

Question 1

Prove that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

Question 1 - Solution

We compare the growth rates of the numerator and denominator.

For $x > 0$, both $\ln x$ and x are differentiable. Apply L'Hôpital's Rule to the limit

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(x) = 1.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x}.$$

From basic limits,

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

$$\boxed{0}$$

Question 2

Prove that

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0 \quad \text{for any positive integer } n.$$

Question 2 - Solution

We compare the growth rates of the polynomial x^n and the exponential function e^x .

Both x^n and e^x are differentiable for all real x . Apply L'Hôpital's Rule repeatedly to the limit

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x}.$$

Differentiate the numerator and denominator once:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(e^x) = e^x.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{nx^{n-1}}{e^x}.$$

Apply L'Hôpital's Rule again. After n applications, the numerator becomes a constant:

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x}.$$

Since e^x grows without bound as $x \rightarrow \infty$,

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

$$\boxed{0}$$

Question 3

Prove that

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x = \frac{1}{2}.$$

Question 3 - Solution

We begin by algebraically simplifying the expression.

Consider

$$\sqrt{x^2 + x} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + x} - x = \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x}.$$

Simplify the numerator:

$$(x^2 + x) - x^2 = x.$$

Thus,

$$\sqrt{x^2 + x} - x = \frac{x}{\sqrt{x^2 + x} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + x} = x\sqrt{1 + \frac{1}{x}}.$$

Substitute:

$$\frac{x}{x\sqrt{1 + \frac{1}{x}} + x} = \frac{x}{x\left(\sqrt{1 + \frac{1}{x}} + 1\right)}.$$

Cancel x :

$$\frac{1}{\sqrt{1 + \frac{1}{x}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{\sqrt{1 + 0} + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 4

Prove that

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 3x}) = \frac{3}{2}.$$

Question 4 - Solution

We simplify the expression using algebraic manipulation.

Consider

$$x - \sqrt{x^2 - 3x}.$$

Multiply and divide by the conjugate:

$$x - \sqrt{x^2 - 3x} = \frac{(x - \sqrt{x^2 - 3x})(x + \sqrt{x^2 - 3x})}{x + \sqrt{x^2 - 3x}}.$$

Simplify the numerator:

$$x^2 - (x^2 - 3x) = 3x.$$

Thus,

$$x - \sqrt{x^2 - 3x} = \frac{3x}{x + \sqrt{x^2 - 3x}}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 - 3x} = x\sqrt{1 - \frac{3}{x}}.$$

Substitute into the expression:

$$\frac{3x}{x + x\sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1 - \frac{3}{x}}}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{3}{1 + \sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1}} = \frac{3}{2}.$$

$$\boxed{\frac{3}{2}}$$

Question 5

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x} - \sqrt{x^2 + x} \right) = 2.$$

Question 5 - Solution

We simplify the expression by factoring out x from each square root.

Consider

$$\sqrt{x^2 + 5x} - \sqrt{x^2 + x}.$$

Factor x^2 inside each square root:

$$\sqrt{x^2 \left(1 + \frac{5}{x}\right)} - \sqrt{x^2 \left(1 + \frac{1}{x}\right)}.$$

Since $x > 0$ for large x , this becomes

$$x \left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}} \right).$$

Multiply and divide by the conjugate:

$$x \cdot \frac{\left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}}\right) \left(\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}\right)}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Simplify the numerator:

$$\left(1 + \frac{5}{x}\right) - \left(1 + \frac{1}{x}\right) = \frac{4}{x}.$$

Thus the expression becomes

$$x \cdot \frac{\frac{4}{x}}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{1 + 1} = 2.$$

2

Question 6

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 4x + 1} - x \right) = 2.$$

Question 6 - Solution

Start with the expression

$$\sqrt{x^2 + 4x + 1} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + 4x + 1} - x = \frac{(\sqrt{x^2 + 4x + 1} - x)(\sqrt{x^2 + 4x + 1} + x)}{\sqrt{x^2 + 4x + 1} + x}.$$

Simplify the numerator:

$$(x^2 + 4x + 1) - x^2 = 4x + 1.$$

Thus,

$$\sqrt{x^2 + 4x + 1} - x = \frac{4x + 1}{\sqrt{x^2 + 4x + 1} + x}.$$

Factor x from the square root in the denominator:

$$\sqrt{x^2 + 4x + 1} = x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}}.$$

Substitute:

$$\frac{4x + 1}{x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + x} = \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1}.$$

Take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1} = \frac{4}{1 + 1} = 2.$$

2

Question 7

Prove that

$$\lim_{x \rightarrow \infty} x(\sqrt{x^2 + 1} - x) = \frac{1}{2}.$$

Question 7 - Solution

Begin with the expression

$$x(\sqrt{x^2 + 1} - x).$$

Factor the expression inside the parentheses using a conjugate:

$$\sqrt{x^2 + 1} - x = \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x} = \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \frac{1}{\sqrt{x^2 + 1} + x}.$$

Substitute this back:

$$x(\sqrt{x^2 + 1} - x) = \frac{x}{\sqrt{x^2 + 1} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + 1} = x\sqrt{1 + \frac{1}{x^2}}.$$

Thus,

$$\frac{x}{x\sqrt{1 + \frac{1}{x^2}} + x} = \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 8

Prove that

$$\lim_{x \rightarrow \infty} (\ln(x+1) - \ln x) = 0.$$

Question 8 - Solution

Combine the logarithms using logarithmic properties:

$$\ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right).$$

Rewrite the fraction:

$$\frac{x+1}{x} = 1 + \frac{1}{x}.$$

Thus, the expression becomes

$$\ln\left(1 + \frac{1}{x}\right).$$

Now take the limit as $x \rightarrow \infty$. Since

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

and the natural logarithm function is continuous at 1, we obtain

$$\lim_{x \rightarrow \infty} \ln\left(1 + \frac{1}{x}\right) = \ln(1) = 0.$$

0

Question 9

Prove that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0.$$

Question 9 - Solution

We compare the growth rates of the logarithmic function and the power function.

Both $\ln x$ and $\sqrt{x}$ are positive and differentiable for $x > 0$. Apply L'Hôpital's Rule to the limit

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x}.$$

Simplify:

$$\frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}.$$

Since

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0,$$

it follows that

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0.$$

0

Question 10

Prove that

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x = \frac{1}{e}.$$

Question 10 - Solution

Let

$$L = \lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x.$$

Take the natural logarithm:

$$\ln L = \lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+1} \right).$$

Rewrite the logarithm:

$$\ln \left(\frac{x}{x+1} \right) = \ln \left(\frac{1}{1 + \frac{1}{x}} \right) = -\ln \left(1 + \frac{1}{x} \right).$$

Thus,

$$\ln L = - \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Consider the limit

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Rewrite it as a quotient:

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}}.$$

This is an indeterminate form $\frac{0}{0}$, so apply L'Hôpital's Rule.

Differentiate numerator and denominator:

$$\frac{d}{dx} \ln \left(1 + \frac{1}{x} \right) = \frac{-1}{x(x+1)}, \quad \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{-1}{x(x+1)}}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1.$$

Therefore,

$$\ln L = -1.$$

Exponentiating both sides:

$$L = e^{-1} = \frac{1}{e}.$$

$$\boxed{\frac{1}{e}}$$