

Question 1

Suppose that F and G are antiderivatives of the same function f on an interval I . Prove that there exists a constant C such that

$$F(x) = G(x) + C \quad \text{for all } x \in I.$$

Question 1 - Solution

Since F and G are both antiderivatives of f on I , we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I .

From the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = C.$$

Rewriting,

$$F(x) = G(x) + C.$$

$$\boxed{F(x) = G(x) + C}$$

Question 2

Suppose that $F'(x) = 0$ for all x in an interval I . Prove that F is constant on I .

Question 2 - Solution

Let $x_1, x_2 \in I$ with $x_1 < x_2$. We will show that

$$F(x_1) = F(x_2).$$

Since $F'(x) = 0$ for all $x \in I$, the function F is differentiable on I and continuous on the closed interval $[x_1, x_2]$.

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$F'(c) = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

But $F'(c) = 0$ by assumption, so

$$0 = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$F(x_2) - F(x_1) = 0.$$

Thus,

$$F(x_2) = F(x_1).$$

Because x_1 and x_2 were arbitrary points in I , the function F has the same value at every point of I .

$F \text{ is constant on } I$

Question 3

Let f be a continuous function on an interval I . Suppose that

$$\int f(x) \, dx = F(x) + C_1 \quad \text{and} \quad \int f(x) \, dx = G(x) + C_2$$

are two indefinite integrals of f . Prove that

$$F(x) - G(x) = \text{constant}.$$

Question 3 - Solution

Since both expressions represent antiderivatives of f , we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the difference of the two functions:

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I .

From the Mean Value Theorem, a function whose derivative is zero everywhere on an interval must be constant. Therefore, there exists a real number K such that

$$H(x) = K \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = K.$$

$F(x) - G(x) = \text{constant}$

Question 4

Suppose that f is continuous on an interval I and that

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Show that every solution of the differential equation

$$y' = f(x)$$

has the form

$$y = F(x) + C$$

for some constant C .

Question 4 - Solution

Let y be any solution of the differential equation

$$y' = f(x)$$

on the interval I .

Since $F'(x) = f(x)$ and $y'(x) = f(x)$ for all $x \in I$, we have

$$y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = y(x) - F(x).$$

Differentiate $H(x)$:

$$H'(x) = y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Thus, $H'(x) = 0$ on I .

By the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back gives

$$y(x) - F(x) = C.$$

Rewriting,

$$y = F(x) + C.$$

$$\boxed{y = F(x) + C}$$

Question 5

Suppose that f and g are continuous on a nondegenerate interval I , that $a \in I$, and that

$$\int_a^x f(t) \, dt = \int_a^x g(t) \, dt \quad \text{for all } x \in I.$$

Prove that

$$f(x) = g(x) \quad \text{for all } x \in I.$$

Question 5 - Solution

Set $F(x) = \int_a^x f(t) dt$ and $G(x) = \int_a^x g(t) dt$.

The hypothesis gives $F = G$ on I . At every interior point x of I , continuity of both integrands allows the Fundamental Theorem of Calculus:

$$f(x) = F'(x) = G'(x) = g(x).$$

At an included endpoint, take limits from the interior and use continuity of f and g . Thus

$$\boxed{f(x) = g(x) \quad \text{for every } x \in I.}$$

Question 6

Let f be continuous on an interval I , and fix one antiderivative F . Write the general antiderivative as

$$y(x) = F(x) + C.$$

Show that the condition

$$y(x_0) = y_0, \quad x_0 \in I,$$

uniquely determines C .

Question 6 - Solution

Fix an antiderivative F of f on I . Every other antiderivative has the form $y(x) = F(x) + C$.

Imposing $y(x_0) = y_0$ yields $F(x_0) + C = y_0$, so

$$\boxed{C = y_0 - F(x_0)}.$$

This value is unique because F , x_0 , and y_0 are fixed.

Question 7

Suppose that f is continuous on an interval I and that

$$\int_a^x f(t) dt = F(x) + C$$

for some constant C . Show that choosing a different lower limit $b \in I$ changes the antiderivative only by a constant.

Question 7 - Solution

Consider the two functions

$$F_1(x) = \int_a^x f(t) dt, \quad F_2(x) = \int_b^x f(t) dt.$$

Both F_1 and F_2 are antiderivatives of f on I by the Fundamental Theorem of Calculus.

Evaluate the difference between them:

$$F_1(x) - F_2(x) = \int_a^x f(t) dt - \int_b^x f(t) dt.$$

Using properties of definite integrals,

$$\int_a^x f(t) dt - \int_b^x f(t) dt = \int_a^b f(t) dt.$$

The right-hand side is a fixed real number that does not depend on x .

Therefore,

$$F_1(x) - F_2(x) = \text{constant}.$$

This shows that changing the lower limit of integration alters the antiderivative only by an additive constant.

$$\boxed{\int_a^x f(t) dt = \int_b^x f(t) dt + \text{constant}}$$

Question 8

Suppose that f is continuous on an interval I and that

$$\int f(x) \, dx = F(x) + C.$$

Prove that if

$$F'(x) = 0 \quad \text{for all } x \in I,$$

then

$$\int f(x) \, dx = C.$$

Question 8 - Solution

Since

$$\int f(x) dx = F(x) + C,$$

the function $F(x)$ is an antiderivative of f .

Given that

$$F'(x) = 0 \quad \text{for all } x \in I,$$

it follows from the Mean Value Theorem that F is constant on the interval I .

Thus, there exists a real number K such that

$$F(x) = K \quad \text{for all } x \in I.$$

Substitute this into the expression for the indefinite integral:

$$\int f(x) dx = K + C.$$

Since $K + C$ is itself a constant, we may write

$$\int f(x) dx = C'.$$

Therefore, if the derivative of an antiderivative is zero on an interval, the indefinite integral reduces to a constant.

$$\boxed{\int f(x) dx = C}$$

Question 9

Suppose that f is continuous on an interval I and that

$$\int f(x) dx = F(x) + C.$$

Show that if two antiderivatives of f agree at a single point, then they are identical on the entire interval.

Question 9 - Solution

Let F and G be two antiderivatives of f on the interval I . Then

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Assume that there exists a point $x_0 \in I$ such that

$$F(x_0) = G(x_0).$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, H has zero derivative on I , so H is constant on I . That is,

$$H(x) = K \quad \text{for all } x \in I.$$

Evaluate H at x_0 :

$$H(x_0) = F(x_0) - G(x_0) = 0.$$

Hence, $K = 0$, and therefore

$$F(x) - G(x) = 0 \quad \text{for all } x \in I.$$

This implies

$$F(x) = G(x) \quad \text{for all } x \in I.$$

$$\boxed{F(x) \equiv G(x)}$$

Question 10

Suppose that f is continuous on an interval I and that

$$\int f(x) dx = F(x) + C.$$

Show that the family of curves

$$y = F(x) + C$$

consists of vertical translations of a single curve.

Question 10 - Solution

Consider two members of the family of antiderivatives:

$$y_1 = F(x) + C_1, \quad y_2 = F(x) + C_2,$$

where C_1 and C_2 are constants.

Subtract the equations:

$$y_2 - y_1 = (F(x) + C_2) - (F(x) + C_1) = C_2 - C_1.$$

The difference $C_2 - C_1$ is a constant independent of x . Thus, for every value of x , the vertical distance between the two curves is the same.

This means that changing the constant of integration does not alter the shape of the graph, but only shifts it upward or downward by a fixed amount.

Therefore, the family of functions

$$y = F(x) + C$$

represents all vertical translations of a single curve.

Different values of C produce vertical shifts of the same graph