$ and $\vec{d}_2 = \langle -2, 6 \rangle$. Find possible adjacent side vectors.

Question 7 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 If adjacent sides are $\vec{a}, \vec{b}$, then $\vec{d}_1 = \vec{a} + \vec{b}$ and $\vec{d}_2 = \vec{a} - \vec{b}$.

Step 2 Add and divide by two: $\vec{a} = (\vec{d}_1 + \vec{d}_2)/2 = \langle 3, 4 \rangle$.

Step 3 Subtract and divide by two: $\vec{b} = (\vec{d}_1 - \vec{d}_2)/2 = \boxed{\langle 5, -2 \rangle}$. Thus one ordered pair is $\boxed{\langle 3, 4 \rangle, \langle 5, -2 \rangle}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

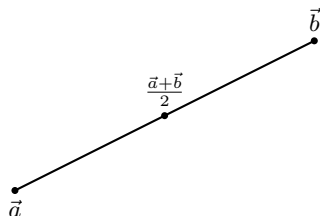
Key idea Vector equations behave like ordinary algebra provided that every operation is performed component by component. A final component check is often the fastest verification.

Question 8

Prove using components that the midpoint of points with position vectors $\vec{a}$ and $\vec{b}$ has position vector $(\vec{a} + \vec{b})/2$.

Question 8 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Write $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Step 2 Their average is $\frac{1}{2}(\vec{a} + \vec{b}) = \left\langle \frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2} \right\rangle$.

Step 3 Each coordinate is the ordinary midpoint of the corresponding coordinates, so this is exactly the midpoint position vector. Proved

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

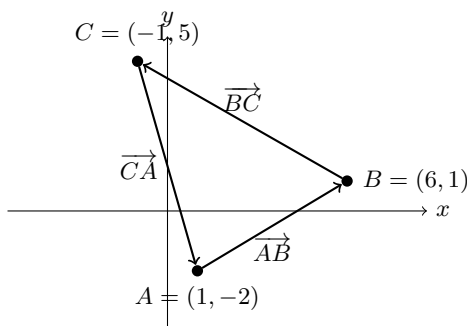
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Question 9

A triangle has vertices $A = (1, -2)$, $B = (6, 1)$, and $C = (-1, 5)$. Verify the closed-path identity $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.

Question 9 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 Compute $\overrightarrow{AB} = \langle 5, 3 \rangle$, $\overrightarrow{BC} = \langle -7, 4 \rangle$, and $\overrightarrow{CA} = \langle 2, -7 \rangle$.

Step 2 Add the three: $\langle 5 - 7 + 2, 3 + 4 - 7 \rangle = \langle 0, 0 \rangle$.

Step 3 Therefore $\boxed{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}}$, as any closed displacement path must.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

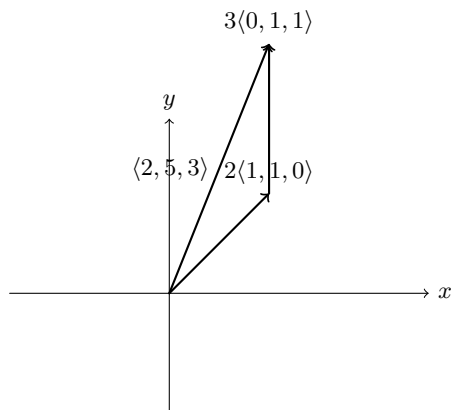
Key idea Vector equations behave like ordinary algebra provided that every operation is performed component by component. A final component check is often the fastest verification.

Question 10

Find constants a, b so that $a\langle 1, 1, 0 \rangle + b\langle 0, 1, 1 \rangle = \langle 2, 5, 3 \rangle$, or prove it is impossible.

Question 10 – Solution

Strategy Write every vector in matching component order. Perform scalar multiplication before addition or subtraction, then interpret the resulting components in the context of the problem.



Step 1 The linear combination is $\langle a, a + b, b \rangle$.

Step 2 The first and third components force $a = 2$ and $b = 3$.

Step 3 Then $a + b = 5$, matching the middle component. Hence $\boxed{a = 2, b = 3}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

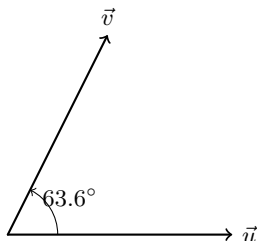
Key idea Vector equations behave like ordinary algebra provided that every operation is performed component by component. A final component check is often the fastest verification.

Question 1

Find the angle between $\vec{u} = \langle 1, 2, 2 \rangle$ and $\vec{v} = \langle 2, -1, 2 \rangle$.

Question 1 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Compute $\vec{u} \cdot \vec{v} = 2 - 2 + 4 = 4$.

Step 2 Both magnitudes are 3, so $\cos \theta = 4/(3 \cdot 3) = 4/9$.

Step 3 Therefore $\theta = \cos^{-1}(4/9) \approx 63.6^\circ$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

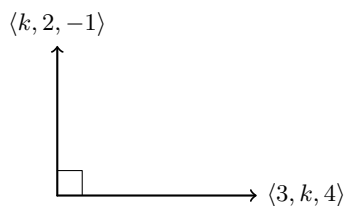
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 2

Find all k such that $\langle k, 2, -1 \rangle$ is perpendicular to $\langle 3, k, 4 \rangle$.

Question 2 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Perpendicular vectors have zero dot product.

Step 2 Set $3k + 2k - 4 = 0$, which simplifies to $5k = 4$.

Step 3 Thus $k = 4/5$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

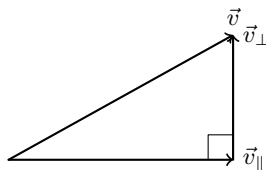
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 3

Decompose $\vec{v} = \langle 5, 1, -2 \rangle$ into components parallel and perpendicular to $\vec{u} = \langle 1, 2, 2 \rangle$.

Question 3 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 The parallel component is $\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$. Here $\vec{v} \cdot \vec{u} = 3$ and $\vec{u} \cdot \vec{u} = 9$.

Step 2 So $\vec{v}_{\parallel} = \frac{1}{3} \langle 1, 2, 2 \rangle = \langle 1/3, 2/3, 2/3 \rangle$.

Step 3 Subtract: $\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = \boxed{\langle 14/3, 1/3, -8/3 \rangle}$. Their dot product is zero.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

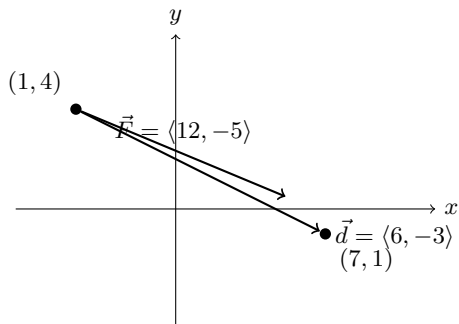
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 4

A force $\vec{F} = \langle 12, -5 \rangle$ moves an object from $(1, 4)$ to $(7, 1)$. Find the work done.

Question 4 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 The displacement is $\vec{d} = \langle 7 - 1, 1 - 4 \rangle = \langle 6, -3 \rangle$.

Step 2 Work is the dot product: $W = \vec{F} \cdot \vec{d}$.

Step 3 Thus $W = 12(6) + (-5)(-3) = 72 + 15 = \boxed{87}$ force-distance units.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

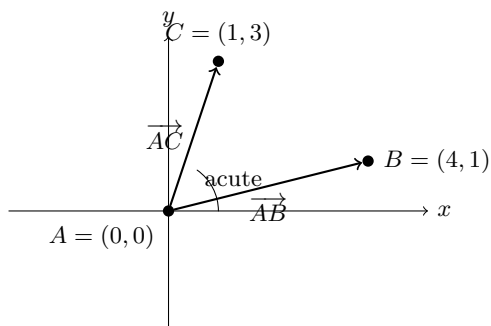
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 5

Determine whether the triangle with vertices $A = (0, 0)$, $B = (4, 1)$, and $C = (1, 3)$ has an acute, right, or obtuse angle at A .

Question 5 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 The sides from A are $\overrightarrow{AB} = \langle 4, 1 \rangle$ and $\overrightarrow{AC} = \langle 1, 3 \rangle$.

Step 2 Their dot product is $4(1) + 1(3) = 7 > 0$.

Step 3 A positive dot product means the included angle has positive cosine, so the angle at A is acute.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

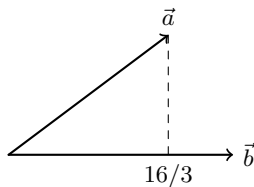
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 6

Find the scalar projection of $\vec{a} = \langle -2, 5, 4 \rangle$ onto $\vec{b} = \langle 1, 2, 2 \rangle$, and explain the sign.

Question 6 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Compute $\vec{a} \cdot \vec{b} = -2 + 10 + 8 = 16$ and $\|\vec{b}\| = 3$.

Step 2 The scalar projection is $\text{comp}_{\vec{b}} \vec{a} = (\vec{a} \cdot \vec{b}) / \|\vec{b}\| = \boxed{16/3}$.

Step 3 It is positive because $\vec{a}$ has a net component in the same direction as $\vec{b}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

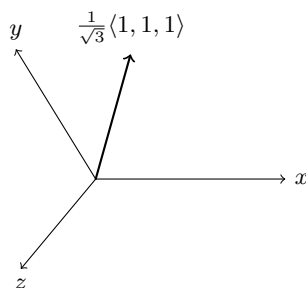
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 7

A unit vector $\vec{n}$ makes equal acute angles with the positive coordinate axes in $\mathbb{R}^3$. Find $\vec{n}$ and the common angle.

Question 7 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Equal direction cosines imply $\vec{n} = \langle c, c, c \rangle$ with $c > 0$.

Step 2 The unit condition gives $3c^2 = 1$, so $c = 1/\sqrt{3}$.

Step 3 Hence $\vec{n} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$ and $\theta = \cos^{-1}(1/\sqrt{3})$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

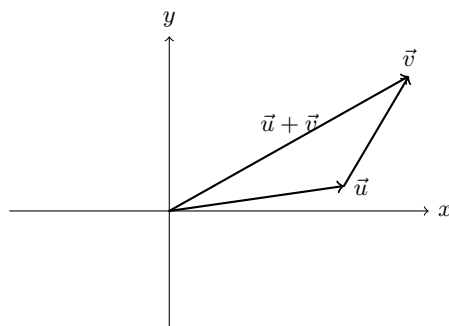
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 8

Prove the identity $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$.

Question 8 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Start with the squared norm: $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$.

Step 2 Distribute the dot product to obtain $\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$.

Step 3 Use symmetry and $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$ to get the stated identity. Proved

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

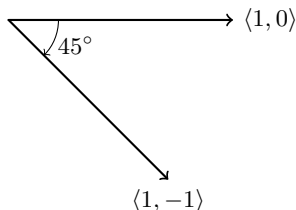
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 9

For which real t is the angle between $\langle 1, t \rangle$ and $\langle 1, -1 \rangle$ equal to 45° ?

Question 9 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Use the cosine formula: $(1 - t)/(\sqrt{1 + t^2}\sqrt{2}) = \sqrt{2}/2$.

Step 2 This simplifies to $1 - t = \sqrt{1 + t^2}$. The right side is nonnegative, so $t \leq 1$. Squaring gives $1 - 2t + t^2 = 1 + t^2$, hence $t = 0$.

Step 3 It satisfies the original equation, so $\boxed{t = 0}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

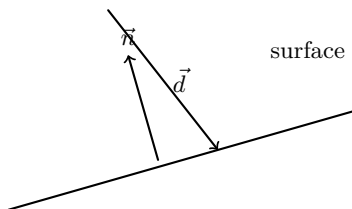
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 10

A spotlight beam points along $\vec{d} = \langle 2, -1, 2 \rangle$. A surface has unit normal $\vec{n} = \frac{1}{3}\langle 1, 2, 2 \rangle$. Find the magnitude of the component of $\vec{d}$ normal to the surface.

Question 10 – Solution

Strategy Compute the dot product carefully, then choose the formula that matches the question. A zero, positive, or negative dot product also gives immediate geometric information.



Step 1 Because $\vec{n}$ is a unit vector, the normal component magnitude is $|\vec{d} \cdot \vec{n}|$.

Step 2 Compute $\vec{d} \cdot \vec{n} = (2 - 2 + 4)/3 = 4/3$.

Step 3 Therefore the requested magnitude is $\boxed{4/3}$. The absolute value makes the result independent of the chosen normal orientation.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

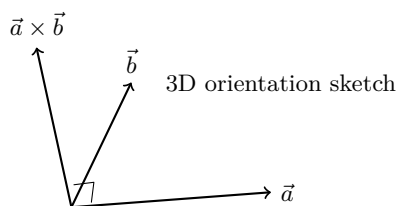
Key idea The dot product converts two vectors into a scalar measuring directional agreement: positive means generally aligned, zero means perpendicular, and negative means generally opposed.

Question 1

Compute $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, 2, -1 \rangle$ and $\vec{b} = \langle 3, 0, 4 \rangle$, then verify it is perpendicular to both vectors.

Question 1 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Expand the determinant: $\vec{a} \times \vec{b} = \langle 2(4) - (-1)(0), (-1)(3) - 1(4), 1(0) - 2(3) \rangle = \langle 8, -7, -6 \rangle$.

Step 2 Check $(\vec{a} \times \vec{b}) \cdot \vec{a} = 8 - 14 + 6 = 0$.

Step 3 Also $(\vec{a} \times \vec{b}) \cdot \vec{b} = 24 - 24 = 0$, so $\boxed{\vec{a} \times \vec{b} = \langle 8, -7, -6 \rangle}$ is perpendicular to both.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

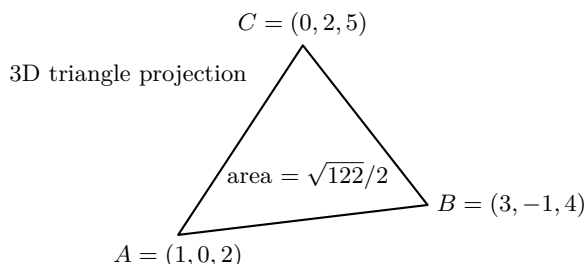
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 2

Find the area of the triangle with vertices $A = (1, 0, 2)$, $B = (3, -1, 4)$, and $C = (0, 2, 5)$.

Question 2 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Form two sides: $\overrightarrow{AB} = \langle 2, -1, 2 \rangle$ and $\overrightarrow{AC} = \langle -1, 2, 3 \rangle$.

Step 2 Their cross product is $\langle -7, -8, 3 \rangle$, whose magnitude is $\sqrt{49 + 64 + 9} = \sqrt{122}$.

Step 3 A triangle has half the parallelogram area, so $A = \sqrt{122}/2$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

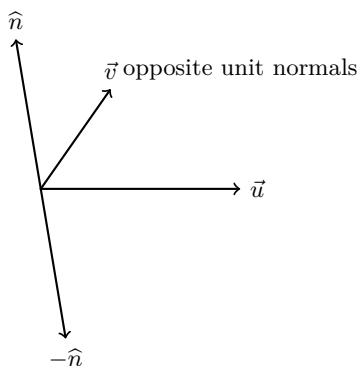
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 3

Find both unit vectors perpendicular to $\langle 2, 1, 0 \rangle$ and $\langle 1, -1, 3 \rangle$.

Question 3 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Take a cross product: $\langle 2, 1, 0 \rangle \times \langle 1, -1, 3 \rangle = \langle 3, -6, -3 \rangle = 3\langle 1, -2, -1 \rangle$.

Step 2 Its magnitude is $3\sqrt{6}$.

Step 3 Normalize and include both orientations: $\pm \frac{1}{\sqrt{6}} \langle 1, -2, -1 \rangle$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

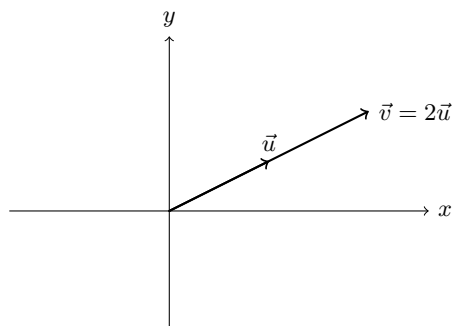
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 4

For what values of k are $\vec{u} = \langle 1, k, 2 \rangle$ and $\vec{v} = \langle 2, 2k, 4 \rangle$ parallel? What is their cross product then?

Question 4 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Observe directly that $\vec{v} = 2\vec{u}$ for every real k .

Step 2 Therefore the vectors are parallel for $\boxed{\text{all } k \in \mathbb{R}}$.

Step 3 The cross product of parallel vectors is the zero vector, so $\boxed{\vec{u} \times \vec{v} = \vec{0}}$ for every k .

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

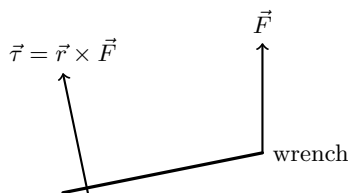
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 5

A wrench extends from the origin along $\vec{r} = \langle 0.30, 0.10, 0 \rangle$ meters. A force $\vec{F} = \langle 0, 40, -20 \rangle$ newtons is applied. Find the torque vector and its magnitude.

Question 5 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Torque is $\vec{\tau} = \vec{r} \times \vec{F}$.

Step 2 Compute $\vec{\tau} = \langle 0.10(-20), -0.30(-20), 0.30(40) \rangle = \langle -2, 6, 12 \rangle \text{ N}\cdot\text{m}$.

Step 3 Its magnitude is $\sqrt{4 + 36 + 144} = \boxed{2\sqrt{46} \text{ N}\cdot\text{m}}$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

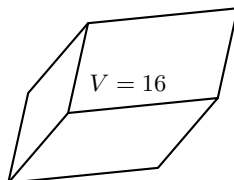
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 6

Find the volume of the parallelepiped determined by $\vec{a} = \langle 1, 2, 0 \rangle$, $\vec{b} = \langle 0, 1, 3 \rangle$, and $\vec{c} = \langle 2, -1, 1 \rangle$.

Question 6 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Volume is the absolute scalar triple product $|\vec{a} \cdot (\vec{b} \times \vec{c})|$.

Step 2 Compute $\vec{b} \times \vec{c} = \langle 4, 6, -2 \rangle$.

Step 3 Then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 4 + 12 = 16$, so $V = 16$ cubic units.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

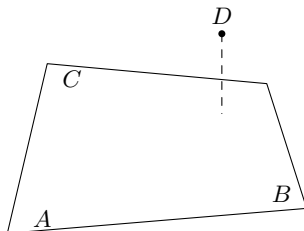
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 7

Determine whether the points $A = (1, 2, 0)$, $B = (3, 1, 4)$, $C = (-1, 5, 1)$, and $D = (5, -2, 7)$ are coplanar.

Question 7 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Use vectors from A: $\overrightarrow{AB} = \langle 2, -1, 4 \rangle$, $\overrightarrow{AC} = \langle -2, 3, 1 \rangle$, $\overrightarrow{AD} = \langle 4, -4, 7 \rangle$.

Step 2 Compute $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -13, -10, 4 \rangle$.

Step 3 Dot with $\overrightarrow{AD}$: $-52 + 40 + 28 = 16 \neq 0$. Therefore the points are not coplanar.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

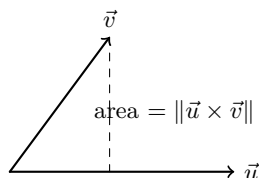
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 8

Prove the area identity $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ using the angle between the vectors.

Question 8 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 If either vector is zero, both sides are zero. Otherwise, let θ be the angle between $\vec{u}$ and $\vec{v}$. Then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$.

Step 2 Squaring and using $\sin^2\theta = 1 - \cos^2\theta$ gives $\|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta)$.

Step 3 Since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$, substitution yields the claimed identity. Proved

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

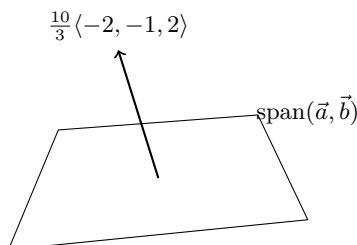
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 9

Find a vector of magnitude 10 perpendicular to the plane spanned by $\vec{a} = \langle 1, 0, 1 \rangle$ and $\vec{b} = \langle 0, 2, 1 \rangle$, with positive z -component.

Question 9 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 A normal direction is $\vec{a} \times \vec{b} = \langle -2, -1, 2 \rangle$.

Step 2 Its magnitude is 3, and its z -component is positive.

Step 3 Scale it to length 10: $\vec{n} = \frac{10}{3} \langle -2, -1, 2 \rangle$.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

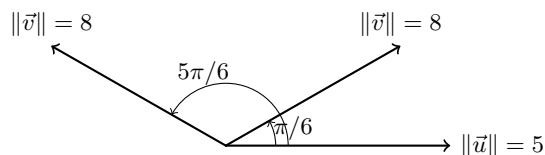
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 10

Vectors $\vec{u}$ and $\vec{v}$ have lengths 5 and 8, and $\|\vec{u} \times \vec{v}\| = 20$. Find every possible angle between them in $[0, \pi]$.

Question 10 – Solution

Strategy Keep the component order and signs organized when expanding the determinant. After computing the cross product, use a dot-product check or the relevant magnitude formula to interpret it.



Step 1 Use $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$: $20 = 40\sin\theta$.

Step 2 Thus $\sin\theta = 1/2$.

Step 3 On $[0, \pi]$, the two solutions are $\theta = \pi/6$ or $5\pi/6$. The cross-product magnitude cannot distinguish acute from obtuse orientation.

Verification The result follows from the defining vector formulas used above, and each component, magnitude, or scalar condition has been checked against the information in the question.

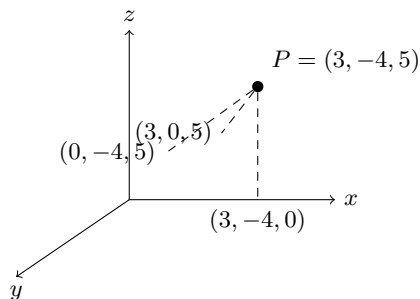
Key idea The cross product produces a vector perpendicular to both inputs. Its direction follows the right-hand rule, while its magnitude records the area of the spanned parallelogram.

Question 1

A point is $P = (3, -4, 5)$. Find its perpendicular distances to each coordinate plane and to each coordinate axis. Identify the closest plane and the closest axis.

Question 1 – Solution

Strategy A coordinate plane is reached by changing one coordinate to zero. For an axis, two coordinates must become zero, so the distance comes from those two perpendicular components.



Coordinate planes The distances to the yz -, xz -, and xy -planes are $|3| = 3$, $|-4| = 4$, and $|5| = 5$, respectively. Thus the closest plane is the yz -plane.

Coordinate axes For the x -axis, $d = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$. Similarly,

$$d(P, y\text{-axis}) = \sqrt{3^2 + 5^2} = \sqrt{34}, \quad d(P, z\text{-axis}) = \sqrt{3^2 + (-4)^2} = 5.$$

Hence the closest axis is the z -axis, at distance $\boxed{5}$.

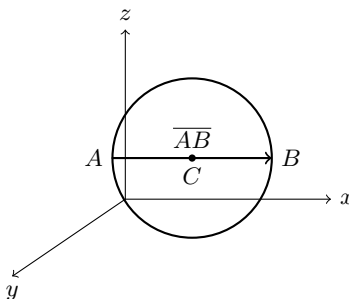
Verification Each plane distance uses exactly the omitted coordinate; each axis distance uses exactly the other two coordinates.

Question 2

The endpoints of a diameter of a sphere are $A = (-2, 1, 4)$ and $B = (6, -3, 0)$. Find the sphere's center, radius, and equation. Determine where the sphere intersects the z -axis.

Question 2 – Solution

Strategy The center is the midpoint of the diameter, and the radius is half its length. Points on the z -axis have the form $(0, 0, z)$.



Center and radius The midpoint is $C = (2, -1, 2)$. Also,

$$|AB| = \sqrt{8^2 + (-4)^2 + (-4)^2} = 4\sqrt{6}, \quad r = 2\sqrt{6}.$$

Therefore the sphere is

$$(x - 2)^2 + (y + 1)^2 + (z - 2)^2 = 24.$$

z -axis intersections Set $x = y = 0$: $4 + 1 + (z - 2)^2 = 24$, so $(z - 2)^2 = 19$. The intersection points are $(0, 0, 2 \pm \sqrt{19})$.

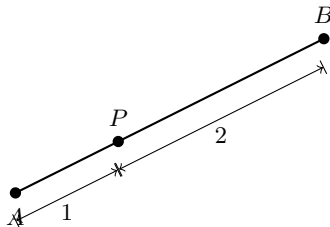
Verification Both A and B are $2\sqrt{6}$ units from C , and their midpoint is C .

Question 3

Point P divides the segment from $A = (-5, 2, 7)$ to $B = (4, -4, 1)$ internally in the ratio $AP : PB = 1 : 2$. Find P , then verify the ratio using distances.

Question 3 – Solution

Strategy Moving one-third of the way from A to B produces the required $1 : 2$ division.



Section formula Since $B - A = \langle 9, -6, -6 \rangle$,

$$P = A + \frac{1}{3}(B - A) = (-5, 2, 7) + (3, -2, -2) = \boxed{(-2, 0, 5)}.$$

Distance check We have $P - A = \langle 3, -2, -2 \rangle$ and $B - P = \langle 6, -4, -4 \rangle = 2(P - A)$. Therefore $AP = \sqrt{17}$ and $PB = 2\sqrt{17}$, giving $AP : PB = 1 : 2$.

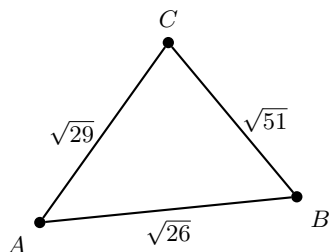
Key idea An internal ratio determines an affine weighted average; the distance check also confirms that P lies between the endpoints.

Question 4

The points $A = (1, 0, 2)$, $B = (4, 4, 1)$, and $C = (-1, 3, 6)$ form a triangle. Classify it as scalene, isosceles, or equilateral, and determine whether it is acute, right, or obtuse.

Question 4 – Solution

Strategy Compare squared side lengths; this avoids unnecessary square roots. The converse of the Pythagorean theorem classifies the largest angle.



Side lengths Direct calculation gives

$$AB^2 = 3^2 + 4^2 + (-1)^2 = 26, \quad AC^2 = (-2)^2 + 3^2 + 4^2 = 29, \quad BC^2 = (-5)^2 + (-1)^2 + 5^2 = 51.$$

The three lengths differ, so the triangle is scalene.

Angle classification The longest side is BC . Since $BC^2 = 51 < 26 + 29 = 55$, the angle opposite BC is acute. The other angles are necessarily smaller, so the triangle is scalene and acute.

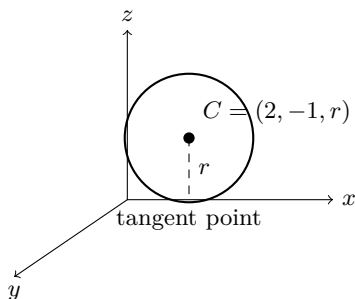
Verification The triangle inequality holds; for example, $\sqrt{26} + \sqrt{29} > \sqrt{51}$.

Question 5

A sphere has center $C = (2, -1, c)$, is tangent to the xy -plane, lies entirely above that plane, and passes through $P = (5, 3, 2)$. Find c and the sphere's equation.

Question 5 – Solution

Strategy Tangency to the xy -plane makes the radius equal to the center's height. The condition that the sphere lies above the plane forces $c > 0$.



Build the equation Here $r = c$. Because P is on the sphere,

$$(5 - 2)^2 + (3 + 1)^2 + (2 - c)^2 = c^2.$$

Thus $25 + (2 - c)^2 = c^2$, so $29 - 4c = 0$ and $c = 29/4$.

Result The radius is $29/4$, and

$$\boxed{(x - 2)^2 + (y + 1)^2 + \left(z - \frac{29}{4}\right)^2 = \left(\frac{29}{4}\right)^2}.$$

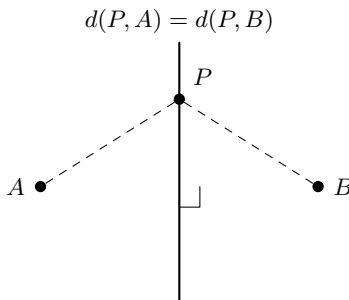
Verification The lowest z -coordinate is $c - r = 0$, so the sphere is tangent to the plane and otherwise lies above it.

Question 6

Describe geometrically the set of all points $P = (x, y, z)$ equidistant from $A = (2, -1, 3)$ and $B = (-4, 5, 1)$. Derive a Cartesian equation for the set and show that the midpoint of AB belongs to it.

Question 6 – Solution

Strategy Equate squared distances. Quadratic terms cancel, revealing the perpendicular-bisector surface.



Derivation Set

$$(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = (x + 4)^2 + (y - 5)^2 + (z - 1)^2.$$

After expanding and simplifying, $\boxed{3x - 3y + z + 7 = 0}$.

Interpretation This is the plane perpendicular to $\overrightarrow{AB} = \langle -6, 6, -2 \rangle$ through the midpoint $M = (-1, 2, 2)$. Indeed, $3(-1) - 3(2) + 2 + 7 = 0$.

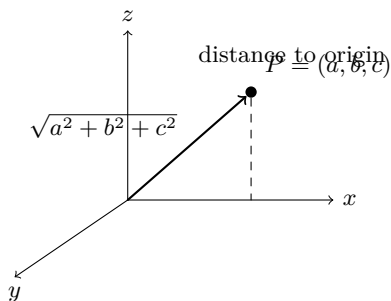
Verification Interchanging A and B does not change the locus, and every point in the plane has equal squared distance to the endpoints.

Question 7

A point $P = (a, b, c)$ is 7 units from the origin. Its distances to the yz - and xz -planes are 2 and 3, respectively, and P lies in the octant $x < 0, y > 0, z < 0$. Find P .

Question 7 – Solution

Strategy Distances to the yz - and xz -planes determine $|a|$ and $|b|$. The octant fixes signs, and the distance to the origin determines c .



Known coordinates We have $|a| = 2$ and $|b| = 3$, so $a = -2$ and $b = 3$.

Use the radial distance Since $a^2 + b^2 + c^2 = 49$,

$$4 + 9 + c^2 = 49 \implies c^2 = 36.$$

The octant condition $z < 0$ selects $c = -6$. Hence $\boxed{P = (-2, 3, -6)}$.

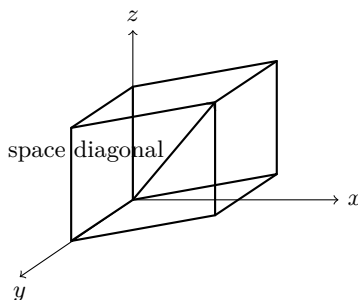
Verification Its distance from the origin is $\sqrt{4 + 9 + 36} = 7$, and its plane distances are 2 and 3 as required.

Question 8

A rectangular box has one vertex at the origin and the opposite vertex $P = (a, b, c)$ in the first octant. Its space diagonal has length 13, its xy -face diagonal has length 5, and the y -edge is longer than the x -edge. If the side lengths are positive integers, determine P .

Question 8 – Solution

Strategy Use the face diagonal to determine two integer side lengths, then use the space diagonal for the third.



Base dimensions The positive integer side lengths satisfying $a^2 + b^2 = 25$ are 3 and 4. Since the y -edge is longer than the x -edge, $a = 3$ and $b = 4$.

Third dimension The space diagonal gives

$$a^2 + b^2 + c^2 = 169 \implies 25 + c^2 = 169,$$

so $c = 12$. Therefore $P = (3, 4, 12)$.

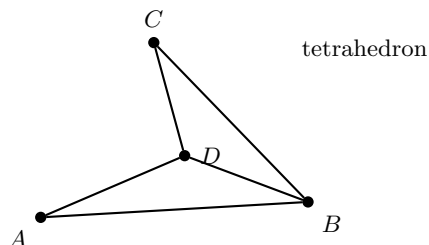
Verification The face diagonal is $\sqrt{3^2 + 4^2} = 5$ and the space diagonal is $\sqrt{3^2 + 4^2 + 12^2} = 13$.

Question 9

Four points are $A = (1, 1, 1)$, $B = (5, 1, 1)$, $C = (3, 1 + 2\sqrt{3}, 1)$, and $D = (3, 1 + 2/\sqrt{3}, 1 + 4\sqrt{2/3})$. Prove that they are the vertices of a regular tetrahedron and find its center.

Question 9 – Solution

Strategy A tetrahedron is regular if all six pairwise distances agree. Symmetry suggests that its center is the average of its four vertices.



Edge lengths Computation gives $AB = AC = BC = 4$. Also,

$$AD^2 = BD^2 = 4 + \frac{4}{3} + \frac{32}{3} = 16, \quad CD^2 = \frac{16}{3} + \frac{32}{3} = 16.$$

Thus all six edges have length 4, proving the tetrahedron is regular.

Center Averaging the vertices gives

$$G = \left(3, 1 + \frac{2\sqrt{3}}{3}, 1 + \sqrt{\frac{2}{3}} \right).$$

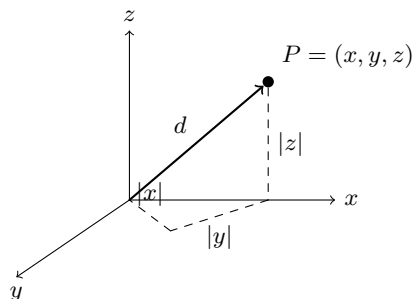
Verification By symmetry—or by direct substitution— G is equidistant from all four vertices.

Question 10

Derive the distance formula between $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ using the Pythagorean theorem twice. Then use it to find all values of t for which the distance from $(t, 2t, -1)$ to $(1, -2, 3)$ is 6.

Question 10 – Solution

Strategy First combine the x - and y -changes into a planar diagonal; then combine that diagonal with the z -change.



Derivation The planar displacement has length $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. A second right triangle therefore gives

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Application Set the squared distance equal to 36:

$$(t - 1)^2 + (2t + 2)^2 + (-4)^2 = 36.$$

This becomes $5t^2 + 6t - 15 = 0$, so

$$t = \frac{-3 \pm 2\sqrt{21}}{5}.$$

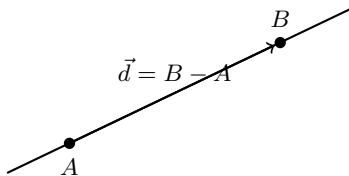
Verification Substitution into the quadratic recovers squared distance 36; both values are valid because distance is nonnegative.

Question 1

Find vector, parametric, and symmetric equations of the line through $A = (-2, 3, 1)$ and $B = (4, -1, 7)$. Determine the point on the line whose x -coordinate is 7.

Question 1 – Solution

Strategy Use $B - A$ as a direction vector and keep one common parameter in all coordinates.



Equations A direction is $\langle 6, -4, 6 \rangle = 2 \langle 3, -2, 3 \rangle$. Thus

$$\vec{r} = \langle -2, 3, 1 \rangle + t \langle 3, -2, 3 \rangle, \quad x = -2 + 3t, \quad y = 3 - 2t, \quad z = 1 + 3t,$$

and $\boxed{(x + 2)/3 = (y - 3)/(-2) = (z - 1)/3}$.

Requested point From $-2 + 3t = 7$, $t = 3$, giving $\boxed{(7, -3, 10)}$.

Verification $t = 0$ gives A and $t = 2$ gives B .

Question 2

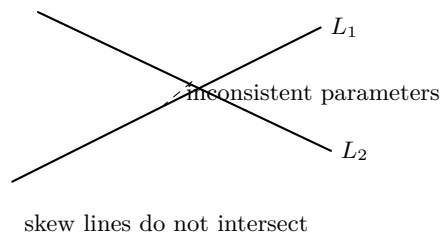
Determine whether the lines

$$L_1 : \vec{r} = \langle 1, 0, 2 \rangle + s \langle 2, -1, 3 \rangle, \quad L_2 : \vec{r} = \langle 3, -1, 1 \rangle + t \langle 1, 1, 2 \rangle$$

intersect, are parallel, or are skew. Justify the classification.

Question 2 – Solution

Strategy Equate coordinates. An intersection requires one pair (s, t) satisfying all three equations.



Coordinate equations The first two equations are $1 + 2s = 3 + t$ and $-s = -1 + t$. The second gives $t = 1 - s$; substitution into the first gives $s = 1$, hence $t = 0$.

Consistency check The third coordinates would be $2 + 3(1) = 5$ and $1 + 2(0) = 1$, so they disagree. The directions are not scalar multiples. Therefore the lines are skew.

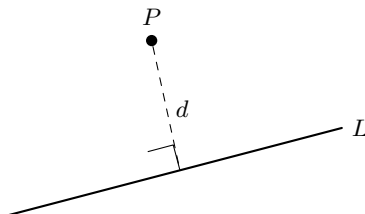
Key idea Solving only two coordinate equations can produce a false intersection; the remaining coordinate is essential.

Question 3

Find the distance from $P = (3, 1, -2)$ to the line $L : \vec{r} = \langle 1, -1, 2 \rangle + t \langle 2, 1, -2 \rangle$ and find the closest point on L .

Question 3 – Solution

Strategy At the closest point Q , the connector $P - Q$ is perpendicular to the line's direction.



Perpendicular condition Write $Q = (1 + 2t, -1 + t, 2 - 2t)$. Then

$$(P - Q) \cdot \langle 2, 1, -2 \rangle = 0.$$

Since $P - Q = \langle 2 - 2t, 2 - t, -4 + 2t \rangle$, the equation is $14 - 9t = 0$, so $t = 14/9$.

Closest point and distance Thus $Q = \boxed{(37/9, 5/9, -10/9)}$. Also $P - Q = \frac{1}{9} \langle -10, 4, -8 \rangle$, so

$$\boxed{d(P, L) = \frac{\sqrt{180}}{9} = \frac{2\sqrt{5}}{3}}.$$

Verification The dot product of $\langle -10, 4, -8 \rangle$ with $\langle 2, 1, -2 \rangle$ is zero.

Question 4

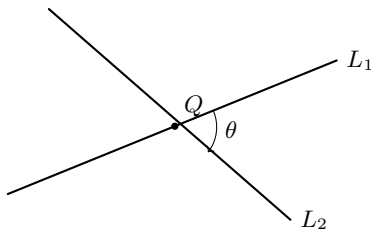
The lines

$$L_1 : \vec{r} = \langle 1, 2, -1 \rangle + s \langle 2, -1, 2 \rangle, \quad L_2 : \vec{r} = \langle 5, 0, 3 \rangle + t \langle 1, 2, -2 \rangle$$

intersect. Find their intersection and the acute angle between them.

Question 4 – Solution

Strategy Solve the coordinate equations for the intersection; use direction vectors for the angle.



Intersection Equating coordinates gives $1 + 2s = 5 + t$, $2 - s = 2t$, and $-1 + 2s = 3 - 2t$. The solution is $s = 2, t = 0$, so $Q = (5, 0, 3)$.

Angle With $\vec{d}_1 = \langle 2, -1, 2 \rangle$ and $\vec{d}_2 = \langle 1, 2, -2 \rangle$,

$$|\vec{d}_1 \cdot \vec{d}_2| = 4, \quad \|\vec{d}_1\| = 3, \quad \|\vec{d}_2\| = 3.$$

Hence $\theta = \cos^{-1}(4/9) \approx 63.6^\circ$.

Verification Substituting $s = 2$ and $t = 0$ gives the same point on both lines.

Question 5

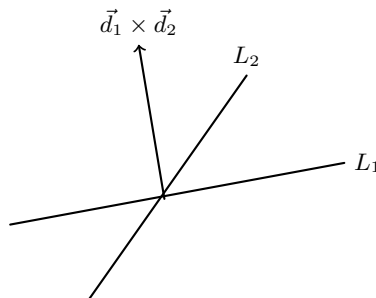
Find the line through $P = (2, -1, 4)$ whose direction is perpendicular to the directions of both lines

$$L_1 : \vec{r} = \langle 0, 1, 2 \rangle + s \langle 1, 2, -1 \rangle, \quad L_2 : \vec{r} = \langle 3, 0, -1 \rangle + t \langle 2, -1, 1 \rangle.$$

Explain why exactly one such direction exists.

Question 5 – Solution

Strategy A vector perpendicular to both directions is their cross product.



Direction Compute

$$\langle 1, 2, -1 \rangle \times \langle 2, -1, 1 \rangle = \langle 1, -3, -5 \rangle.$$

Therefore the requested line is

$$\boxed{\vec{r} = \langle 2, -1, 4 \rangle + u \langle 1, -3, -5 \rangle}.$$

Verification The new direction has dot product 0 with each original direction.

Uniqueness The original directions are not parallel, so their common perpendicular directions form the one-dimensional span of their nonzero cross product; reversing direction describes the same line.

Question 6

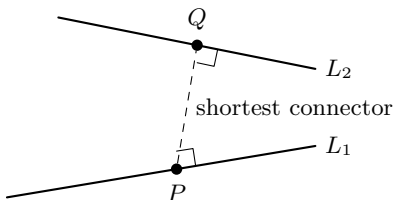
Find the shortest distance between the skew lines

$$L_1 : \vec{r} = \langle 0, 0, 0 \rangle + s \langle 1, 1, 0 \rangle, \quad L_2 : \vec{r} = \langle 0, 1, 1 \rangle + t \langle 1, -1, 0 \rangle,$$

and identify the pair of closest points.

Question 6 – Solution

Strategy The connector between closest points must be perpendicular to both line directions.



Closest points Let $P = (s, s, 0)$ and $Q = (t, 1 - t, 1)$. Requiring $(Q - P) \cdot \langle 1, 1, 0 \rangle = 0$ gives $1 - 2s = 0$, so $s = 1/2$. Requiring $(Q - P) \cdot \langle 1, -1, 0 \rangle = 0$ gives $2t - 1 = 0$, so $t = 1/2$.

Distance Thus $P = (1/2, 1/2, 0)$ and $Q = (1/2, 1/2, 1)$, with

$$\boxed{d(L_1, L_2) = |PQ| = 1}.$$

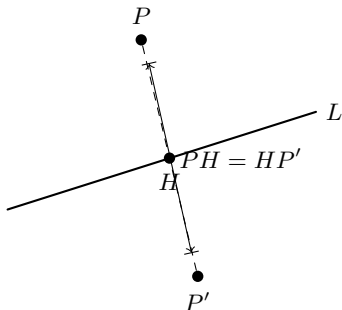
Verification $Q - P = \langle 0, 0, 1 \rangle$ is perpendicular to both direction vectors.

Question 7

Reflect the point $P = (4, 0, 1)$ across the line $L : \vec{r} = \langle 1, 1, 1 \rangle + t \langle 1, -1, 0 \rangle$. That is, find P' such that L perpendicularly bisects $\overline{PP'}$.

Question 7 – Solution

Strategy First find the perpendicular foot H on the line; then use $H = (P + P')/2$.



Projection point Write $H = (1 + t, 1 - t, 1)$. The condition $(P - H) \cdot \langle 1, -1, 0 \rangle = 0$ gives

$$\langle 3 - t, t - 1, 0 \rangle \cdot \langle 1, -1, 0 \rangle = 4 - 2t = 0,$$

so $t = 2$ and $H = (3, -1, 1)$.

Reflection Therefore $P' = 2H - P = \boxed{(2, -2, 1)}$.

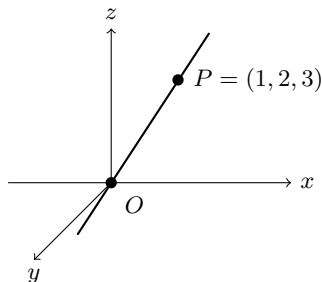
Verification The midpoint of P and P' is H , and $P - H = \langle 1, 1, 0 \rangle$ is perpendicular to the direction of L .

Question 8

Find the line that passes through $P = (1, 2, 3)$ and meets both the x -axis and the y -axis. Determine the two intercept points, or prove that no such line exists.

Question 8 – Solution

Strategy Distinct axis intercepts determine a line in the xy -plane, but both intercepts may be the origin.



Distinct intercepts A line through distinct points $X = (a, 0, 0)$ and $Y = (0, b, 0)$ lies in $z = 0$, so it cannot contain P .

Coincident intercepts The axes share the origin. The line through the origin and P is

$$\vec{r}(t) = t \langle 1, 2, 3 \rangle, \quad t \in \mathbb{R}.$$

It passes through P at $t = 1$ and meets each axis at $t = 0$. Thus both intercepts are $(0, 0, 0)$.

Key idea The distinct-point argument does not rule out coincident intercepts. This is the unique line satisfying the question.

Question 9

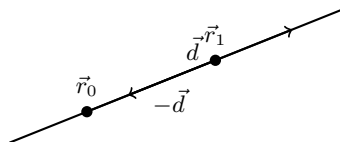
Show that

$$L_1 : \vec{r} = \langle 2, -1, 3 \rangle + t \langle 4, 2, -2 \rangle \quad \text{and} \quad L_2 : \vec{r} = \langle 8, 2, 0 \rangle + s \langle -2, -1, 1 \rangle$$

describe the same geometric line. Find the parameter conversion between t and s .

Question 9 – Solution

Strategy Check parallel directions and show that one base point lies on the other line.



different parameters, same point set

Same point set The second direction is $-\frac{1}{2}\langle 4, 2, -2 \rangle$. Also $\langle 8, 2, 0 \rangle = \langle 2, -1, 3 \rangle + \frac{3}{2}\langle 4, 2, -2 \rangle$, so the second base point lies on L_1 . Thus the lines coincide.

Parameter conversion Equating the representations gives

$$\vec{r} = \vec{r}_0 + t\vec{d} = \vec{r}_0 + \frac{3}{2}\vec{d} - \frac{s}{2}\vec{d},$$

so $\boxed{t = \frac{3}{2} - \frac{s}{2}}$, equivalently $s = 3 - 2t$.

Verification The conversion makes all three coordinates identical.

Question 10

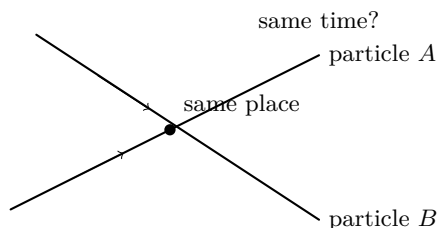
Two particles move according to

$$\vec{r}_A(t) = \langle 1, 0, 2 \rangle + t \langle 2, 1, -1 \rangle, \quad \vec{r}_B(t) = \langle 5, 2, 0 \rangle + t \langle -1, 0, 1 \rangle, \quad t \geq 0.$$

Do their paths intersect? Do the particles collide? Find their minimum separation for $t \geq 0$.

Question 10 – Solution

Strategy Path intersection permits different parameters; collision requires the same time. Minimum separation comes from minimizing the squared relative distance.



Paths Solving $\vec{r}_A(s) = \vec{r}_B(u)$ gives $s = 2, u = 0$, so the paths intersect at $(5, 2, 0)$. The particles do not collide there because they reach it at different times. Directly equating positions at the same t is inconsistent.

Minimum separation The relative position is

$$\vec{r}_A(t) - \vec{r}_B(t) = \langle -4 + 3t, -2 + t, 2 - 2t \rangle.$$

Thus $D^2(t) = 14t^2 - 36t + 24$, minimized at $t = 36/28 = 9/7$. The minimum squared distance is $6/7$, so

$$D_{\min} = \sqrt{6/7} = \frac{\sqrt{42}}{7}.$$

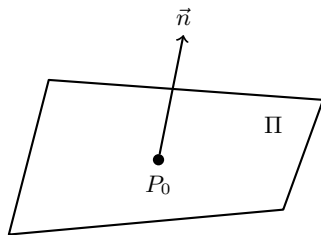
Verification The minimizing time is nonnegative, and the quadratic has positive leading coefficient.

Question 1

Find scalar and vector equations of the plane through $P_0 = (2, -1, 3)$ with normal vector $\vec{n} = \langle 4, 2, -1 \rangle$. Find its intercepts with the coordinate axes.

Question 1 – Solution

Strategy Use $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, then set two coordinates equal to zero for each intercept.



Equations The vector equation is $\langle 4, 2, -1 \rangle \cdot \langle x - 2, y + 1, z - 3 \rangle = 0$, which simplifies to $4x + 2y - z = 3$.

Intercepts The x -, y -, and z -intercepts are

$$(3/4, 0, 0), \quad (0, 3/2, 0), \quad (0, 0, -3).$$

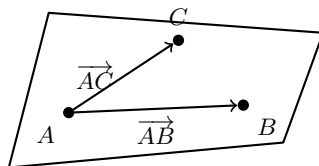
Verification Substituting P_0 gives $8 - 2 - 3 = 3$, and each intercept satisfies the scalar equation.

Question 2

Find an equation of the plane through $A = (1, 0, 2)$, $B = (3, -1, 4)$, and $C = (0, 2, 5)$. Determine whether the origin and $D = (2, 1, 3)$ lie on the same side of the plane.

Question 2 – Solution

Strategy Cross two independent displacement vectors to obtain a normal. Compare signs after substituting test points.



Plane Here $\overrightarrow{AB} = \langle 2, -1, 2 \rangle$ and $\overrightarrow{AC} = \langle -1, 2, 3 \rangle$, so $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -7, -8, 3 \rangle$. Thus

$$\boxed{7x + 8y - 3z = 1}.$$

Side test Write $F(x, y, z) = 7x + 8y - 3z - 1$. Then $F(0, 0, 0) = -1$, while $F(2, 1, 3) = 12$. The signs differ, so the points lie on opposite sides.

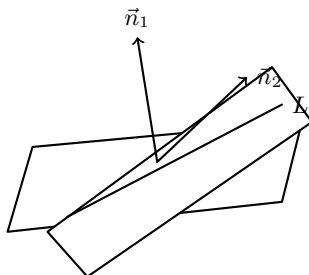
Verification Each of A, B, C satisfies the plane equation.

Question 3

Find the line of intersection of the planes $\Pi_1 : x + 2y - z = 3$ and $\Pi_2 : 2x - y + 2z = 1$. Also find the acute dihedral angle between the planes.

Question 3 – Solution

Strategy The intersection direction is perpendicular to both normals. Obtain one point by fixing a convenient coordinate.



Intersection line With $\vec{n}_1 = \langle 1, 2, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 2 \rangle$,

$$\vec{n}_1 \times \vec{n}_2 = \langle 3, -4, -5 \rangle.$$

Setting $z = 0$ gives $x + 2y = 3$, $2x - y = 1$, hence $(x, y) = (1, 1)$. Therefore

$$\vec{r} = \langle 1, 1, 0 \rangle + t \langle 3, -4, -5 \rangle.$$

Angle Since $\vec{n}_1 \cdot \vec{n}_2 = -2$, $\|\vec{n}_1\| = \sqrt{6}$, and $\|\vec{n}_2\| = 3$,

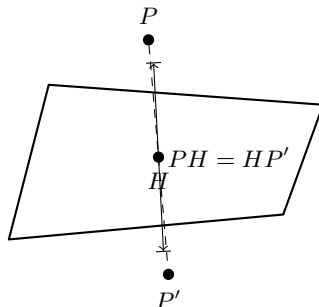
$$\theta = \cos^{-1} \left(\frac{2}{3\sqrt{6}} \right) \approx 74.2^\circ.$$

Question 4

Find the distance from $P = (4, -1, 5)$ to the plane $2x - y + 2z = 3$, the perpendicular foot H , and the reflection P' of P across the plane.

Question 4 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$.



Distance The signed numerator is $2(4) - (-1) + 2(5) - 3 = 16$, and $\|\vec{n}\| = 3$, so $d = 16/3$.

Foot For $H = P - \lambda\vec{n}$, impose the plane equation. Since $\lambda = 16/9$,

$$H = \left\langle 4/9, 7/9, 13/9 \right\rangle.$$

Reflection Since H is the midpoint of PP' , $P' = 2H - P = \left\langle -28/9, 23/9, -19/9 \right\rangle$.

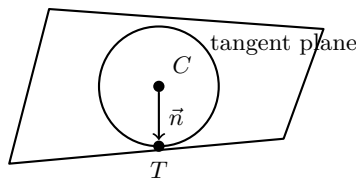
Verification H satisfies the plane equation, and $P - H$ is parallel to $\vec{n}$.

Question 5

The sphere $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 36$ contains the point $T = (3, 2, 7)$. Find the tangent plane at T and the line normal to the sphere there.

Question 5 – Solution

Strategy The radius from the center to T is normal to the tangent plane.



Normal direction The center is $C = (1, -2, 3)$ and $T - C = \langle 2, 4, 4 \rangle = 2\langle 1, 2, 2 \rangle$.

Tangent plane Using normal $\langle 1, 2, 2 \rangle$ through T ,

$$\boxed{(x - 3) + 2(y - 2) + 2(z - 7) = 0}, \quad \text{or} \quad \boxed{x + 2y + 2z = 21}.$$

Normal line $\boxed{\vec{r} = \langle 3, 2, 7 \rangle + t \langle 1, 2, 2 \rangle}.$

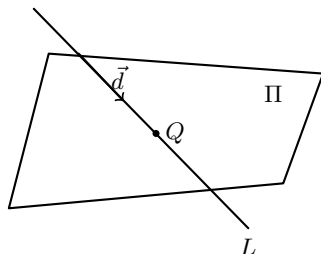
Verification $|T - C| = 6$, so T is indeed on the sphere.

Question 6

Classify the relationship between the line $L : \vec{r} = \langle 1, 2, 0 \rangle + t \langle 2, -1, 3 \rangle$ and the plane $\Pi : 3x + y - z = 5$. If they intersect, find the point and angle at which the line meets the plane.

Question 6 – Solution

Strategy Substitute the line into the plane. The line-plane angle is complementary to the angle between the direction and the normal.



Intersection Substitution gives $3(1 + 2t) + (2 - t) - 3t = 5$, so $2t = 0$ and $t = 0$. Thus $Q = \boxed{(1, 2, 0)}$.

Angle With $\vec{d} = \langle 2, -1, 3 \rangle$ and $\vec{n} = \langle 3, 1, -1 \rangle$,

$$\sin \alpha = \frac{|\vec{d} \cdot \vec{n}|}{\|\vec{d}\| \|\vec{n}\|} = \frac{2}{\sqrt{154}}.$$

Hence $\alpha = \sin^{-1}(2/\sqrt{154}) \approx 9.27^\circ$.

Verification Since $\vec{d} \cdot \vec{n} \neq 0$, the line is not parallel to the plane.

Question 7

Find the plane in the family containing the intersection line of $x + y + z = 2$ and $2x - y + 3z = 1$ that is perpendicular to the plane $x - 2y + z = 4$.

Question 7 – Solution

Strategy Apart from $2x - y + 3z = 1$ itself, every plane through the common line has equation $(x + y + z - 2) + \lambda(2x - y + 3z - 1) = 0$. The omitted plane has normal dot product $7 \neq 0$ with the given normal, so it cannot qualify. Choose λ so the family normal is perpendicular to the given normal.

pencil of planes



Normal condition The family normal is $\langle 1 + 2\lambda, 1 - \lambda, 1 + 3\lambda \rangle$. Dotting with $\langle 1, -2, 1 \rangle$ gives 7λ , so $\lambda = 0$.

Result The required plane is therefore simply

$$\boxed{x + y + z = 2}.$$

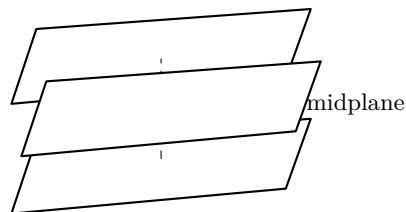
Verification Its normal $\langle 1, 1, 1 \rangle$ has dot product $1 - 2 + 1 = 0$ with $\langle 1, -2, 1 \rangle$, and it contains the stated intersection line.

Question 8

Find the distance between the parallel planes $2x - y + 2z = 3$ and $4x - 2y + 4z = 21$. Find the plane exactly halfway between them.

Question 8 – Solution

Strategy Normalize the equations to identical left sides before comparing constants.



Distance Divide the second equation by 2 to get $2x - y + 2z = 21/2$. Therefore

$$d = \frac{|21/2 - 3|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{15/2}{3} = \boxed{5/2}.$$

Midplane Average the normalized constants: $(3 + 21/2)/2 = 27/4$. Hence

$$\boxed{2x - y + 2z = 27/4}.$$

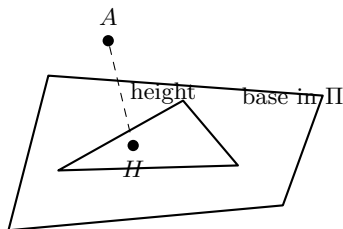
Verification Its constant differs from each neighboring constant by $15/4$, corresponding to distance $5/4$.

Question 9

A tetrahedron has base vertices $B = (0, 0, 0)$, $C = (4, 0, 0)$, $D = (1, 3, 0)$ and apex $A = (2, 1, 6)$. Find the base plane, the height, and the volume. Then find the plane parallel to the base that cuts the tetrahedron at half its height.

Question 9 – Solution

Strategy The base is horizontal. Volume is one third of base area times perpendicular height.



Base and height The base plane is $z = 0$. Its triangular area is $\frac{1}{2}(4)(3) = 6$, and the apex is 6 units from the plane.

Volume Thus $V = \frac{1}{3}(6)(6) = 12$.

Half-height plane The plane parallel to $z = 0$ halfway from the base to the apex is $z = 3$.

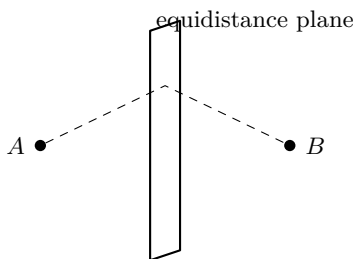
Verification Distance from $z = 3$ to both $z = 0$ and A 's level $z = 6$ is 3.

Question 10

Prove that the set of points equidistant from two distinct points A and B is a plane perpendicular to $\overline{AB}$ through its midpoint. Then find that plane for $A = (1, -2, 4)$ and $B = (5, 0, -2)$.

Question 10 – Solution

Strategy Expand $\|\vec{r} - \vec{a}\|^2 = \|\vec{r} - \vec{b}\|^2$ and cancel $\vec{r} \cdot \vec{r}$.



General proof Expansion gives

$$2(\vec{b} - \vec{a}) \cdot \vec{r} = \|\vec{b}\|^2 - \|\vec{a}\|^2.$$

This is a plane with normal $\vec{b} - \vec{a}$. Substituting $(\vec{a} + \vec{b})/2$ verifies that the midpoint lies on it.

Application Here $\vec{b} - \vec{a} = \langle 4, 2, -6 \rangle$ and the midpoint is $(3, -1, 1)$. Thus

$$4(x - 3) + 2(y + 1) - 6(z - 1) = 0,$$

or $\boxed{2x + y - 3z = 2}$.

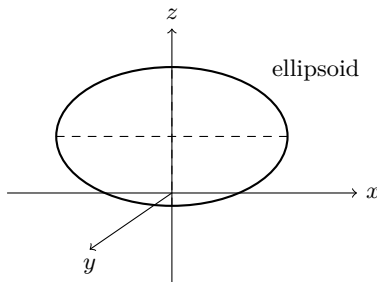
Verification Direct substitution of the midpoint gives $6 - 1 - 3 = 2$.

Question 1

Put $4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0$ in standard form. Classify the surface and give its center, semiaxes, and coordinate-direction extreme points.

Question 1 – Solution

Strategy Complete the square separately in x, y, z , then divide by the resulting constant.



Standard form Completing squares gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45,$$

so

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its axial extreme points are

$$(1 \pm 3\sqrt{5}/2, -3, 2), \quad (1, -3 \pm 3\sqrt{5}, 2), \quad (1, -3, 2 \pm \sqrt{5}).$$

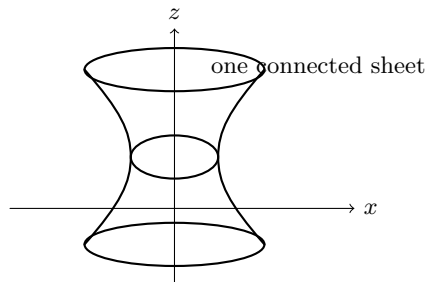
Verification Substitution of any listed endpoint makes exactly one normalized square equal to 1.

Question 2

Analyze $x^2 + 4y^2 - z^2 - 2x + 16y + 4z - 12 = 0$. Put it in standard form, classify it, identify its axis, and describe the traces $z = 2$ and $z = 2 + k$.

Question 2 – Solution

Strategy Complete squares and inspect the signs. Then substitute fixed z -levels.



Standard form Simplification gives

$$(x-1)^2 + 4(y+2)^2 - (z-2)^2 = 25, \quad \boxed{\frac{(x-1)^2}{25} + \frac{(y+2)^2}{25/4} - \frac{(z-2)^2}{25} = 1}.$$

It is a hyperboloid of one sheet centered at $(1, -2, 2)$ with axis parallel to z .

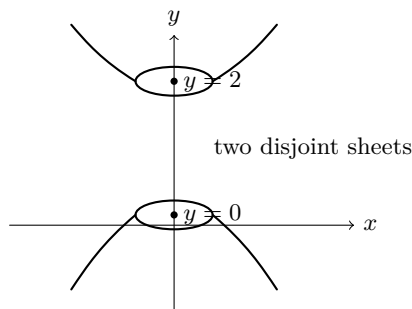
Traces At $z = 2$ the waist ellipse is $(x-1)^2 + 4(y+2)^2 = 25$. At $z = 2+k$ it becomes $(x-1)^2 + 4(y+2)^2 = 25 + k^2$, so ellipses expand as $|k|$ grows.

Question 3

For $9x^2 - 4y^2 + 9z^2 + 36x + 8y - 18z + 45 = 0$, find the standard form, classify the surface, and determine the vertices and which coordinate planes fail to intersect it.

Question 3 – Solution

Strategy Complete squares; a single negative term with right side -1 is more clearly read after multiplying by -1 .



Standard form Completing squares gives

$$9(x+2)^2 - 4(y-1)^2 + 9(z-1)^2 = -4,$$

hence

$$\frac{(y-1)^2}{1} - \frac{(x+2)^2}{4/9} - \frac{(z-1)^2}{4/9} = 1.$$

This is a hyperboloid of two sheets with axis parallel to y .

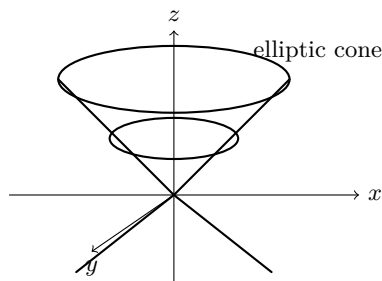
Vertices and coordinate traces The vertices are $(-2, 0, 1), (-2, 2, 1)$. All three coordinate planes intersect the surface: $y = 0$ contains the first vertex, while $x = 0$ and $z = 0$ give hyperbolas. The plane $y = 1$ does not intersect the surface, but it is not a coordinate plane.

Question 4

Classify the surface $4x^2 + y^2 - 9z^2 = 0$. Describe its traces in the planes $z = k$, $x = k$, and $y = k$, and find the lines obtained in the trace $y = 0$.

Question 4 – Solution

Strategy A homogeneous quadratic with two positive squares and one negative square describes a cone.



Classification The equation is $4x^2 + y^2 = 9z^2$, an elliptic cone with vertex at the origin and axis the z -axis.

Traces For $z = k \neq 0$, $4x^2 + y^2 = 9k^2$ is an ellipse; for $z = 0$ it is only the vertex. For fixed $x = k$ or $y = k$, the traces are hyperbolas when the fixed value is nonzero and intersecting lines when it is zero.

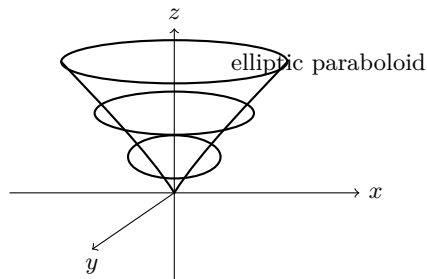
$y = 0$ Then $4x^2 = 9z^2$, so $z = \pm \frac{2}{3}x$.

Question 5

The surface $z = x^2 + 4y^2 - 2x + 8y + 7$ is cut by the plane $z = 10$. Identify the surface, locate its vertex, and find the semiaxes and area of the resulting planar region.

Question 5 – Solution

Strategy Complete squares in x, y ; a horizontal cut of an elliptic paraboloid is an ellipse.



Standard form We obtain

$$z = (x - 1)^2 + 4(y + 1)^2 + 2.$$

Thus the surface is an elliptic paraboloid opening upward with vertex $(1, -1, 2)$.

Cut at $z = 10$ The boundary satisfies $(x - 1)^2 + 4(y + 1)^2 = 8$, or

$$\frac{(x - 1)^2}{8} + \frac{(y + 1)^2}{2} = 1.$$

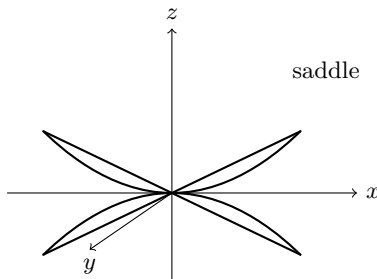
Its semiaxes are $2\sqrt{2}$ and $\sqrt{2}$, so the enclosed area is $\pi(2\sqrt{2})(\sqrt{2}) = 4\pi$.

Question 6

Analyze the hyperbolic paraboloid $z = x^2 - 4y^2$. Find the traces in $x = 0$, $y = 0$, $z = 0$, and $z = k \neq 0$. Explain how these traces establish the saddle shape.

Question 6 – Solution

Strategy Substitute each fixed coordinate and classify the resulting planar curve.



Vertical traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -4y^2$ opens downward. Their opposite concavities create the saddle.

Horizontal traces At $z = 0$, $x^2 - 4y^2 = 0$, so $x = \pm 2y$, two intersecting lines. For $k > 0$, $x^2 - 4y^2 = k$ is a hyperbola opening in the x -direction; for $k < 0$, it opens in the y -direction.

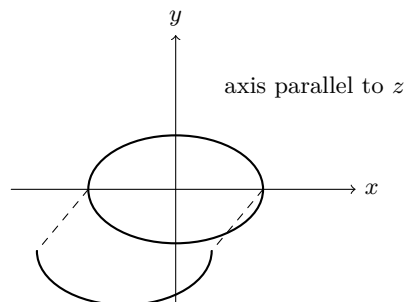
Conclusion The origin is neither a local maximum nor minimum because nearby traces rise in one direction and fall in the perpendicular direction.

Question 7

Classify $9x^2 + 4y^2 - 18x + 16y - 11 = 0$, describe its axis, and find the area of a cross-section perpendicular to that axis. Does the surface intersect the z -axis?

Question 7 – Solution

Strategy Because z is absent, any planar curve in xy is extruded parallel to the z -axis.



Standard form Completing squares yields

$$9(x-1)^2 + 4(y+2)^2 = 36, \quad \boxed{\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1}.$$

This is an elliptic cylinder parallel to the z -axis.

Cross-section Its semiaxes are 2 and 3, so each perpendicular cross-section has area $\boxed{6\pi}$.

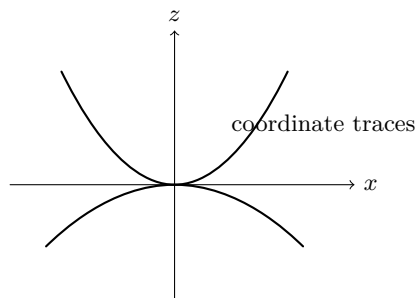
z -axis test At $x = y = 0$, the left side of the original equation is $-11 \neq 0$, so the cylinder does not meet the z -axis.

Question 8

A quadric has the traces $z = x^2$ in the plane $y = 0$ and $z = -y^2$ in the plane $x = 0$. Every horizontal trace $z = k \neq 0$ is a hyperbola. Determine a simplest equation for the surface and explain whether the trace information uniquely fixes all coefficients.

Question 8 – Solution

Strategy Combine the two vertical traces into a quadratic having the required restrictions.



Simplest model The equation

$$z = x^2 - y^2$$

has exactly the stated coordinate-plane traces and horizontal hyperbolas. It is a hyperbolic paraboloid.

Uniqueness issue The given coordinate traces force the coefficients of x^2 and y^2 in a diagonal model, but without an assumption excluding a mixed term, $z = x^2 - y^2 + cxy$ has the same traces in $x = 0$ and $y = 0$. Its horizontal traces remain hyperbolic when the quadratic form is indefinite.

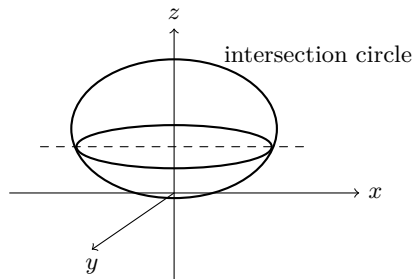
Conclusion The simplest axis-aligned equation is determined, but the data do not uniquely exclude rotated saddles.

Question 9

Find and classify the curve of intersection of the sphere $x^2 + y^2 + z^2 = 25$ with the plane $x + 2y + 2z = 9$. Find the curve's center and radius, and determine whether the plane is tangent, secant, or disjoint.

Question 9 – Solution

Strategy The circle center is the perpendicular projection of the sphere center onto the plane.



Plane distance The origin-to-plane distance is $d = 9/\sqrt{1+4+4} = 3$. Since $3 < 5$, the plane is secant.

Center and radius The foot from the origin lies along $\langle 1, 2, 2 \rangle$ and is

$$C = \frac{9}{9} \langle 1, 2, 2 \rangle = \boxed{(1, 2, 2)}.$$

By the right triangle from sphere center to circle edge, $r^2 = 5^2 - 3^2 = 16$, so $\boxed{r = 4}$.

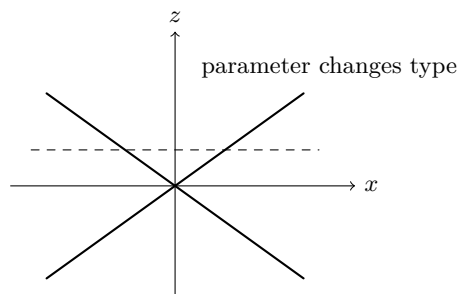
Classification The intersection is a circle in the given plane, centered at $(1, 2, 2)$ with radius 4.

Question 10

Classify the family $x^2 + y^2 - z^2 = \lambda$ for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Describe how the topology changes as λ passes through zero and give the vertices when they exist.

Question 10 – Solution

Strategy Divide by $|\lambda|$ and inspect which side has the single positive term.



$\lambda > 0$ The form $x^2/\lambda + y^2/\lambda - z^2/\lambda = 1$ is a hyperboloid of one sheet, connected, with axis z .

$\lambda = 0$ The equation $x^2 + y^2 = z^2$ is a double circular cone with vertex at the origin.

$\lambda < 0$ Writing $z^2/|\lambda| - x^2/|\lambda| - y^2/|\lambda| = 1$ gives a hyperboloid of two sheets. Its vertices are $(0, 0, \pm\sqrt{|\lambda|})$.

Transition At zero, the one-sheet surface pinches to a cone and separates into two components.

Question 1

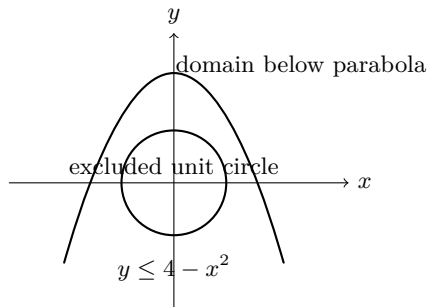
Find and sketch the domain of

$$f(x, y) = \frac{\sqrt{4 - x^2 - y^2}}{\ln(x^2 + y^2)}.$$

State which boundary pieces are included and whether the domain is bounded, open, closed, or neither.

Question 1 – Solution

Strategy Enforce the square-root, logarithm, and denominator restrictions simultaneously.



Restrictions We need $4 - x^2 - y \geq 0$, $x^2 + y^2 > 0$, and $\ln(x^2 + y^2) \neq 0$. Thus

$$D = \{(x, y) : y \leq 4 - x^2, x^2 + y^2 \neq 0, 1\}.$$

Boundary and topology The entire parabola $y = 4 - x^2$ is included; it does not meet the unit circle or the origin. The origin and entire unit circle are excluded. The region is unbounded downward. It is not open because the parabolic boundary is included, and not closed because excluded circle points are limit points.

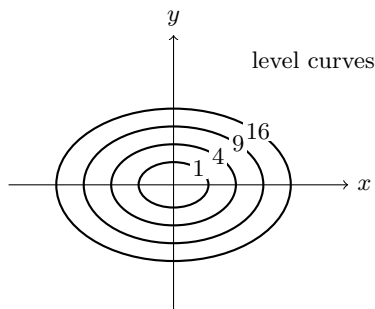
Check Every remaining point makes the radicand nonnegative and the denominator defined and nonzero.

Question 2

For $f(x, y) = 9 - 4(x - 1)^2 - (y + 2)^2$, find the range, global extrema, and level curves $f = c$. Describe how the curves change as c varies.

Question 2 – Solution

Strategy The subtracted squares are nonnegative; solve $f = c$ for the level sets.



Range and extrema The maximum is $\boxed{9}$ at $(1, -2)$. There is no minimum because $f \rightarrow -\infty$ as $\|(x, y)\| \rightarrow \infty$, so the range is $\boxed{(-\infty, 9]}$.

Level curves For $c < 9$,

$$4(x - 1)^2 + (y + 2)^2 = 9 - c,$$

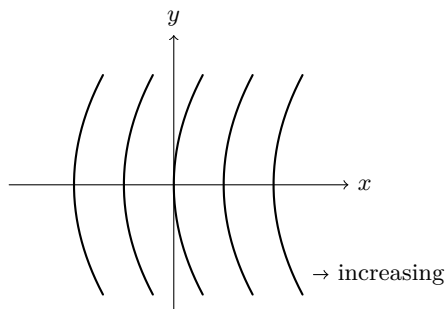
an ellipse centered at $(1, -2)$ with semiaxes $\sqrt{9 - c}/2$ and $\sqrt{9 - c}$. For $c = 9$ the level is one point; for $c > 9$ it is empty.

Question 3

A contour map consists of the curves $x - y^2 = c$ for all real c , labeled so values increase to the right. Reconstruct a simplest function producing the map. Find the level curve through $(3, -1)$ and decide where the function is positive.

Question 3 – Solution

Strategy Read the constant expression directly from the family of contours.



Function A simplest choice is $f(x, y) = x - y^2$. Any strictly increasing relabeling $g(f)$ would preserve the geometric contours but change their numerical labels.

Requested contour At $(3, -1)$, $c = 3 - 1 = 2$, so the curve is $x - y^2 = 2$.

Sign region The function is positive precisely when $x > y^2$, the region to the right of the parabola $x = y^2$.

Question 4

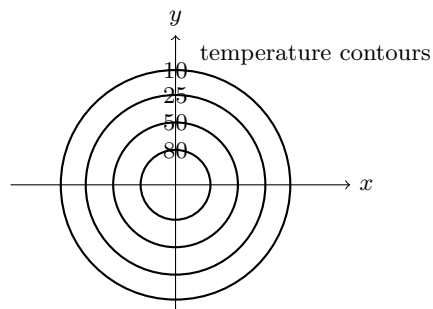
A thin plate has temperature

$$T(x, y) = 100e^{-(x^2+4y^2)/8} \quad (^\circ\text{C}).$$

Find the 50° isotherm, its semiaxes and enclosed area. Determine the warmest point and explain the contour spacing as temperature decreases.

Question 4 – Solution

Strategy Set T equal to a constant and take logarithms.



Isotherm Setting $T = 50$ gives $e^{-(x^2+4y^2)/8} = 1/2$, hence

$$x^2 + 4y^2 = 8 \ln 2.$$

The semiaxes are $\sqrt{8 \ln 2}$ and $\sqrt{2 \ln 2}$, so the area is $\boxed{4\pi \ln 2}$.

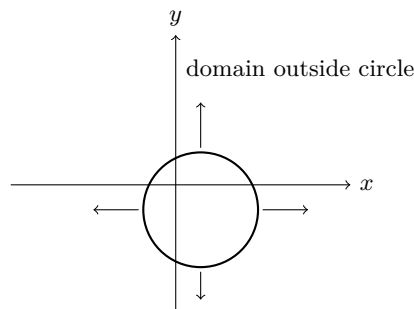
Maximum and spacing The maximum is 100° at $(0,0)$. For level $0 < c < 100$, $x^2 + 4y^2 = 8 \ln(100/c)$; lower temperatures correspond to larger ellipses, with nonlinear spacing caused by exponential decay.

Question 5

Let $f(u, v) = \sqrt{u - v}$ and $g(x, y) = (x^2 + y, x - y^2)$. Find $(f \circ g)(x, y)$ and its domain. Determine the boundary and whether the point $(1, 1)$ is interior, exterior, or on the boundary.

Question 5 – Solution

Strategy Substitute both components of g into f and require a nonnegative radicand.



Composition We obtain

$$(f \circ g)(x, y) = \sqrt{x^2 + y - x + y^2}.$$

Thus the domain is $x^2 - x + y^2 + y \geq 0$, or

$$\boxed{(x - 1/2)^2 + (y + 1/2)^2 \geq 1/2}.$$

It is the exterior of a disk, including its circular boundary.

Point test At $(1, 1)$ the original radicand is $2 > 0$, so the point is interior to the domain.

Question 6

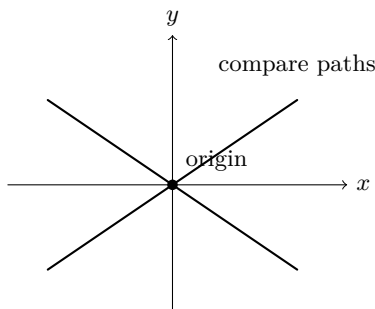
Investigate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Evaluate along at least three families of paths and explain why the evidence is decisive.

Question 6 – Solution

Strategy A multivariable limit must have the same value along every path.



Path tests Along $y = 0$, the expression equals 1; along $x = 0$, it equals -1 ; and along $y = mx$, it equals

$$\frac{1 - m^2}{1 + m^2},$$

which varies continuously with m .

Conclusion Since even two paths give different values, the limit does not exist.

Why decisive Path agreement can only suggest a limit, but one disagreement proves nonexistence because the definition requires uniform behavior over all approaches.

Question 7

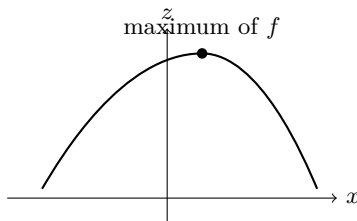
Without using derivatives, find the range and all global extrema of

$$f(x, y) = 7 - 2x^2 - 2xy - y^2 + 8x + 6y.$$

Hint: rewrite the quadratic as a sum of squares.

Question 7 – Solution

Strategy Complete squares in a way that respects the mixed term.



Rewrite Since $2x^2 + 2xy + y^2 = (x + y)^2 + x^2$,

$$f = 7 - (x + y)^2 - x^2 + 8x + 6y.$$

Set $u = x + y$. Then $y = u - x$ and $f = 7 - u^2 - x^2 + 2x + 6u = 17 - (x - 1)^2 - (u - 3)^2$.

Conclusion Therefore

$$\boxed{f(x, y) = 17 - (x - 1)^2 - (x + y - 3)^2}.$$

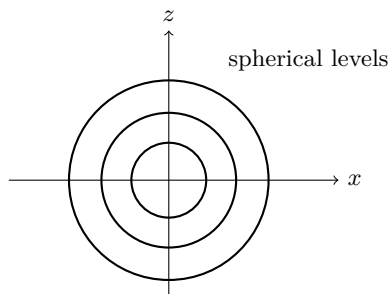
The global maximum is 17 at $x = 1$, $x + y = 3$, hence $(1, 2)$. There is no minimum, and the range is $(-\infty, 17]$.

Question 8

For $F(x, y, z) = \sqrt{(x-1)^2 + (y+2)^2 + (z-3)^2}$, interpret F geometrically. Describe every level surface $F = c$, determine the range, and find the level surface through $(4, 2, 3)$.

Question 8 – Solution

Strategy Recognize the three-dimensional distance formula.



Interpretation F is distance from $C = (1, -2, 3)$, so its range is $[0, \infty)$.

Levels For $c > 0$, $F = c$ is the sphere

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = c^2.$$

For $c = 0$ it is the single point C ; for $c < 0$ it is empty.

Requested surface The point $(4, 2, 3)$ is distance $\sqrt{3^2 + 4^2} = 5$ from C , so its level surface is the sphere of radius 5 .

Question 9

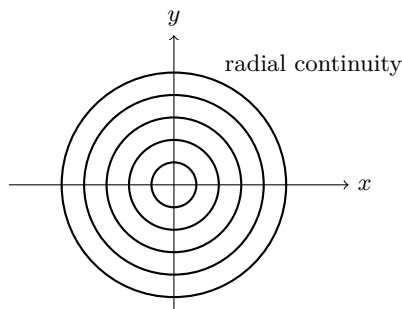
Define

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove directly that f is continuous at the origin, despite the direction-dependent factor in its polar form.

Question 9 – Solution

Strategy Bound the absolute value by a constant times the distance to the origin.



Estimate For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = |x| \frac{|x^2 - y^2|}{x^2 + y^2} \leq |x| \leq \sqrt{x^2 + y^2}.$$

Therefore $|f(x, y) - 0| \leq r$, where r is the distance to the origin.

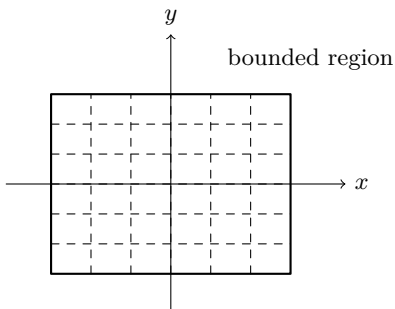
Epsilon argument Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $0 < r < \delta$, then $|f(x, y) - f(0, 0)| < \varepsilon$. Hence $\lim f = 0 = f(0, 0)$ and f is continuous.

Question 10

Let $f(x, y) = \frac{xy}{1 + x^2 + y^2}$ on the closed rectangle $R = [-2, 2] \times [-3, 3]$. Prove that $|f| < 1/2$ everywhere, and explain why f must attain a maximum and minimum on R even without finding them.

Question 10 – Solution

Strategy Compare $2|xy|$ with $x^2 + y^2$, then use continuity on a closed bounded domain.



Bound From $(|x| - |y|)^2 \geq 0$, $2|xy| \leq x^2 + y^2$. Hence

$$|f(x, y)| \leq \frac{x^2 + y^2}{2(1 + x^2 + y^2)} < \boxed{\frac{1}{2}}.$$

The strict inequality comes from the added 1 in the denominator.

Existence of extrema The denominator never vanishes, so f is continuous. The rectangle R is closed and bounded (compact). By the Extreme Value Theorem, f attains both an absolute maximum and absolute minimum on R .

Question 1

Find the domain of

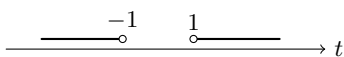
$$\vec{r}(t) = \left\langle \sqrt{t+2}, \ln(1-t), \frac{1}{t^2-1} \right\rangle.$$

Evaluate $\vec{r}(-1/2)$ and determine whether $\lim_{t \rightarrow -1^+} \vec{r}(t)$ exists in $\mathbb{R}^3$.

Question 1 – Solution

Strategy Intersect the domains of all three scalar components; vector limits exist exactly when every component limit is finite.

common component domain



Domain The restrictions are $t \geq -2$, $t < 1$, and $t \neq \pm 1$. Hence

$$\boxed{[-2, -1) \cup (-1, 1]}.$$

Also $\vec{r}(-1/2) = \langle \sqrt{3/2}, \ln(3/2), -4/3 \rangle$.

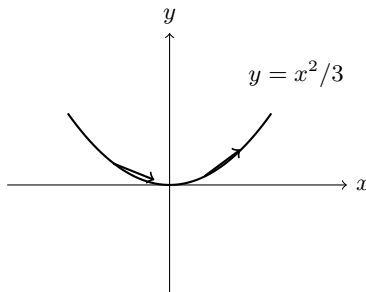
Limit As $t \rightarrow -1^+$, $1/(t^2 - 1) \rightarrow -\infty$, so no finite vector limit exists in $\mathbb{R}^3$.

Question 2

For $\vec{r}(t) = \langle 3t, t^2, 2 - t \rangle$, eliminate the parameter to describe the curve as the intersection of two surfaces. State its orientation as t increases and locate the point corresponding to $t = -1$.

Question 2 – Solution

Strategy Solve the simplest component for t and substitute into the others.



Elimination Since $x = 3t$, $t = x/3$. Therefore

$$\boxed{y = x^2/9, \quad z = 2 - x/3}.$$

The curve is the intersection of a parabolic cylinder and a plane.

Orientation and point As t increases, x increases and z decreases. At $t = -1$, the point is $\boxed{(-3, 1, 3)}$.

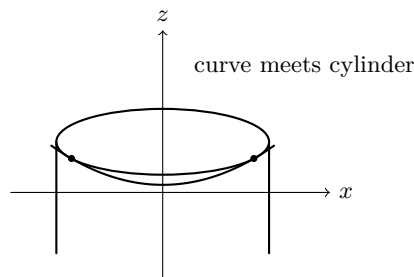
Verification Substituting these coordinates into both surface equations confirms membership.

Question 3

The curve $\vec{r}(t) = \langle t, t^2, 2t - 1 \rangle$ meets the cylinder $x^2 + y^2 = 2$. Find every intersection point and the corresponding parameter values.

Question 3 – Solution

Strategy Substitute the curve's coordinate functions into the surface equation.



Parameter equation We need $t^2 + t^4 = 2$. Let $u = t^2 \geq 0$; then $u^2 + u - 2 = 0$, so $(u - 1)(u + 2) = 0$. Thus $t^2 = 1$ and $t = \pm 1$.

Points At $t = 1$, $\vec{r} = (1, 1, 1)$; at $t = -1$, $\vec{r} = (-1, 1, -3)$. Hence the intersections are

$$\boxed{(1, 1, 1) \quad \text{and} \quad (-1, 1, -3)}.$$

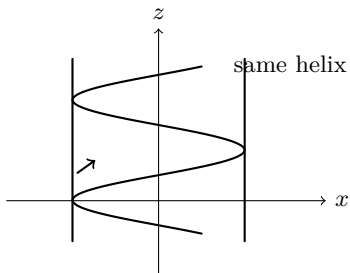
Verification Both points satisfy $x^2 + y^2 = 2$.

Question 4

Show that $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$ and $\vec{r}_2(u) = \langle \cos(2u), \sin(2u), 2u \rangle$ have the same image. Compare their orientations and speeds of traversal at matching points. Then give a reparameterization with the opposite orientation.

Question 4 – Solution

Strategy Relate parameters by matching all three components.



Same image Setting $t = 2u$ makes $\vec{r}_1(t) = \vec{r}_2(u)$, so both trace the same helix. As each parameter increases, z increases, so orientations agree.

Traversal rate A change Δu produces twice the angular and vertical change of the same Δt ; equivalently, $\vec{r}_2(u) = \vec{r}_1(2u)$ traverses the image twice as fast with respect to its parameter.

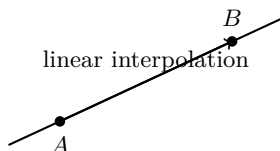
Reverse orientation For example, $\boxed{\vec{q}(s) = \langle \cos s, -\sin s, -s \rangle = \vec{r}_1(-s)}$.

Question 5

Construct a vector function that travels from $A = (-2, 1, 4)$ to $B = (3, -1, 0)$ during $0 \leq t \leq 1$, but does so nonuniformly: it starts and ends at rest while remaining on the line segment.

Question 5 – Solution

Strategy Use $\vec{r} = A + s(t)(B - A)$ with a scalar progress function satisfying $s(0) = 0$, $s(1) = 1$, and $s'(0) = s'(1) = 0$.



Progress function The cubic $s(t) = 3t^2 - 2t^3$ has the required endpoint values and zero endpoint derivatives. Since $0 \leq s(t) \leq 1$ on $[0, 1]$, the path stays on the segment.

Vector function Because $B - A = \langle 5, -2, -4 \rangle$,

$$\boxed{\vec{r}(t) = \langle -2, 1, 4 \rangle + (3t^2 - 2t^3) \langle 5, -2, -4 \rangle}, \quad 0 \leq t \leq 1.$$

Verification Substitution gives the correct endpoints, and $\vec{r}'(0) = \vec{r}'(1) = \vec{0}$.

Question 6

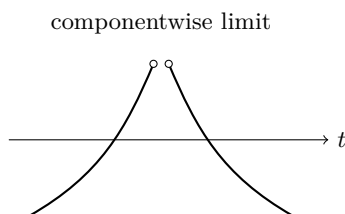
Find constants a, b, c so that

$$\lim_{t \rightarrow 0} \left\langle \frac{\sqrt{1+at}-1}{t}, \frac{\sin(bt)}{t}, \frac{e^{ct}-1}{t} \right\rangle = \langle 2, 3, -4 \rangle.$$

Justify the limit component by component.

Question 6 – Solution

Strategy A vector limit equals the target precisely when all component limits equal their corresponding target components.



First component Rationalizing gives

$$\frac{\sqrt{1+at}-1}{t} = \frac{a}{\sqrt{1+at}+1} \rightarrow \frac{a}{2},$$

so $a = 4$.

Other components Standard limits give $\sin(bt)/t \rightarrow b$ and $(e^{ct}-1)/t \rightarrow c$. Hence $a = 4, b = 3, c = -4$.

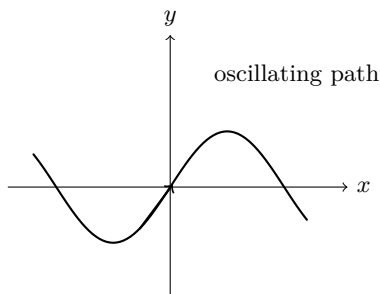
Verification The three component limits are respectively $2, 3, -4$.

Question 7

Analyze the planar vector function $\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ for $0 \leq t \leq 4\pi$. Find the points for $t = 0, \pi, 2\pi, 3\pi, 4\pi$, identify repeated heights, and describe the geometric motion.

Question 7 – Solution

Strategy Evaluate at multiples of π and use the periodicity of sine and cosine.



Key points The points are

$$(0, 0), (\pi, 2), (2\pi, 0), (3\pi, 2), (4\pi, 0).$$

The height oscillates between 0 and 2 while $x = t - \sin t$ increases from 0 to 4π .

Geometry The curve is two arches of a cycloid, moving left to right, with cusps at $(0, 0), (2\pi, 0), (4\pi, 0)$.

Verification Since $1 - \cos t \in [0, 2]$ and $x'(t) = 1 - \cos t \geq 0$, the description is consistent.

Question 8

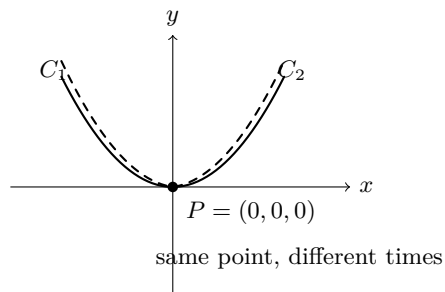
Curves C_1 and C_2 are given by

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \vec{r}_2(s) = \langle 1 - s, (1 - s)^2, s - 1 \rangle.$$

Determine all geometric intersection points. If particles use their displayed parameters as time, determine whether they collide.

Question 8 – Solution

Strategy Geometric intersection permits $t \neq s$; collision requires $t = s$ as well as equal positions.



Intersection From the first coordinate, set $u = 1 - s = t$. The second coordinates then agree automatically. The third coordinates require $t^3 = -u = -t$, so $t(t^2 + 1) = 0$. Thus $t = 0$ and $s = 1$, giving the sole intersection $\boxed{(0, 0, 0)}$.

Collision The required times differ, so particles using the displayed parameters do not collide.

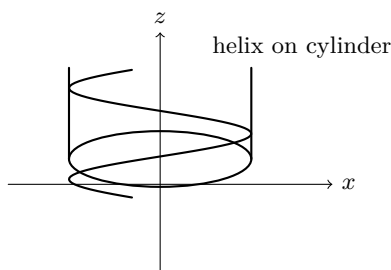
Check At $t = 0$, $\vec{r}_1 = \vec{0}$; at $s = 1$, $\vec{r}_2 = \vec{0}$.

Question 9

Construct a vector function for a helix that lies on the cylinder $(x - 2)^2 + (y + 1)^2 = 9$, begins at $(5, -1, 0)$ when $t = 0$, rises 4 units during each counterclockwise revolution, and makes two revolutions.

Question 9 – Solution

Strategy Use the cylinder center and radius for the circular components, then choose a linear height matching the pitch.



Construction Counterclockwise motion from the rightmost point is

$$x = 2 + 3 \cos t, \quad y = -1 + 3 \sin t.$$

One revolution is $\Delta t = 2\pi$ and should raise z by 4, so $z = (2/\pi)t$. Two revolutions require $0 \leq t \leq 4\pi$. Thus

$$\boxed{\vec{r}(t) = \langle 2 + 3 \cos t, -1 + 3 \sin t, 2t/\pi \rangle}, \quad 0 \leq t \leq 4\pi.$$

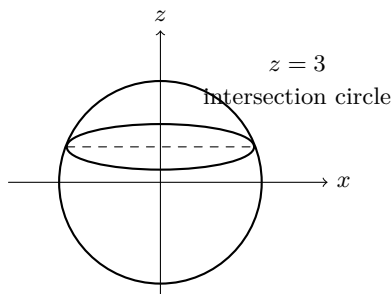
Verification The cylinder equation is identically 9, and the final height is 8.

Question 10

Find a parametrization of the curve where the sphere $x^2 + y^2 + z^2 = 25$ meets the plane $z = 3$. Give both orientations and explain why the parametrization covers the entire intersection exactly once on your chosen interval.

Question 10 – Solution

Strategy A horizontal plane cuts the sphere in a circle; determine its radius.



Circle Substituting $z = 3$ gives $x^2 + y^2 = 16$, a circle of radius 4 centered at $(0, 0, 3)$. A counterclockwise parametrization viewed from positive z is

$$\vec{r}(t) = \langle 4 \cos t, 4 \sin t, 3 \rangle, \quad 0 \leq t < 2\pi.$$

The opposite orientation is $\vec{q}(t) = \langle 4 \cos t, -4 \sin t, 3 \rangle$.

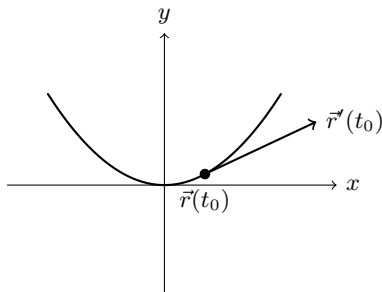
Coverage Sine and cosine generate every radius-4 circle point; the half-open interval prevents duplication of the initial point at 2π .

Question 1

For $\vec{r}(t) = \langle t^2 - 1, e^t, \sin(2t) \rangle$, find $\vec{r}'(t)$ and $\vec{r}''(t)$. Find the tangent line at $t = 0$ and a second-order approximation to $\vec{r}(0.1)$.

Question 1 – Solution

Strategy Differentiate componentwise; use the vector Taylor polynomial through second order.



Derivatives $\vec{r}'(t) = \langle 2t, e^t, 2\cos(2t) \rangle$ and $\vec{r}''(t) = \langle 2, e^t, -4\sin(2t) \rangle$. At $t = 0$, $\vec{r}(0) = \langle -1, 1, 0 \rangle$ and $\vec{r}'(0) = \langle 0, 1, 2 \rangle$.

Tangent line $\vec{L}(s) = \langle -1, 1, 0 \rangle + s \langle 0, 1, 2 \rangle$.

Approximation Since $\vec{r}''(0) = \langle 2, 1, 0 \rangle$,

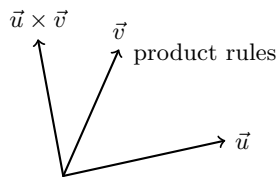
$$\vec{r}(0.1) \approx \vec{r}(0) + 0.1\vec{r}'(0) + \frac{0.1^2}{2}\vec{r}''(0) = \langle -0.99, 1.105, 0.2 \rangle.$$

Question 2

Let $\vec{u}(t) = \langle t, t^2, 1 \rangle$ and $\vec{v}(t) = \langle e^t, \cos t, t \rangle$. Compute $\frac{d}{dt}(\vec{u} \cdot \vec{v})$ and $\frac{d}{dt}(\vec{u} \times \vec{v})$ at $t = 0$ using product rules, then verify the dot-product result by expanding first.

Question 2 – Solution

Strategy Use $(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$ and $(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$.



Values at zero $\vec{u}(0) = \langle 0, 0, 1 \rangle$, $\vec{u}'(0) = \langle 1, 0, 0 \rangle$, $\vec{v}(0) = \langle 1, 1, 0 \rangle$, and $\vec{v}'(0) = \langle 1, 0, 1 \rangle$.

Results Thus

$$(\vec{u} \cdot \vec{v})'(0) = 1 + 1 = \boxed{2},$$

and

$$(\vec{u} \times \vec{v})'(0) = \langle 0, 0, 1 \rangle + \langle 0, 1, 0 \rangle = \boxed{\langle 0, 1, 1 \rangle}.$$

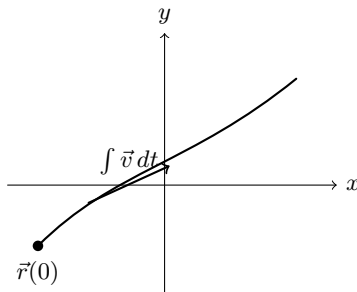
Expanding $\vec{u} \cdot \vec{v} = te^t + t^2 \cos t + t$ and differentiating also gives 2 at zero.

Question 3

A curve has velocity $\vec{r}'(t) = \langle 2t, 3t^2, e^{-t} \rangle$ and passes through $P = (1, -2, 4)$ at $t = 0$. Find $\vec{r}(t)$ and the net displacement from $t = 0$ to $t = 2$.

Question 3 – Solution

Strategy Integrate each velocity component and use the initial condition to determine the constant vector.



Antiderivative $\vec{r}(t) = \langle t^2, t^3, -e^{-t} \rangle + \vec{C}$. Since $\vec{r}(0) = \langle 0, 0, -1 \rangle + \vec{C} = P$, $\vec{C} = \langle 1, -2, 5 \rangle$. Therefore

$$\boxed{\vec{r}(t) = \langle t^2 + 1, t^3 - 2, 5 - e^{-t} \rangle}.$$

Displacement $\vec{r}(2) - \vec{r}(0) = \int_0^2 \vec{r}'(t) dt = \boxed{\langle 4, 8, 1 - e^{-2} \rangle}$.

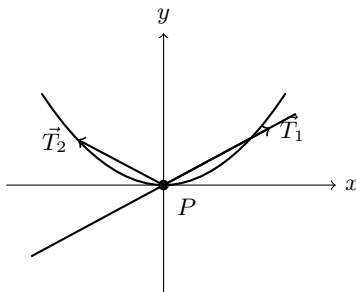
Verification Differentiation recovers the given velocity and substitution gives P .

Question 4

The curves $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(s) = \langle s^2, s, s^3 \rangle$ meet at the origin and at $(1, 1, 1)$. Find the smaller angle between their tangent lines at each intersection.

Question 4 – Solution

Strategy Differentiate each parametrization and evaluate using its own parameter at the shared point.



At the origin Both use parameter 0. The tangent vectors are $\vec{r}_1'(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}_2'(0) = \langle 0, 1, 0 \rangle$, so the angle is $\boxed{90^\circ}$.

At $(1, 1, 1)$ Both use parameter 1. The tangents are $\langle 1, 2, 3 \rangle$ and $\langle 2, 1, 3 \rangle$. Their dot product is 13 and both magnitudes are $\sqrt{14}$, so

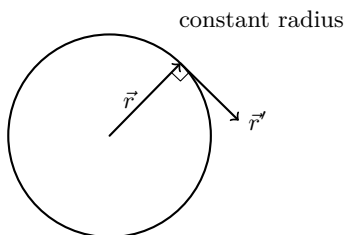
$$\boxed{\theta = \cos^{-1}(13/14) \approx 21.8^\circ}.$$

Question 5

Suppose a differentiable vector function satisfies $\|\vec{r}(t)\| = 5$ for every t . Prove that $\vec{r}(t) \perp \vec{r}'(t)$. Apply the result to $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle$ and verify directly.

Question 5 – Solution

Strategy Differentiate the squared magnitude rather than the square root.



Proof Since $\vec{r} \cdot \vec{r} = 25$, differentiation gives $2\vec{r} \cdot \vec{r}' = 0$. Thus $\boxed{\vec{r} \cdot \vec{r}' = 0}$, proving orthogonality wherever $\vec{r}'$ is nonzero.

Application Here $\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 0 \rangle$ and

$$\vec{r} \cdot \vec{r}' = -9 \cos t \sin t + 9 \sin t \cos t = 0.$$

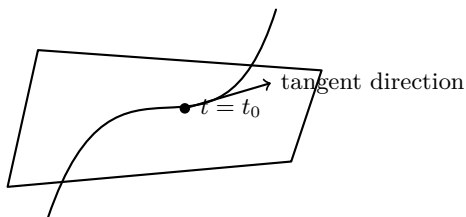
Also $\|\vec{r}'(t)\| = \sqrt{9 + 16} = 5$, as required.

Question 6

For $\vec{r}(t) = \langle t, t^2, t^3 - 3t \rangle$, find all points where the tangent line is parallel to the plane $2x - y + z = 4$. Determine whether any tangent line is perpendicular to that plane.

Question 6 – Solution

Strategy A tangent is parallel to a plane when its direction is orthogonal to the plane normal; it is perpendicular when its direction is parallel to the normal.



Parallel condition $\vec{r}'(t) = \langle 1, 2t, 3t^2 - 3 \rangle$ and $\vec{n} = \langle 2, -1, 1 \rangle$. Thus

$$\vec{r}'(t) \cdot \vec{n} = 3t^2 - 2t - 1 = (3t + 1)(t - 1).$$

So $t = 1$ or $t = -1/3$, giving points $\boxed{(1, 1, -2)}$ and $\boxed{(-1/3, 1/9, 26/27)}$.

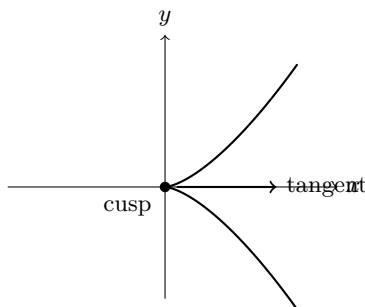
Perpendicular condition Parallelism of $\vec{r}'(t)$ with $\vec{n}$ would force scale $1/2$ from the first component, then $2t = -1/2$, but the third component at $t = -1/4$ is not $1/2$. Hence none exist.

Question 7

Analyze the smoothness and regularity of $\vec{r}(t) = \langle t^2, t^3 \rangle$ near $t = 0$. Does the curve have a tangent line at the origin even though $\vec{r}'(0) = \vec{0}$? Justify using a reparameterization or limiting secant directions.

Question 7 – Solution

Strategy A zero derivative makes the given parametrization nonregular, but the geometric curve may still have a tangent.



Regularity $\vec{r}'(t) = \langle 2t, 3t^2 \rangle$, so $\vec{r}'(0) = \vec{0}$ and this polynomial parametrization is smooth but not regular at 0. Eliminating t gives $y^2 = x^3$, a semicubical cusp.

Tangent direction For $t \neq 0$, the secant direction from the origin is proportional to $\langle t^2, t^3 \rangle = t^2 \langle 1, t \rangle$, which approaches $\langle 1, 0 \rangle$ from both sides. Thus the geometric tangent line exists and is $y = 0$.

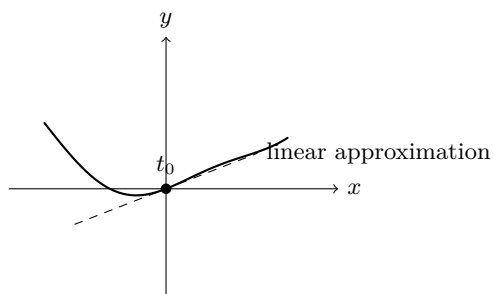
Distinction The curve has a tangent at the cusp, but no nonzero velocity there.

Question 8

For $\vec{r}(t) = \langle \cos t, \sin t, e^t \rangle$, form the second-order Taylor approximation about $t = 0$. Use it to approximate $\vec{r}(0.05)$ and state the order of the neglected error.

Question 8 – Solution

Strategy Use $\vec{r}(0) + t\vec{r}'(0) + \frac{1}{2}t^2\vec{r}''(0)$.



Data $\vec{r}(0) = \langle 1, 0, 1 \rangle$, $\vec{r}'(0) = \langle 0, 1, 1 \rangle$, and $\vec{r}''(0) = \langle -1, 0, 1 \rangle$. Hence

$$\vec{P}_2(t) = \langle 1 - t^2/2, t, 1 + t + t^2/2 \rangle.$$

At $t = 0.05$, $\vec{r}(0.05) \approx \langle 0.99875, 0.05, 1.05125 \rangle$.

Error Because the components have bounded third derivatives near zero, the remainder is $O(t^3)$, here on the scale of 0.05^3 .

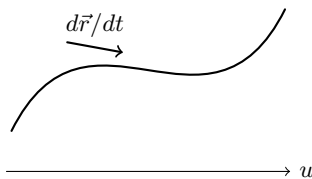
Question 9

Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ and reparameterize by $t = u^3 - u$. Find $d\vec{r}/du$. Identify every u where the reparameterized curve momentarily stops, and explain why the geometric helix itself remains smooth there.

Question 9 – Solution

Strategy Apply the vector chain rule and distinguish the image curve from the chosen parameter clock.

speed changes, image fixed



Chain rule With $t = g(u) = u^3 - u$,

$$\frac{d\vec{r}}{du} = \vec{r}'(g(u))g'(u) = (3u^2 - 1) \langle -\sin(u^3 - u), \cos(u^3 - u), 1 \rangle.$$

It vanishes when $3u^2 - 1 = 0$, so $\boxed{u = \pm 1/\sqrt{3}}$.

Interpretation The original derivative $\langle -\sin t, \cos t, 1 \rangle$ never vanishes. The stops occur because $dt/du = 0$; they reflect a nonregular reparameterization, not a cusp in the helix.

Question 10

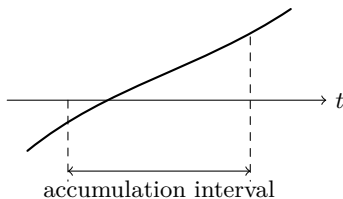
Evaluate

$$\int_0^{\pi} \langle t \sin t, t \cos t, \sin^2 t \rangle dt.$$

Then interpret the result as displacement if the integrand is velocity, and find the average velocity on $[0, \pi]$.

Question 10 – Solution

Strategy Integrate componentwise, using integration by parts for the first two components.



Integral We have

$$\int_0^\pi t \sin t \, dt = \pi, \quad \int_0^\pi t \cos t \, dt = -2, \quad \int_0^\pi \sin^2 t \, dt = \frac{\pi}{2}.$$

Therefore the displacement is $\langle \pi, -2, \pi/2 \rangle$.

Average velocity Divide displacement by the elapsed time π :

$$\vec{v}_{\text{avg}} = \langle 1, -2/\pi, 1/2 \rangle.$$

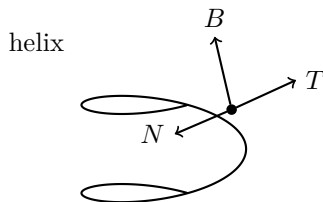
Verification Multiplying the average velocity by π recovers the net displacement.

Question 1

For the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, find the unit tangent $\mathbf{T}$, principal unit normal $\mathbf{N}$, and binormal $\mathbf{B}$ for all t . Evaluate the frame at $t = 0$.

Question 1 – Solution

Strategy Normalize $\vec{r}'$, normalize $\mathbf{T}'$, then use $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.



Frame Since $\|\vec{r}'\| = \sqrt{2}$,

$$\mathbf{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle,$$

and

$$\mathbf{B} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle.$$

At $t = 0$ $\mathbf{T} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle, \mathbf{N} = \langle -1, 0, 0 \rangle, \mathbf{B} = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle.$

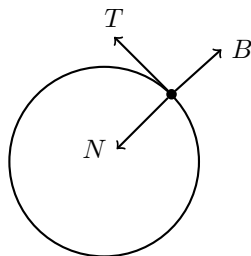
Verification The three vectors are unit, mutually orthogonal, and $\mathbf{T} \times \mathbf{N} = \mathbf{B}$.

Question 2

For the circle $\vec{r}(t) = \langle a \cos t, a \sin t, h \rangle$ with $a > 0$, compute $\mathbf{T}, \mathbf{N}, \mathbf{B}$. Explain why $\mathbf{N}$ is independent of a and why $\mathbf{B}$ is constant.

Question 2 – Solution

Strategy The parameter changes angular position while a changes only the circle's scale.



Computation $\vec{r}' = a \langle -\sin t, \cos t, 0 \rangle$, so

$$\mathbf{T} = \langle -\sin t, \cos t, 0 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

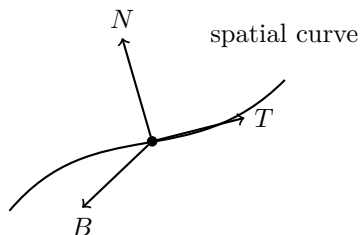
Interpretation The principal normal points radially inward regardless of radius. The curve lies in the fixed horizontal plane $z = h$, so the right-handed binormal is its constant upward normal.

Question 3

Find the Frenet frame for $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Use the cross-product method for $\mathbf{B}$ and verify the orientation.

Question 3 – Solution

Strategy Use $\mathbf{T} = \vec{r}' / \|\vec{r}'\|$, $\mathbf{B} = (\vec{r}' \times \vec{r}'') / \|\vec{r}' \times \vec{r}''\|$, and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.



Derivatives At $t = 1$, $\vec{r}' = \langle 1, 2, 3 \rangle$ and $\vec{r}'' = \langle 0, 2, 6 \rangle$. Thus

$$\mathbf{T} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle, \quad \mathbf{B} = \frac{1}{\sqrt{19}} \langle 3, -3, 1 \rangle.$$

Normal Therefore

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle.$$

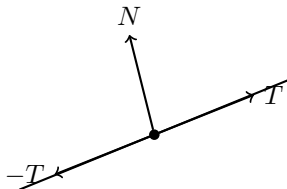
Verification Direct cross multiplication gives $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ and all norms equal 1.

Question 4

A regular curve $\vec{r}(t)$ with nonzero curvature is reparameterized with the opposite orientation by $\vec{q}(u) = \vec{r}(\phi(u))$, where $\phi'(u) < 0$. Determine how $\mathbf{T}, \mathbf{N}, \mathbf{B}$ change. Give a geometric explanation.

Question 4 – Solution

Strategy Apply the chain rule and track the sign introduced by the decreasing parameter.



Tangent Since $\vec{q}' = \vec{r}'(\phi)\phi'$, normalization gives $\mathbf{T}_q = -\mathbf{T}_r$.

Normal Differentiating $\mathbf{T}_q = -\mathbf{T}_r(\phi)$ with respect to u introduces another negative factor ϕ' , so its direction agrees with $\mathbf{N}_r$. Hence $\mathbf{N}_q = \mathbf{N}_r$.

Binormal $\mathbf{B}_q = \mathbf{T}_q \times \mathbf{N}_q = -\mathbf{B}_r$. Thus

$$(\mathbf{T}, \mathbf{N}, \mathbf{B}) \mapsto (-\mathbf{T}, \mathbf{N}, -\mathbf{B}).$$

The bending side is geometric, while forward and right-handed directions reverse.

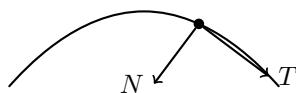
Question 5

Let $\vec{r}(t) = \langle t, t^2, 0 \rangle$. Find $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for all t . Explain why the binormal is constant and identify the direction in which the curve bends.

Question 5 – Solution

Strategy Normalize $\vec{r}'$, differentiate $\mathbf{T}$, and use the planar geometry as a check.

B perpendicular to curve's plane



Tangent and normal With $w = \sqrt{1 + 4t^2}$,

$$\mathbf{T} = \frac{1}{w} \langle 1, 2t, 0 \rangle, \quad \mathbf{N} = \frac{1}{w} \langle -2t, 1, 0 \rangle.$$

Consequently $\boxed{\mathbf{B} = \langle 0, 0, 1 \rangle}$.

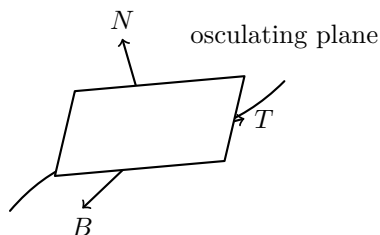
Interpretation The curve is contained in the xy -plane, whose oriented unit normal is constant. The normal has positive y -component for all t , so the upward-opening parabola bends toward its concave side.

Question 6

For $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at $t = 0$, find equations of the normal, osculating, and rectifying planes. State which Frenet vectors span each plane.

Question 6 – Solution

Strategy Use the frame from the helix and choose the Frenet vector perpendicular to each named plane.



Frame and point At $t = 0$, $P = (1, 0, 0)$, $\mathbf{T} \parallel \langle 0, 1, 1 \rangle$, $\mathbf{N} = \langle -1, 0, 0 \rangle$, and $\mathbf{B} \parallel \langle 0, -1, 1 \rangle$.

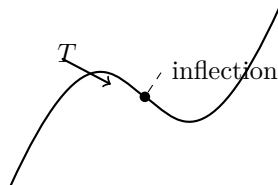
Planes The normal plane is perpendicular to $\mathbf{T}$: $y + z = 0$, and is spanned by $\mathbf{N}, \mathbf{B}$. The osculating plane is perpendicular to $\mathbf{B}$: $-y + z = 0$, spanned by $\mathbf{T}, \mathbf{N}$. The rectifying plane is perpendicular to $\mathbf{N}$: $x = 1$, spanned by $\mathbf{T}, \mathbf{B}$.

Question 7

For $\vec{r}(t) = \langle t, t^3, 0 \rangle$, determine where the Frenet principal normal is undefined. Does the curve still have a tangent there? Analyze the limiting normals on the two sides.

Question 7 – Solution

Strategy Compute $\mathbf{T}'$ and distinguish loss of curvature from loss of tangent direction.



Tangent $\vec{r}'(t) = \langle 1, 3t^2, 0 \rangle$ never vanishes, so the curve is regular and has tangent $\mathbf{T}(0) = \langle 1, 0, 0 \rangle$.

Normal failure Differentiating the normalized tangent shows $\mathbf{T}'(0) = \vec{0}$, so $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$ is undefined at 0. For $t > 0$, the normal tends to $\langle 0, 1, 0 \rangle$; for $t < 0$, it tends to $\langle 0, -1, 0 \rangle$.

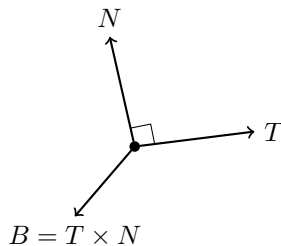
Interpretation The origin is an inflection point: curvature vanishes and the bending side changes.

Question 8

Assume $\mathbf{T}$ is a differentiable unit tangent and $\mathbf{T}' \neq \vec{0}$. Prove that the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a right-handed orthonormal basis.

Question 8 – Solution

Strategy Differentiate $\mathbf{T} \cdot \mathbf{T} = 1$, normalize $\mathbf{T}'$, and use cross-product identities.



Orthogonality Differentiation gives $2\mathbf{T} \cdot \mathbf{T}' = 0$. Since $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$, $\mathbf{T} \perp \mathbf{N}$, and both are unit.

Binormal Define $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The cross product of perpendicular unit vectors is unit and perpendicular to both.

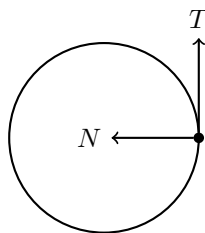
Orientation By definition $\mathbf{T} \times \mathbf{N} = \mathbf{B}$, so the ordered frame is right-handed. Hence its Gram matrix is the identity and its orientation determinant is $+1$.

Question 9

The circle $x^2 + y^2 = 4$ is parametrized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3), 0 \rangle$ for $t > 0$. Find its Frenet frame and explain which parts depend on the varying speed.

Question 9 – Solution

Strategy Let $\theta = t^3$ and separate angular position from the positive rate $d\theta/dt$.



same frame, nonuniform speed

Frame Because $3t^2 > 0$,

$$\mathbf{T} = \langle -\sin(t^3), \cos(t^3), 0 \rangle, \quad \mathbf{N} = \langle -\cos(t^3), -\sin(t^3), 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

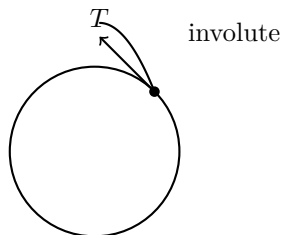
Dependence The frame at a geometric point depends on position and orientation, not on how fast that point is reached. The speed $\|\dot{\mathbf{r}}\| = 6t^2$ varies, but normalization removes it. At $t = 0$ the parametrization stops, so the usual Frenet construction from velocity is not regular there.

Question 10

For the unit circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$, define its involute by $\vec{q}(t) = \vec{r}(t) - t\mathbf{T}(t)$ for $t > 0$. Find $\mathbf{T}_q(t)$ and show that it is parallel to the circle's principal normal at the corresponding point.

Question 10 – Solution

Strategy Differentiate the involute and use the circle's simple Frenet derivatives.



Circle frame $\mathbf{T}(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{T}'(t) = \langle -\cos t, -\sin t \rangle = \mathbf{N}(t)$.

Involute derivative Differentiating,

$$\vec{q}' = \vec{r}' - \mathbf{T} - t\mathbf{T}' = \mathbf{T} - \mathbf{T} - t\mathbf{N} = -t\mathbf{N}.$$

Since $t > 0$, $\|\vec{q}'\| = t$, so $\boxed{\mathbf{T}_q = -\mathbf{N} = \langle \cos t, \sin t \rangle}$.

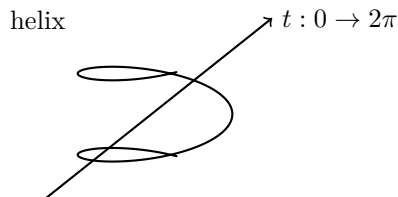
Interpretation The taut string leaves the circle tangentially, while the involute's tangent points along the outward radial direction.

Question 1

A particle follows the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ for $0 \leq t \leq 2\pi$. Find the exact distance traveled. Then explain why the vertical rise alone is not the distance.

Question 1 – Solution

Strategy Distance traveled is the integral of speed, not the net change in any one coordinate.



Speed

$$\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\vec{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Arc length Therefore

$$L = \int_0^{2\pi} 5 \, dt = 10\pi.$$

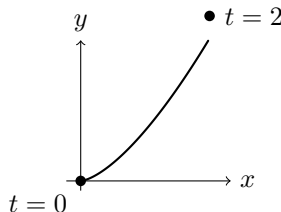
Interpretation The vertical rise is $4(2\pi) = 8\pi$, but the particle simultaneously travels around the cylinder. The velocity has perpendicular horizontal and vertical components of magnitudes 3 and 4, producing total speed 5.

Question 2

Find the exact arc length of $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3 \rangle$ on $0 \leq t \leq 2$. Show the algebraic step that makes the integral elementary.

Question 2 – Solution

Strategy Differentiate first, factor the speed, and use $u = 1 + t^2$.



Speed

$$\vec{r}'(t) = \langle 2t, 2t^2 \rangle, \quad \|\vec{r}'(t)\| = 2t\sqrt{1+t^2},$$

where $t \geq 0$ allows $\sqrt{t^2} = t$.

Integration

$$L = \int_0^2 2t\sqrt{1+t^2} dt = \frac{2}{3} \left[(1+t^2)^{3/2} \right]_0^2 = \boxed{\frac{2}{3}(5\sqrt{5} - 1)}.$$

Check Both coordinate functions increase. The straight-line distance between the endpoints is

$$\sqrt{4^2 + (16/3)^2} = \frac{20}{3}.$$

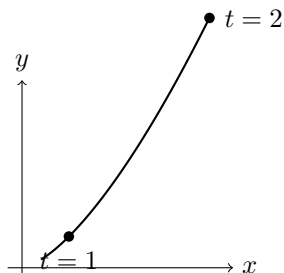
The computed curve length is larger, as it must be.

Question 3

For $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, 0 \rangle$, find the arc length from the point corresponding to $t = 1$ to the point corresponding to $t = 2$. Also find the chord length and compare the two quantities.

Question 3 – Solution

Strategy Use the same speed structure as in Question 2, but retain the nonzero lower limit.



Arc length

$$L = \int_1^2 2t\sqrt{1+t^2} dt = \boxed{\frac{2}{3}(5\sqrt{5} - 2\sqrt{2})}.$$

Chord The endpoints are $P = (1, 2/3, 0)$ and $Q = (4, 16/3, 0)$, so

$$\|Q - P\| = \sqrt{3^2 + \left(\frac{14}{3}\right)^2} = \boxed{\frac{\sqrt{277}}{3}}.$$

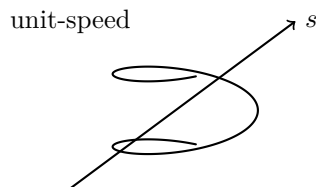
Comparison Numerically, $L \approx 5.568$ while the chord is about 5.548. The curve length is slightly larger, as required; equality would occur only for a straight segment traversed without reversal.

Question 4

Let $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ with starting point at $t = 0$. Find its arc-length function $s(t)$ and reparameterize the helix by arc length. Verify directly that the new parameterization has unit speed.

Question 4 – Solution

Strategy Integrate speed from the stated base point, solve for t , and substitute.



Arc-length coordinate From Question 1, $\|\vec{r}'(t)\| = 5$. Hence $s(t) = \int_0^t 5 \, du = 5t$, so $t = s/5$.

Reparameterization

$$\boxed{\vec{R}(s) = \langle 3 \cos(s/5), 3 \sin(s/5), 4s/5 \rangle}, \quad s \geq 0.$$

Verification

$$\vec{R}'(s) = \left\langle -\frac{3}{5} \sin(s/5), \frac{3}{5} \cos(s/5), \frac{4}{5} \right\rangle, \quad \|\vec{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

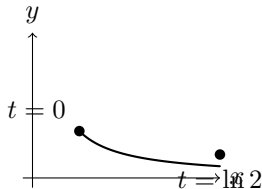
Thus an increase of one unit in s represents exactly one unit of distance along the helix.

Question 5

Evaluate exactly the length of $\vec{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$ for $0 \leq t \leq \ln 2$. The square root in the speed appears complicated; identify its hidden perfect square.

Question 5 – Solution

Strategy Expand the squared speed and recognize $(e^t + e^{-t})^2$.



Speed

$$\|\vec{r}'(t)\|^2 = e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2.$$

Both terms are positive, so $\|\vec{r}'(t)\| = e^t + e^{-t}$.

Length

$$L = \int_0^{\ln 2} (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^{\ln 2} = \boxed{\frac{3}{2}}.$$

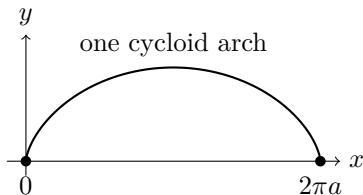
Insight Simplifying the norm before choosing a substitution is decisive here; treating the radical as generic would hide the elementary antiderivative.

Question 6

One arch of a cycloid is parameterized by $\vec{r}(t) = \langle a(t - \sin t), a(1 - \cos t) \rangle$, $0 \leq t \leq 2\pi$, where $a > 0$. Prove that its length is $8a$. Explain the absolute-value issue in simplifying the speed.

Question 6 – Solution

Strategy Use half-angle identities and determine the sign of $\sin(t/2)$ on the interval.



Speed

$$\vec{r}' = a \langle 1 - \cos t, \sin t \rangle, \quad \|\vec{r}'\| = a\sqrt{2 - 2\cos t} = 2a \left| \sin \frac{t}{2} \right|.$$

For $0 \leq t \leq 2\pi$, $\sin(t/2) \geq 0$, so the absolute value may be removed.

Length

$$L = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = \left[-4a \cos \frac{t}{2} \right]_0^{2\pi} = \boxed{8a}.$$

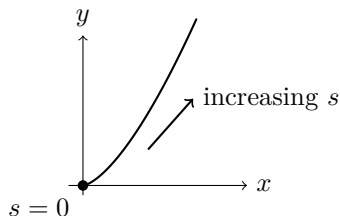
Endpoint note The velocity vanishes at the two cusps, but the speed is continuous and integrable, so the arch still has a well-defined finite length.

Question 7

For $t \geq 0$, let $\vec{r}(t) = \langle \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$. Construct the arc-length coordinate measured from $t = 0$, invert it explicitly, and give a unit-speed parameterization $\vec{R}(s)$.

Question 7 – Solution

Strategy The factor $t \geq 0$ makes the speed easy to integrate; then solve algebraically for t^2 .



Arc-length function

$$\|\vec{r}'(t)\| = \|\langle t, t^2 \rangle\| = t\sqrt{1+t^2}, \quad s(t) = \frac{1}{3}((1+t^2)^{3/2} - 1).$$

Inverse From $(1+t^2)^{3/2} = 3s+1$,

$$t(s) = \sqrt{(3s+1)^{2/3} - 1}.$$

Unit-speed form With $u = \sqrt{(3s+1)^{2/3} - 1}$,

$$\boxed{\vec{R}(s) = \left\langle \frac{1}{2}u^2, \frac{1}{3}u^3 \right\rangle}, \quad s \geq 0.$$

Because s was defined by integrating speed and is strictly increasing for $t > 0$, the chain rule gives $\|d\vec{R}/ds\| = 1$ there; the right-hand unit-speed limit also exists at $s = 0$.

Question 8

Suppose $\vec{q}(u) = \vec{r}(\phi(u))$ traces the same regular curve, where ϕ is continuously differentiable and strictly monotone. Prove that the arc length is unchanged by this reparameterization, including the case in which orientation is reversed.

Question 8 – Solution

Strategy Apply the chain rule to speed, then use substitution while carefully retaining the absolute value.

same trace, opposite orientations



Transformed speed

$$\vec{q}'(u) = \vec{r}'(\phi(u))\phi'(u), \quad \|\vec{q}'(u)\| = \|\vec{r}'(\phi(u))\| |\phi'(u)|.$$

Increasing case If $\phi' \geq 0$ and $\phi(\alpha) = a, \phi(\beta) = b$, substitution gives

$$\int_{\alpha}^{\beta} \|\vec{q}'(u)\| du = \int_a^b \|\vec{r}'(t)\| dt.$$

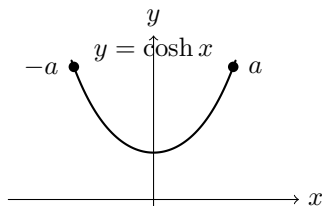
Decreasing case If $\phi' \leq 0$, then $\phi(\alpha) = b$ and $\phi(\beta) = a$. Since $|\phi'| = -\phi'$, reversing the substituted limits again produces $\int_a^b \|\vec{r}'(t)\| dt$. Thus length depends on the traced curve, not its orientation or timing.

Question 9

The catenary $\vec{r}(t) = \langle t, \cosh t \rangle$ is restricted to $-a \leq t \leq a$, where $a > 0$. Find its exact length in terms of a . Then determine the value of a that makes the length equal to 6.

Question 9 – Solution

Strategy Use the identity $1 + \sinh^2 t = \cosh^2 t$ and the positivity of $\cosh t$.



Length

$$\|\vec{r}'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t, \quad L = \int_{-a}^a \cosh t \, dt = 2 \sinh a.$$

Therefore $L = 2 \sinh a$.

Prescribed length Setting $2 \sinh a = 6$ gives $\sinh a = 3$, hence

$$a = \operatorname{arsinh}(3) = \ln(3 + \sqrt{10}).$$

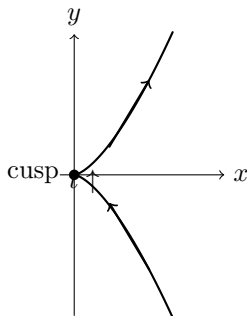
Symmetry The even speed makes the two halves have equal length; equivalently, $L = 2 \int_0^a \cosh t \, dt$.

Question 10

Consider $\vec{r}(t) = \langle t^2, t^3 \rangle$ on $-1 \leq t \leq 1$. The parameterization is not regular at $t = 0$. Determine the total arc length exactly and explain why one must split the integral or preserve an absolute value.

Question 10 – Solution

Strategy The speed contains $\sqrt{t^2} = |t|$; use symmetry across the cusp.



Speed

$$\vec{r}'(t) = \langle 2t, 3t^2 \rangle, \quad \|\vec{r}'(t)\| = |t|\sqrt{4 + 9t^2}.$$

Length

$$L = 2 \int_0^1 t\sqrt{4 + 9t^2} dt = \frac{2}{27} \left[(4 + 9t^2)^{3/2} \right]_0^1 = \boxed{\frac{2}{27}(13\sqrt{13} - 8)}.$$

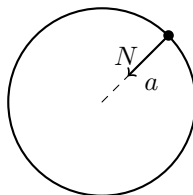
Why care about $|t|$? Replacing $|t|$ by t on the full symmetric interval would make the integrand odd and falsely give zero. Although velocity vanishes at the cusp, each half is smooth and their nonnegative lengths add.

Question 1

Starting from $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$, where $a > 0$, derive the curvature of a circle. Interpret the result in terms of turning sharpness and radius of curvature.

Question 1 – Solution

Strategy Use $\kappa = \|\vec{r}' \times \vec{r}''\| / \|\vec{r}'\|^3$, viewing planar vectors as three-dimensional vectors with zero third component.



Calculation $\vec{r}' = \langle -a \sin t, a \cos t, 0 \rangle$ and $\vec{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle$. Thus $\|\vec{r}'\| = a$ and $\|\vec{r}' \times \vec{r}''\| = a^2$. Therefore

$$\boxed{\kappa = \frac{a^2}{a^3} = \frac{1}{a}}, \quad \boxed{\rho = \frac{1}{\kappa} = a}.$$

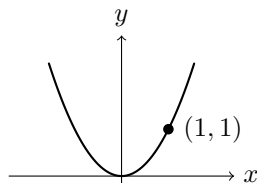
Interpretation Small circles turn more sharply and have larger curvature; the radius of curvature ρ is exactly the circle's geometric radius.

Question 2

Find the curvature of the parabola $\vec{r}(t) = \langle t, t^2 \rangle$ for general t , and evaluate it at the point $(1, 1)$. Where is the curvature largest?

Question 2 – Solution

Strategy Apply the planar curvature formula and then analyze its denominator.



Calculation With $x' = 1$, $y' = 2t$, $x'' = 0$, and $y'' = 2$,

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \boxed{\frac{2}{(1 + 4t^2)^{3/2}}}.$$

At $t = 1$, $\boxed{\kappa = 2/(5\sqrt{5})}$.

Maximum The numerator is constant and the denominator is minimized at $t = 0$. Hence the maximum curvature is $\kappa(0) = 2$, at the vertex.

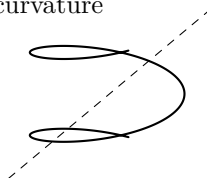
Question 3

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, compute the curvature for all t . Explain geometrically why a constant answer is reasonable.

Question 3 – Solution

Strategy Compute speed and the magnitude of $\vec{r}' \times \vec{r}''$ separately.

constant curvature



Calculation

$$\vec{r}' = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{r}'' = \langle -3 \cos t, -3 \sin t, 0 \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 12 \sin t, -12 \cos t, 9 \rangle.$$

Thus $\|\vec{r}'\| = 5$ and $\|\vec{r}' \times \vec{r}''\| = 15$, giving

$$\kappa = \frac{15}{5^3} = \frac{3}{25}.$$

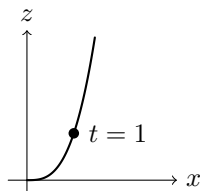
Geometry Every point of a circular helix has the same local shape: rotation about and translation along its axis carry one point to another, so its bending is uniform.

Question 4

Compute the curvature of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Show the cross product and simplify the exact value.

Question 4 – Solution

Strategy Use the parameter-independent cross-product formula.



Derivatives

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}'' = \langle 0, 2, 6t \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 6t^2, -6t, 2 \rangle.$$

At $t = 1$, its magnitude is $2\sqrt{19}$, while $\|\vec{r}'(1)\| = \sqrt{14}$. Therefore

$$\kappa(1) = \frac{2\sqrt{19}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}}.$$

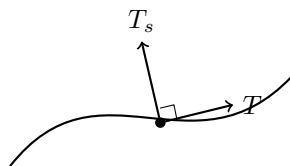
Check The cross product is nonzero, so the tangent is genuinely changing direction and the curvature is positive.

Question 5

Let $\vec{r}(s)$ be a twice-differentiable unit-speed curve. Prove that $\kappa(s) = \|\vec{r}''(s)\|$ and that the acceleration $\vec{r}''(s)$ is perpendicular to the tangent.

Question 5 – Solution

Strategy Differentiate the unit-speed identity $\vec{r}' \cdot \vec{r}' = 1$.



Orthogonality Because $\|\vec{r}'\| = 1$,

$$\vec{r}' \cdot \vec{r}' = 1 \implies 2\vec{r}' \cdot \vec{r}'' = 0.$$

Thus $\vec{r}'' \perp \vec{r}' = \mathbf{T}$.

Curvature For an arc-length parameter, $\mathbf{T} = d\vec{r}/ds$. By definition,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\|.$$

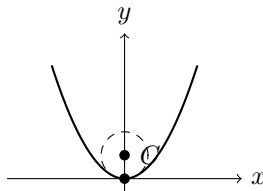
Interpretation Unit speed removes tangential acceleration. Any remaining second derivative measures only the rate at which the direction turns.

Question 6

Find the osculating circle of the parabola $y = x^2$ at its vertex. State its center, radius, and equation, and explain which side of the tangent contains the center.

Question 6 – Solution

Strategy Use curvature for the radius and the principal normal for the center direction.



Curvature and normal From Question 2, $\kappa(0) = 2$, so $\rho = 1/\kappa = 1/2$. At the vertex, the unit tangent points right and the parabola bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Center and equation

$$C = (0, 0) + \frac{1}{2} \langle 0, 1 \rangle = \left(0, \frac{1}{2} \right),$$

$$\boxed{x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4}}.$$

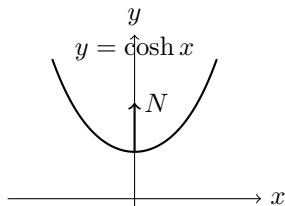
Geometry The center lies on the concave side of the curve, above the horizontal tangent. The circle has second-order contact with the parabola at the vertex.

Question 7

For the catenary $y = \cosh x$, find $\kappa(x)$ and the radius of curvature $\rho(x)$. Where is the curve most sharply bent, and what happens as $|x| \rightarrow \infty$?

Question 7 – Solution

Strategy Use the graph formula $\kappa = |y''|/(1 + y'^2)^{3/2}$ and a hyperbolic identity.



Calculation Since $y' = \sinh x$, $y'' = \cosh x$, and $1 + \sinh^2 x = \cosh^2 x$,

$$\boxed{\kappa(x) = \frac{\cosh x}{\cosh^3 x} = \operatorname{sech}^2 x}, \quad \boxed{\rho(x) = \cosh^2 x}.$$

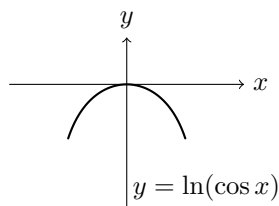
Analysis Curvature is largest at $x = 0$, where $\kappa = 1$. As $|x| \rightarrow \infty$, $\kappa \rightarrow 0$ and $\rho \rightarrow \infty$, so the arms become locally less sharply bent.

Question 8

On $-\frac{\pi}{2} < x < \frac{\pi}{2}$, determine the curvature of $y = \ln(\cos x)$. Explain why the stated domain makes the simplification unambiguous.

Question 8 – Solution

Strategy Differentiate twice and reduce the denominator using $1 + \tan^2 x = \sec^2 x$.



Calculation

$$y' = -\tan x, \quad y'' = -\sec^2 x.$$

Therefore

$$\kappa(x) = \frac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = \frac{\sec^2 x}{|\sec x|^3}.$$

On the stated interval, $\cos x > 0$ and $\sec x > 0$, so

$$\boxed{\kappa(x) = \cos x}.$$

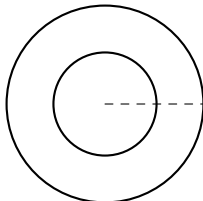
Behavior The maximum curvature is 1 at $x = 0$, while $\kappa \rightarrow 0$ as the graph approaches either vertical asymptote.

Question 9

A regular curve $\vec{r}(t)$ is enlarged by a factor $c > 0$ to form $\vec{q}(t) = c\vec{r}(t)$. Prove that $\kappa_q = \kappa_r/c$. What happens to the radius of curvature?

Question 9 – Solution

Strategy Track how first derivatives, second derivatives, cross products, and norms scale.



dilation by c lowers curvature by $1/c$

Scaling

$$\vec{q}' = c\vec{r}', \quad \vec{q}'' = c\vec{r}'', \quad \|\vec{q}' \times \vec{q}''\| = c^2 \|\vec{r}' \times \vec{r}''\|.$$

Also $\|\vec{q}'\|^3 = c^3 \|\vec{r}'\|^3$. Hence

$$\kappa_q = \frac{c^2}{c^3} \kappa_r = \frac{\kappa_r}{c}.$$

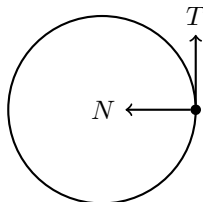
Radius Since $\rho = 1/\kappa$, $\boxed{\rho_q = c\rho_r}$. Enlargement makes every local osculating circle c times larger and the curve correspondingly less sharply bent.

Question 10

The circle of radius 2 is parameterized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3) \rangle$ for $t > 0$. Compute its curvature directly and explain why the changing speed does not affect the result.

Question 10 – Solution

Strategy Let $\theta = t^3$ and use the turning-angle relation $\kappa = \|d\mathbf{T}/dt\|/(ds/dt)$.



nonuniform parameter, same curvature

Tangent and rates Since $\theta' = 3t^2 > 0$,

$$\mathbf{T} = \langle -\sin \theta, \cos \theta \rangle, \quad \left\| \frac{d\mathbf{T}}{dt} \right\| = 3t^2.$$

The speed is $ds/dt = \|\vec{r}'\| = 2(3t^2) = 6t^2$.

Curvature

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{3t^2}{6t^2} = \frac{1}{2}.$$

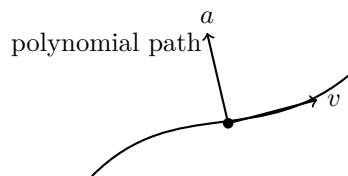
Interpretation Curvature measures direction change per unit distance, not per unit time. Both direction-change rate and speed acquire the same parameter factor, which cancels.

Question 1

A particle has position $\vec{r}(t) = \langle t^2 - 2t, t^3 - 3t, 2t \rangle$, $t \geq 0$. Find velocity, acceleration, and speed at $t = 2$. Determine whether it is speeding up or slowing down then.

Question 1 – Solution

Strategy Differentiate componentwise and use the sign of $\vec{v} \cdot \vec{a}$.



Kinematics

$$\vec{v} = \langle 2t - 2, 3t^2 - 3, 2 \rangle, \quad \vec{a} = \langle 2, 6t, 0 \rangle.$$

At $t = 2$, $\vec{v} = \langle 2, 9, 2 \rangle$, $\vec{a} = \langle 2, 12, 0 \rangle$, and

$\text{speed} = \|\vec{v}\| = \sqrt{89}.$

Speeding test $\vec{v} \cdot \vec{a} = 4 + 108 = 112 > 0$, so speed is increasing. The particle is

speeding up

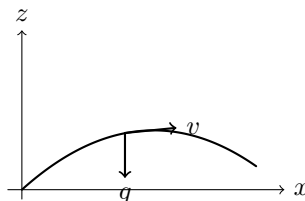
 at $t = 2$.

Question 2

A projectile is launched from $\langle 0, 0, 5 \rangle$ with initial velocity $\langle 12, 5, 20 \rangle$ ft/s. With gravity $\vec{a} = \langle 0, 0, -32 \rangle$ ft/s², find its position, time of impact with $z = 0$, and horizontal range vector.

Question 2 – Solution

Strategy Integrate constant acceleration and impose the ground condition on the vertical coordinate.



Position

$$\vec{r}(t) = \langle 12t, 5t, 5 + 20t - 16t^2 \rangle.$$

Impact Solve $5 + 20t - 16t^2 = 0$. The positive root is

$$t_* = \frac{5 + \sqrt{45}}{8} = \frac{5 + 3\sqrt{5}}{8}.$$

Horizontal range At impact,

$$\langle x(t_*), y(t_*) \rangle = \frac{5 + 3\sqrt{5}}{8} \langle 12, 5 \rangle.$$

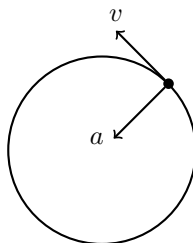
The horizontal velocity is constant because gravity has no horizontal component.

Question 3

A particle moves on the circle $\vec{r}(t) = \langle R \cos(\omega t), R \sin(\omega t) \rangle$, where $R, \omega > 0$. Find velocity, acceleration, speed, and show that acceleration points toward the center with magnitude v^2/R .

Question 3 – Solution

Strategy Differentiate twice and compare acceleration with the position vector.



Derivatives

$$\vec{v} = R\omega \langle -\sin \omega t, \cos \omega t \rangle, \quad \vec{a} = -R\omega^2 \langle \cos \omega t, \sin \omega t \rangle = -\omega^2 \vec{r}.$$

Thus $\boxed{v = R\omega}$ and acceleration is radially inward.

Magnitude

$$\|\vec{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \boxed{\frac{v^2}{R}}.$$

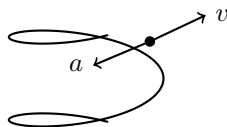
Velocity is tangent to the circle and perpendicular to the centripetal acceleration.

Question 4

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, find velocity and acceleration. Prove that its speed is constant and decide whether acceleration is entirely normal.

Question 4 – Solution

Strategy Compute $\vec{v} \cdot \vec{a}$ after differentiating.



Derivatives

$$\vec{v} = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{a} = \langle -3 \cos t, -3 \sin t, 0 \rangle.$$

The speed is $\|\vec{v}\| = 5$, independent of t .

Decomposition Since

$$\vec{v} \cdot \vec{a} = 9 \sin t \cos t - 9 \sin t \cos t = 0,$$

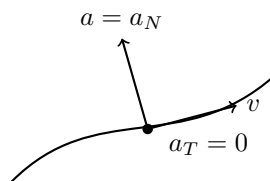
acceleration is perpendicular to velocity. Hence $a_T = 0$ and all acceleration is normal, with $a_N = \|\vec{a}\| = 3$.

Question 5

At an instant, a particle has velocity $\vec{v} = \langle 2, 1, 2 \rangle$ and acceleration $\vec{a} = \langle 1, 2, -2 \rangle$. Find the scalar tangential and normal components a_T, a_N . State what each says about the motion.

Question 5 – Solution

Strategy Use $a_T = (\vec{v} \cdot \vec{a})/\|\vec{v}\|$ and $a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$.



Components Here $\|\vec{v}\| = 3$, $\|\vec{a}\| = 3$, and $\vec{v} \cdot \vec{a} = 0$. Thus

$$\boxed{a_T = 0}, \quad \boxed{a_N = 3}.$$

Interpretation The instantaneous speed is not changing because the tangential component is zero. The full acceleration changes the direction of motion; it is perpendicular to the velocity.

Question 6

Prove that a particle with nonzero velocity is speeding up exactly when $\vec{v} \cdot \vec{a} > 0$, and slowing down exactly when $\vec{v} \cdot \vec{a} < 0$.

Question 6 – Solution

Strategy Differentiate speed through its square to avoid differentiating a square root prematurely.

$$v \cdot a > 0$$



Derivation Let $v = \|\vec{v}\| > 0$. Then

$$v^2 = \vec{v} \cdot \vec{v}.$$

Differentiating gives $2v \, dv/dt = 2\vec{v} \cdot \vec{a}$, so

$$\boxed{\frac{dv}{dt} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}}.$$

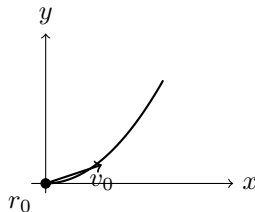
Conclusion The denominator is positive. Therefore the sign of the speed derivative is exactly the sign of $\vec{v} \cdot \vec{a}$: positive means speeding up, negative means slowing down, and zero means instantaneously constant speed.

Question 7

A particle has acceleration $\vec{a}(t) = \langle 2, 6t, e^t \rangle$, initial velocity $\vec{v}(0) = \langle 1, -1, 0 \rangle$, and initial position $\vec{r}(0) = \langle 2, 0, 3 \rangle$. Determine $\vec{v}(t)$ and $\vec{r}(t)$.

Question 7 – Solution

Strategy Integrate twice and use each vector initial condition to determine constants.



Velocity

$$\vec{v}(t) = \langle 2t + C_1, 3t^2 + C_2, e^t + C_3 \rangle.$$

Using $\vec{v}(0) = \langle 1, -1, 0 \rangle$ gives

$$\boxed{\vec{v}(t) = \langle 2t + 1, 3t^2 - 1, e^t - 1 \rangle}.$$

Position Integrating again and applying $\vec{r}(0) = \langle 2, 0, 3 \rangle$,

$$\boxed{\vec{r}(t) = \langle t^2 + t + 2, t^3 - t, e^t - t + 2 \rangle}.$$

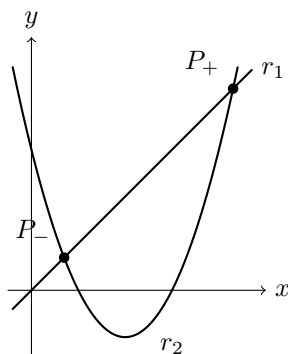
Verification Differentiating the final position recovers $\vec{v}$, and differentiating once more recovers the given acceleration.

Question 8

Particles follow $\vec{r}_1(t) = \langle t, t, 0 \rangle$ and $\vec{r}_2(t) = \langle 2 - t, t^2 - 1, 0 \rangle$. Do their paths intersect? Do the particles collide for $t \geq 0$? Clearly distinguish the two questions.

Question 8 – Solution

Strategy Path intersection may use different parameter values; collision requires equal positions at the same time.



Path intersection Write the first path as $y = x$. On the second, $x = 2 - u$ and $y = u^2 - 1$. Thus

$$u^2 + u - 3 = 0, \quad u = \frac{-1 \pm \sqrt{13}}{2}.$$

The full paths intersect at

$$\left(\frac{5 \mp \sqrt{13}}{2}, \frac{5 \mp \sqrt{13}}{2}, 0 \right).$$

If both paths are restricted to nonnegative times, only the point with $5 - \sqrt{13}$ remains.

Collision test At the same time t , equality of x -coordinates requires $t = 2 - t$, so $t = 1$. But

$$\vec{r}_1(1) = \langle 1, 1, 0 \rangle, \quad \vec{r}_2(1) = \langle 1, 0, 0 \rangle.$$

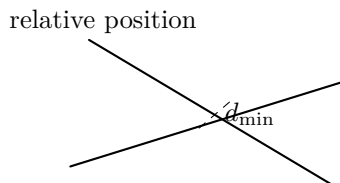
Therefore the particles do not collide. The geometric intersections occur at different parameter times.

Question 9

Two particles move according to $\vec{r}_1(t) = \langle t, 2t, 0 \rangle$ and $\vec{r}_2(t) = \langle 4 - t, 1, 2t \rangle$ for $t \geq 0$. Find the time and value of their minimum separation.

Question 9 – Solution

Strategy Minimize squared distance; it has the same minimizer as distance and avoids a square root.



Relative position

$$\vec{d}(t) = \vec{r}_1 - \vec{r}_2 = \langle 2t - 4, 2t - 1, -2t \rangle.$$

Thus

$$D^2(t) = (2t - 4)^2 + (2t - 1)^2 + (-2t)^2 = 12t^2 - 20t + 17.$$

Minimum The vertex occurs at $t = 20/24 = 5/6$. Then

$$D^2\left(\frac{5}{6}\right) = \frac{26}{3}, \quad \boxed{D_{\min} = \sqrt{\frac{26}{3}}}.$$

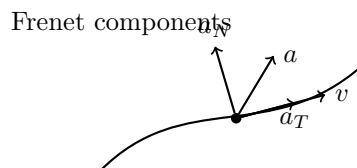
Because the quadratic opens upward and $5/6 \geq 0$, this is the required global minimum.

Question 10

For a regular motion with speed $v = ds/dt$, prove the Frenet acceleration formula $\vec{a} = (dv/dt)\mathbf{T} + \kappa v^2\mathbf{N}$. Identify the geometric role of each term.

Question 10 – Solution

Strategy Write velocity as speed times unit tangent and apply the product and chain rules.



Derivation

$$\vec{v} = v\mathbf{T}.$$

Therefore

$$\vec{a} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Since $d\mathbf{T}/ds = \kappa\mathbf{N}$ and $ds/dt = v$,

$$\boxed{\vec{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}}.$$

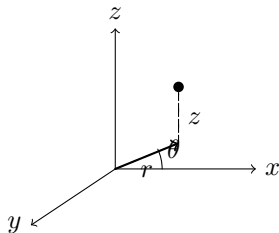
Meaning The tangential term changes speed. The normal term changes direction and points toward the local center of curvature; its magnitude grows quadratically with speed.

Question 1

Convert the rectangular point $(-3, 3\sqrt{3}, 5)$ to cylindrical coordinates using the convention $r \geq 0$ and $0 \leq \theta < 2\pi$. Verify by converting back.

Question 1 – Solution

Strategy Compute r first, then place the angle in the correct quadrant.



Conversion

$$r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = 6.$$

Because $x < 0$ and $y > 0$, the point lies in Quadrant II. With $\tan \theta = -\sqrt{3}$, $\theta = 2\pi/3$. Hence

$$(r, \theta, z) = \left(6, \frac{2\pi}{3}, 5\right).$$

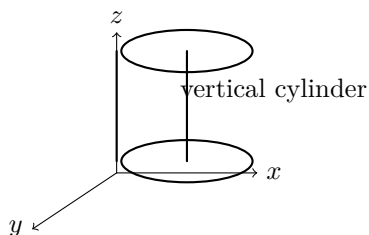
Verification $x = 6 \cos(2\pi/3) = -3$, $y = 6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 5$.

Question 2

Rewrite the surface $x^2 + y^2 = 4x$ in cylindrical coordinates. Identify the surface geometrically, including its axis and radius.

Question 2 – Solution

Strategy Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Cylindrical equation

$$r^2 = 4r \cos \theta.$$

The equation $r = 4 \cos \theta$ describes the non-axis points; $r = 0$ is already included, so

$$\boxed{r = 4 \cos \theta}.$$

Geometry In rectangular form,

$$(x - 2)^2 + y^2 = 4.$$

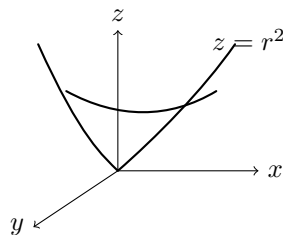
Thus the surface is a vertical circular cylinder of radius 2, whose axis is the line $\boxed{x = 2, y = 0}$ parallel to the z -axis.

Question 3

Convert the paraboloid $z = x^2 + y^2$ to cylindrical coordinates. Describe its traces for fixed z and for fixed θ , and explain why the new equation reveals its symmetry.

Question 3 – Solution

Strategy Use $x^2 + y^2 = r^2$ and interpret one coordinate at a time.



Equation

$$\boxed{z = r^2}, \quad r \geq 0.$$

Horizontal traces For $z = c \geq 0$, $r = \sqrt{c}$, a circle centered on the z -axis. There are no points for $c < 0$.

Vertical traces Fixing θ selects a vertical half-plane through the z -axis; within it, $z = r^2$ is one half of an upward-opening parabola (including its vertex).

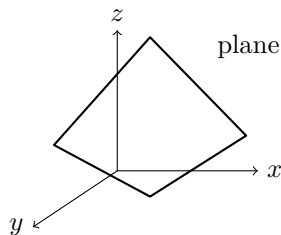
Symmetry The equation contains no θ , so rotating any point about the z -axis preserves the surface.

Question 4

Express the plane $x + y + z = 4$ in cylindrical coordinates. Solve for r where possible and state the angles at which that solved form breaks down.

Question 4 – Solution

Strategy Substitute $x = r \cos \theta$ and $y = r \sin \theta$.



Equation

$$r(\cos \theta + \sin \theta) + z = 4.$$

If $\cos \theta + \sin \theta \neq 0$, then

$$r = \frac{4 - z}{\cos \theta + \sin \theta}.$$

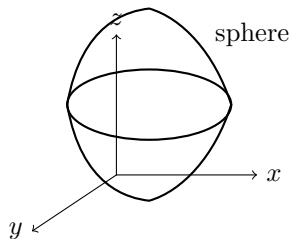
Exceptional angles $\cos \theta + \sin \theta = 0$ when $\theta = 3\pi/4$ or $7\pi/4$ modulo 2π . At those directions, the original cylindrical equation reduces to $z = 4$; dividing by the zero angular factor would incorrectly discard or invent points.

Question 5

Convert $x^2 + y^2 + z^2 = 6z$ to cylindrical coordinates and identify the surface. Describe the ranges of r and z on it.

Question 5 – Solution

Strategy Complete the square in z after replacing $x^2 + y^2$ by r^2 .



Equation

$$r^2 + z^2 = 6z \iff \boxed{r^2 + (z - 3)^2 = 9}.$$

Geometry This is a sphere of radius 3 centered at $(0, 0, 3)$. It is independent of θ , reflecting rotational symmetry about the z -axis.

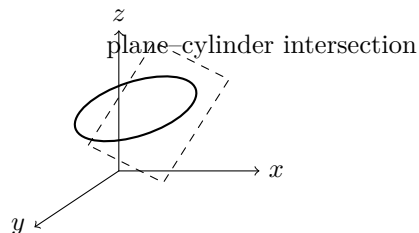
Ranges Since $r^2 \leq 9$, $\boxed{0 \leq r \leq 3}$. The vertical extent is $\boxed{0 \leq z \leq 6}$. At a fixed height z , the horizontal trace has radius $\sqrt{9 - (z - 3)^2}$.

Question 6

Find a one-parameter vector representation for the curve where the cylinder $x^2 + y^2 = 4$ meets the plane $z = x + y$. State an interval that traces the curve exactly once.

Question 6 – Solution

Strategy The cylinder fixes $r = 2$; use $\theta = t$ and substitute into the plane.



Coordinates

$$x = 2 \cos t, \quad y = 2 \sin t, \quad z = x + y = 2 \cos t + 2 \sin t.$$

Therefore

$$\boxed{\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 2 \cos t + 2 \sin t \rangle}, \quad 0 \leq t < 2\pi.$$

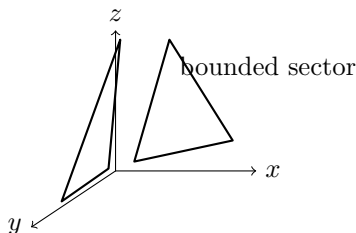
Verification The first two components satisfy $x^2 + y^2 = 4$, and the third is their sum. The half-open interval prevents duplicating the starting point.

Question 7

Describe in cylindrical inequalities the solid in the first octant that lies between the cylinders $r = 1$ and $r = 3$, above $z = 0$, and below the cone $z = 4 - r$. Explain each bound.

Question 7 – Solution

Strategy Translate the geometric restrictions directly into angular, radial, and vertical bounds.



Bounds First octant in the xy -plane gives $0 \leq \theta \leq \pi/2$. Between the cylinders gives $1 \leq r \leq 3$. At each (r, θ) , the solid lies from the xy -plane up to the cone:

$$\boxed{0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 3, \quad 0 \leq z \leq 4 - r.}$$

Consistency On $1 \leq r \leq 3$, the upper bound $4 - r$ remains positive. The cone is rotationally symmetric, while the first-octant restriction selects only a quarter-sector.

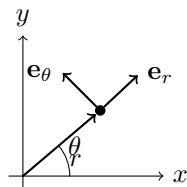
Question 8

A moving point has cylindrical coordinates $r(t), \theta(t), z(t)$. Derive its rectangular velocity and prove that its speed satisfies

$$\|\vec{v}\|^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2.$$

Question 8 – Solution

Strategy Differentiate $x = r \cos \theta$ and $y = r \sin \theta$, then simplify the sum of squares.



Velocity

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta, \quad \dot{z} = \dot{z}.$$

Speed identity Expanding $\dot{x}^2 + \dot{y}^2$ makes the cross terms cancel, while $\sin^2 \theta + \cos^2 \theta = 1$. Therefore

$$\|\vec{v}\|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

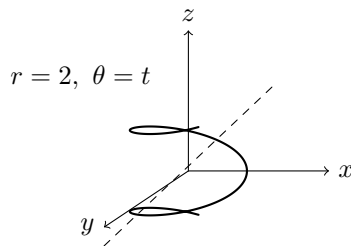
The radial, angular, and vertical velocity directions are mutually perpendicular.

Question 9

A curve is given in cylindrical form by $r = 2$, $\theta = t$, and $z = 3t$. Convert it to rectangular vector form, identify the curve, and find the vertical rise per complete revolution.

Question 9 – Solution

Strategy Substitute into $x = r \cos \theta$, $y = r \sin \theta$ and examine the change when t increases by 2π .



Rectangular form

$$\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle.$$

It is a circular helix of radius 2 around the z -axis.

Pitch One complete revolution corresponds to $\Delta t = 2\pi$. Therefore

$$\Delta z = 3(2\pi) = 6\pi.$$

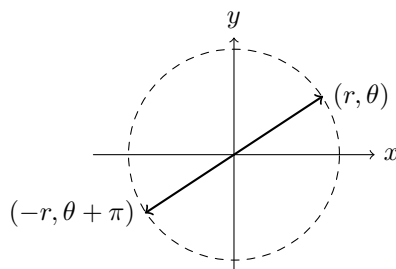
The constant r keeps the curve on a cylinder, while the linear z -coordinate produces uniform vertical advance.

Question 10

Explain why cylindrical coordinates are not unique if negative values of r are allowed. Prove the identity relating (r, θ, z) and $(-r, \theta + \pi, z)$, then state a standard uniqueness convention and its remaining exception.

Question 10 – Solution

Strategy Compare the rectangular coordinates produced by the two triples.



Identity

$$\begin{aligned}(-r) \cos(\theta + \pi) &= (-r)(-\cos \theta) = r \cos \theta, \\(-r) \sin(\theta + \pi) &= (-r)(-\sin \theta) = r \sin \theta.\end{aligned}$$

The z -coordinate is unchanged, so

$$\boxed{(r, \theta, z) \sim (-r, \theta + \pi, z)}.$$

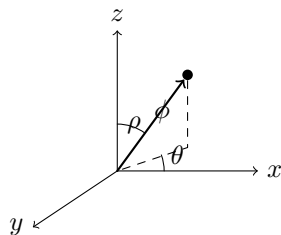
Convention Requiring $r \geq 0$ and $0 \leq \theta < 2\pi$ makes coordinates unique whenever $r > 0$. On the z -axis, $r = 0$ and every angle represents the same point, so θ remains intrinsically undetermined there.

Question 1

Convert the point $(1, \sqrt{3}, 2\sqrt{3})$ to spherical coordinates, using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. Verify the result.

Question 1 – Solution

Strategy Find distance, azimuth, and polar angle in that order.



Coordinates

$$\rho = \sqrt{1 + 3 + 12} = 4, \quad \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Also $\cos \phi = z/\rho = \sqrt{3}/2$, so $\phi = \pi/6$. Thus

$$\boxed{(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)}.$$

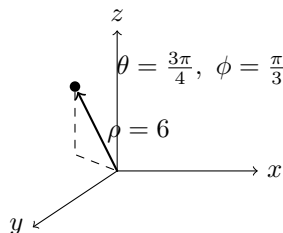
Verification Using $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ gives $1, \sqrt{3}, 2\sqrt{3}$, respectively.

Question 2

Convert the spherical point $(\rho, \theta, \phi) = (6, 3\pi/4, \pi/3)$ to rectangular and cylindrical coordinates. Keep all values exact.

Question 2 – Solution

Strategy First compute the cylindrical radius $r = \rho \sin \phi$ and height $z = \rho \cos \phi$.



Cylindrical form

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}, \quad z = 6 \cos \frac{\pi}{3} = 3.$$

Hence $(r, \theta, z) = (3\sqrt{3}, 3\pi/4, 3)$.

Rectangular form

$$x = 3\sqrt{3} \cos \frac{3\pi}{4} = -\frac{3\sqrt{6}}{2}, \quad y = 3\sqrt{3} \sin \frac{3\pi}{4} = \frac{3\sqrt{6}}{2}.$$

Therefore

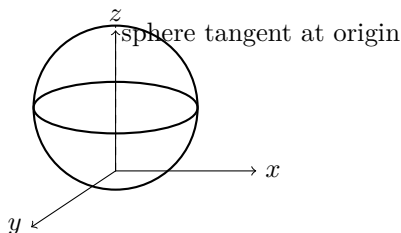
$$(x, y, z) = \left(-\frac{3\sqrt{6}}{2}, \frac{3\sqrt{6}}{2}, 3 \right).$$

Question 3

Convert the spherical equation $\rho = 4 \cos \phi$ to rectangular coordinates and identify the surface, including its center and radius. Explain why the origin belongs to it.

Question 3 – Solution

Strategy Multiply by ρ and use $\rho^2 = x^2 + y^2 + z^2$ and $\rho \cos \phi = z$.



Conversion

$$\rho^2 = 4\rho \cos \phi \implies x^2 + y^2 + z^2 = 4z.$$

Completing the square,

$$\boxed{x^2 + y^2 + (z - 2)^2 = 4}.$$

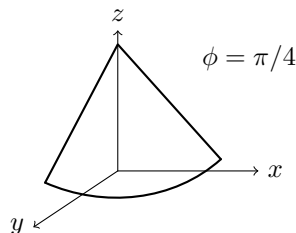
Geometry The surface is a sphere of radius 2 centered at $(0, 0, 2)$. It contains the origin because the distance from the origin to the center is exactly the radius. In the original spherical equation, $\rho = 0$ and $\phi = \pi/2$ represent the origin and satisfy $0 = 4 \cos(\pi/2)$.

Question 4

Identify and convert the surface $\phi = \pi/4$ to rectangular and cylindrical equations. Explain why it is only one nappe of a cone.

Question 4 – Solution

Strategy Relate the polar angle to cylindrical radius and height: $r = \rho \sin \phi$, $z = \rho \cos \phi$.



Conversion At $\phi = \pi/4$,

$$\frac{r}{z} = \tan \phi = 1,$$

so $\boxed{z = r}$ in cylindrical coordinates. Therefore

$$\boxed{z = \sqrt{x^2 + y^2}}.$$

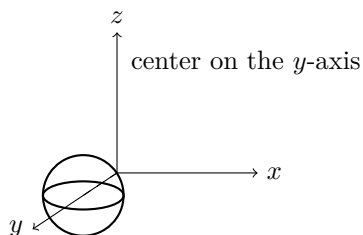
Geometry Because $\phi = \pi/4 < \pi/2$, $z = \rho \cos \phi \geq 0$. Thus this is the upper nappe only. Squaring to $z^2 = x^2 + y^2$ without retaining $z \geq 0$ would incorrectly add the lower nappe.

Question 5

Convert $\rho = 2 \sin \phi \sin \theta$ to rectangular coordinates and identify the surface. Which coordinate axis contains its center?

Question 5 – Solution

Strategy Recognize $\rho \sin \phi \sin \theta = y$, multiply the equation by ρ , and complete a square.



Conversion

$$\rho^2 = 2\rho \sin \phi \sin \theta = 2y.$$

Thus

$$x^2 + y^2 + z^2 = 2y \quad \Longleftrightarrow \quad \boxed{x^2 + (y - 1)^2 + z^2 = 1}.$$

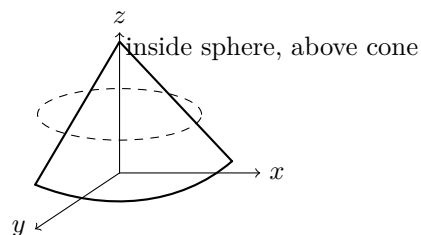
Geometry This is a sphere of radius 1 centered at $\boxed{(0, 1, 0)}$, so its center lies on the positive y -axis. The sphere is tangent to the origin.

Question 6

Describe with spherical inequalities the solid inside the sphere $\rho = 4$ and above the cone $\phi = \pi/3$. Include the full azimuthal range and explain the meaning of “above.”

Question 6 – Solution

Strategy A smaller polar angle lies closer to the positive z -axis.



Bounds Inside the sphere gives $0 \leq \rho \leq 4$. The full rotational sweep is $0 \leq \theta < 2\pi$. Above the cone means between the positive z -axis and the conical surface, so $0 \leq \phi \leq \pi/3$. Hence

$$0 \leq \rho \leq 4, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{3}.$$

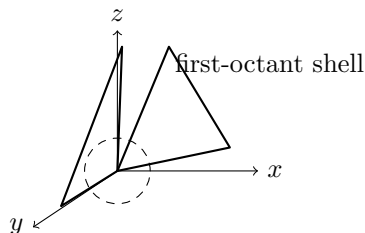
Boundary note At $\rho = 0$, angles are not unique, but including all angle values does not create additional geometric points.

Question 7

Write spherical-coordinate bounds for the first-octant portion of the shell between the spheres of radii 1 and 3. Justify the angular bounds.

Question 7 – Solution

Strategy Separate radial shell bounds from the two angular restrictions defining the first octant.



Bounds Between the spheres,

$$1 \leq \rho \leq 3.$$

The first quadrant in the xy -plane requires $0 \leq \theta \leq \pi/2$. Being above the xy -plane requires $0 \leq \phi \leq \pi/2$. Therefore

$$\boxed{1 \leq \rho \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}.}$$

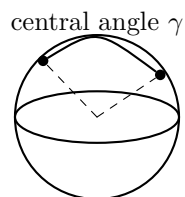
These bounds impose $x, y, z \geq 0$ while retaining every point of the radial shell in that octant.

Question 8

Two points have spherical coordinates $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$. Derive a formula for the square of their distance using their central angle.

Question 8 – Solution

Strategy Write the spherical unit direction vectors and use the dot-product form of the law of cosines.



Direction dot product If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Distance With position vectors $\vec{p}_1, \vec{p}_2$,

$$d^2 = \|\vec{p}_1 - \vec{p}_2\|^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma.$$

Therefore

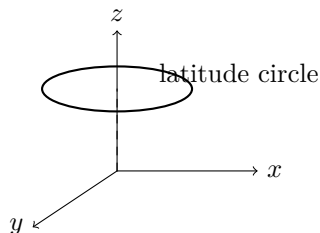
$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 [\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2)].$$

Question 9

A curve on the sphere $\rho = R$ has constant polar angle $\phi = \alpha$, while $\theta = t$ varies from 0 to 2π . Find a rectangular parameterization and the exact length of the curve.

Question 9 – Solution

Strategy Convert the fixed spherical coordinates and recognize a horizontal circle.



Parameterization

$$\vec{r}(t) = \langle R \sin \alpha \cos t, R \sin \alpha \sin t, R \cos \alpha \rangle, \quad 0 \leq t \leq 2\pi.$$

Length Differentiation gives

$$\|\vec{r}'(t)\| = R \sin \alpha$$

for $0 \leq \alpha \leq \pi$. Hence

$$L = 2\pi R \sin \alpha.$$

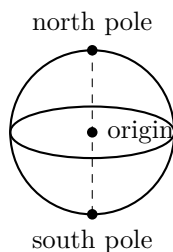
Geometry The latitude circle has Euclidean radius $R \sin \alpha$ and lies in the horizontal plane $z = R \cos \alpha$. At either pole, this radius and length are zero.

Question 10

Under the standard ranges $\rho \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$, discuss exactly where spherical coordinates fail to be unique and why.

Question 10 – Solution

Strategy Examine the conversion formulas at the origin and on the polar axis.



Origin If $\rho = 0$, then $x = y = z = 0$ for every θ and ϕ . Thus both angles are arbitrary at the origin.

Polar axis At $\phi = 0$ or $\phi = \pi$, $\sin \phi = 0$, so $x = y = 0$ regardless of θ . Therefore azimuth is arbitrary at the north and south polar rays.

Elsewhere For $\rho > 0$ and $0 < \phi < \pi$, the height uniquely determines $\phi = \arccos(z/\rho)$, and the nonzero projection (x, y) uniquely determines θ in $[0, 2\pi)$. Thus coordinates are unique away from the origin and z -axis.