

Question 1

Prove that for any positive integer n ,

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Question 1 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k = 1, \quad \frac{1(1+1)}{2} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^n k + (n+1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + (n+1).$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k = (n+1) \left(\frac{n}{2} + 1 \right) = (n+1) \frac{n+2}{2}.$$

Thus,

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k = \frac{n(n+1)}{2}}$$

Question 2

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Question 2 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^2 = 1^2 = 1,$$

and

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^n k^2 + (n+1)^2.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k^2 = (n+1) \left(\frac{n(2n+1)}{6} + (n+1) \right).$$

Combine terms inside the parentheses:

$$\frac{n(2n+1) + 6(n+1)}{6} = \frac{2n^2 + 7n + 6}{6} = \frac{(2n+3)(n+2)}{6}.$$

Thus,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Question 3

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Question 3 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^3 = 1^3 = 1, \quad \left(\frac{1(1+1)}{2} \right)^2 = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3.$$

Factor out $(n+1)^2$:

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1 \right).$$

Combine terms:

$$\frac{n^2 + 4n + 4}{4} = \frac{(n+2)^2}{4}.$$

Thus,

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \cdot \frac{(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2} \right)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2}$$

Question 4

Prove that for any positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Question 4 - Solution

We prove the identity using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 (2k - 1) = 2(1) - 1 = 1,$$

and

$$1^2 = 1.$$

Thus, the statement holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^n (2k - 1) + (2(n + 1) - 1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} (2k - 1) = n^2 + (2n + 1).$$

Simplify:

$$n^2 + 2n + 1 = (n + 1)^2.$$

Thus,

$$\sum_{k=1}^{n+1} (2k - 1) = (n + 1)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the identity holds for all positive integers n .

$$\boxed{\sum_{k=1}^n (2k - 1) = n^2}$$

Question 5

Prove that for any positive integer n ,

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n.$$

Question 5 - Solution

We prove the identity using properties of summations.

Start with the given sum:

$$\sum_{k=1}^n (3k + 2).$$

Split the sum using linearity:

$$\sum_{k=1}^n (3k + 2) = \sum_{k=1}^n 3k + \sum_{k=1}^n 2.$$

Factor out constants:

$$\sum_{k=1}^n 3k = 3 \sum_{k=1}^n k, \quad \sum_{k=1}^n 2 = 2 \sum_{k=1}^n 1.$$

Use known summation formulas:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n 1 = n.$$

Substitute these results:

$$\sum_{k=1}^n (3k + 2) = 3 \cdot \frac{n(n+1)}{2} + 2n.$$

Thus,

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n.$$

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n$$

Question 6

Prove that for any positive integer n ,

$$\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{n}{n+1}.$$

Question 6 - Solution

Write out the first few terms of the sum:

$$\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

Notice that consecutive terms cancel:

$$-\frac{1}{2} + \frac{1}{2} = 0, \quad -\frac{1}{3} + \frac{1}{3} = 0, \quad \dots$$

After all cancellations, only the first and last terms remain:

$$1 - \frac{1}{n+1}.$$

Simplify:

$$1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{n}{n+1}}$$

Question 7

Prove that for any positive integer n ,

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Question 7 - Solution

Begin by rewriting the general term using partial fractions.

For each $k \geq 1$,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Substitute this into the sum:

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Write out the first few terms:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

All intermediate terms cancel, leaving

$$1 - \frac{1}{n+1}.$$

Simplify:

$$1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}}$$

Question 8

Prove that for any positive integer n and real number $r \neq 1$,

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}.$$

Question 8 - Solution

Let

$$S = \sum_{k=0}^n r^k = 1 + r + r^2 + \cdots + r^n.$$

Multiply both sides by r :

$$rS = r + r^2 + r^3 + \cdots + r^{n+1}.$$

Subtract the second equation from the first:

$$S - rS = (1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1}).$$

All intermediate terms cancel, leaving

$$S - rS = 1 - r^{n+1}.$$

Factor the left-hand side:

$$(1 - r)S = 1 - r^{n+1}.$$

Since $r \neq 1$, divide both sides by $1 - r$:

$$S = \frac{1 - r^{n+1}}{1 - r}.$$

Thus,

$$\boxed{\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}}$$

Question 9

Prove that for every positive integer n ,

$$\sum_{k=1}^n \frac{k}{k+1} = n - \sum_{k=1}^n \frac{1}{k+1}.$$

Question 9 - Solution

For every $k \geq 1$,

$$\frac{k}{k+1} = 1 - \frac{1}{k+1}.$$

Summing all n terms gives

$$\boxed{\sum_{k=1}^n \frac{k}{k+1} = n - \sum_{k=1}^n \frac{1}{k+1}.$$

Equivalently, the result is $n + 1 - H_{n+1}$, where $H_{n+1} = \sum_{j=1}^{n+1} 1/j$.
For $n = 1$, both sides of the corrected identity equal $1/2$.

Question 10

Prove that for any positive integer n ,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

Question 10 - Solution

Write both sums explicitly:

$$\sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$
$$\sum_{k=1}^n \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}.$$

Subtract the second sum from the first:

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

All common terms cancel:

$$\frac{1}{2} - \frac{1}{2} = 0, \quad \frac{1}{3} - \frac{1}{3} = 0, \quad \dots$$

Only the first term of the first sum and the last term of the second sum remain:

$$1 - \frac{1}{n+1}.$$

Thus,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

$$\boxed{1 - \frac{1}{n+1}}$$