

Question 1

Assume that f is a continuous, nonnegative function on the interval $[a, b]$. Prove that the area of the region bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) \, dx.$$

Question 1 - Solution

We interpret area using approximating rectangles.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Since $f(x) \geq 0$, the area of the rectangle over $[x_{i-1}, x_i]$ with height $f(x_i^*)$ is

$$\text{Area}_i = f(x_i^*) \Delta x_i.$$

The total area of all rectangles is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

As the partition is refined (that is, as $\|P\| \rightarrow 0$), the rectangles better approximate the region under the curve. Because f is continuous on $[a, b]$, it is integrable, and the limit of these sums exists.

Taking the limit gives

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

By definition, this limit equals the exact area of the region under the curve.

Therefore, the area of the region bounded by $y = f(x)$, the x -axis, and $x = a$, $x = b$ is

$$A = \int_a^b f(x) dx.$$

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Question 2

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 2 - Solution

We approximate the solid of revolution using thin disks.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval.

Over the interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ generates a disk when revolved about the x -axis.

The radius of this disk is

$$r_i = f(x_i^*).$$

Thus, the volume of the disk is

$$\Delta V_i = \pi r_i^2 \Delta x_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

The total volume of all disks is approximated by the sum

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, this sum approaches the exact volume of the solid. Since f is continuous, the limit exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

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Question 3

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b].$$

Prove that the area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b (f(x) - g(x)) \, dx.$$

Question 3 - Solution

We approximate the region using vertical rectangles.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

At the point x_i^* , the height of the region between the curves is

$$f(x_i^*) - g(x_i^*).$$

Thus, the area of the corresponding rectangle is

$$\Delta A_i = (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

The total area of all rectangles is approximated by the sum

$$\sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, these rectangles more accurately approximate the region between the curves. Since f and g are continuous, the limit of the sums exists.

Taking the limit gives

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i = \int_a^b (f(x) - g(x)) \, dx.$$

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Question 4

Assume $0 \leq a < b$ and that f is continuous and nonnegative on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the y -axis is

$$V = 2\pi \int_a^b x f(x) dx.$$

Question 4 - Solution

Take a partition $a = x_0 < \cdots < x_n = b$ and let m_i, M_i be the minimum and maximum of f on $[x_{i-1}, x_i]$.

Since $a \geq 0$, the annular cylinders occupy disjoint radial intervals. Their volumes bound the desired volume:

$$\sum_i \pi(x_i^2 - x_{i-1}^2)m_i \leq V \leq \sum_i \pi(x_i^2 - x_{i-1}^2)M_i.$$

Let $c_i = (x_i + x_{i-1})/2$. Then $\pi(x_i^2 - x_{i-1}^2) = 2\pi c_i \Delta x_i$.

Uniform continuity of f makes $\max_i (M_i - m_i) \rightarrow 0$ as the mesh tends to zero, so the difference between the upper and lower bounds tends to zero.

Both bounds approach the Riemann integral of $2\pi x f(x)$. Hence

$$V = 2\pi \int_a^b x f(x) dx.$$

Question 5

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 5 - Solution

We derive the formula using the idea of approximating the solid with thin disks.

Partition the interval $[a, b]$ into subintervals

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ is approximately constant at height $f(x_i^*)$. When this vertical strip is revolved about the x -axis, it forms a thin circular disk.

The radius of the disk is

$$r_i = f(x_i^*).$$

The area of the circular face of the disk is

$$A_i = \pi r_i^2 = \pi [f(x_i^*)]^2.$$

Since the thickness of the disk is Δx_i , its volume is approximately

$$\Delta V_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

Adding the volumes of all disks gives an approximation to the total volume:

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and the maximum subinterval length approaches zero, these disks more accurately fill the solid. Because f is continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

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Question 6

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

Question 6 - Solution

We derive the formula using washers (disks with holes).

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval near x_i^* , the region between the curves has an outer radius

$$R_i = f(x_i^*)$$

and an inner radius

$$r_i = g(x_i^*).$$

When this vertical strip is revolved about the x -axis, it forms a thin washer.

The area of the outer disk is

$$\pi R_i^2 = \pi [f(x_i^*)]^2,$$

and the area of the inner disk (the hole) is

$$\pi r_i^2 = \pi [g(x_i^*)]^2.$$

Thus, the area of the washer is

$$\pi ([f(x_i^*)]^2 - [g(x_i^*)]^2).$$

Multiplying by the thickness Δx_i , the volume of the washer is

$$\Delta V_i = \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

Adding the volumes of all washers gives an approximation to the total volume:

$$\sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, the washers fill the solid more accurately. Since f and g are continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

Question 7

Prove that the area of a circle of radius R is

$$A = \pi R^2.$$

Question 7 - Solution

We derive the area formula using integration.

Consider a circle of radius R centered at the origin. Its equation is

$$x^2 + y^2 = R^2.$$

Solving for y gives the upper semicircle:

$$y = \sqrt{R^2 - x^2}.$$

The area of the entire circle is twice the area of the upper semicircle:

$$A = 2 \int_{-R}^R \sqrt{R^2 - x^2} dx.$$

To evaluate the integral, use the substitution

$$x = R \sin \theta, \quad dx = R \cos \theta d\theta.$$

When $x = -R$, $\theta = -\frac{\pi}{2}$, and when $x = R$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} \sqrt{R^2 - R^2 \sin^2 \theta} (R \cos \theta) d\theta.$$

Simplify:

$$\sqrt{R^2(1 - \sin^2 \theta)} = R \cos \theta.$$

Thus,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta = \frac{R^2}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta)) d\theta.$$

Evaluate:

$$\int_{-\pi/2}^{\pi/2} 1 d\theta = \pi, \quad \int_{-\pi/2}^{\pi/2} \cos(2\theta) d\theta = 0.$$

So,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \frac{R^2}{2} \pi.$$

Finally, multiply by 2 to get the area of the full circle:

$$A = 2 \cdot \frac{R^2}{2} \pi = \pi R^2.$$

$$\boxed{A = \pi R^2}$$

Question 8

Prove that the volume of a right circular cylinder with radius R and height h is

$$V = \pi R^2 h.$$

Question 8 - Solution

We derive the volume formula using cross-sectional areas and integration.

Place the cylinder so that its base lies in the xy -plane and its axis is along the x -axis. Then the cylinder extends from $x = 0$ to $x = h$.

At any position x with $0 \leq x \leq h$, a cross-section perpendicular to the x -axis is a circle of radius R .

The area of this circular cross-section is constant and equal to

$$A(x) = \pi R^2.$$

The volume of the cylinder can be approximated by slicing it into thin slabs of thickness dx . Each slab has volume approximately

$$dV = A(x) dx = \pi R^2 dx.$$

Adding the volumes of all such slabs gives

$$V = \int_0^h \pi R^2 dx.$$

Since πR^2 is constant, evaluate the integral:

$$V = \pi R^2 \int_0^h dx = \pi R^2 [x]_0^h = \pi R^2 h.$$

$$\boxed{V = \pi R^2 h}$$

Question 9

Prove that the volume of a right circular cone with base radius R and height h is

$$V = \frac{1}{3}\pi R^2 h.$$

Question 9 - Solution

We derive the formula using cross-sectional disks and integration.

Place the cone so that its tip is at the origin and its base lies in the plane $x = h$. The axis of the cone lies along the x -axis.

At a distance x from the tip, where $0 \leq x \leq h$, the radius of the cone is determined by similar triangles.

Since the radius grows linearly from 0 to R over the height h , the radius at position x is

$$r(x) = \frac{R}{h}x.$$

A cross-section perpendicular to the x -axis at position x is a circle of radius $r(x)$. Its area is

$$A(x) = \pi[r(x)]^2 = \pi \left(\frac{R}{h}x \right)^2 = \pi \frac{R^2}{h^2}x^2.$$

An infinitesimal slice of thickness dx has volume

$$dV = A(x) dx = \pi \frac{R^2}{h^2}x^2 dx.$$

Integrate from $x = 0$ to $x = h$ to obtain the total volume:

$$V = \int_0^h \pi \frac{R^2}{h^2}x^2 dx.$$

Factor out the constants:

$$V = \pi \frac{R^2}{h^2} \int_0^h x^2 dx.$$

Evaluate the integral:

$$\int_0^h x^2 dx = \left[\frac{x^3}{3} \right]_0^h = \frac{h^3}{3}.$$

Substitute back:

$$V = \pi \frac{R^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi R^2 h.$$

$$\boxed{V = \frac{1}{3}\pi R^2 h}$$

Question 10

Prove that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3.$$

Question 10 - Solution

We derive the formula using the method of cross-sectional disks.

Place the sphere of radius R so that it is centered at the origin. Its equation is

$$x^2 + y^2 + z^2 = R^2.$$

Fix a value of x with $-R \leq x \leq R$. A cross-section perpendicular to the x -axis is a circle whose radius depends on x .

Solving the equation of the sphere for $y^2 + z^2$ gives

$$y^2 + z^2 = R^2 - x^2.$$

Thus, the radius of the cross-sectional disk at position x is

$$r(x) = \sqrt{R^2 - x^2}.$$

The area of this circular cross-section is

$$A(x) = \pi[r(x)]^2 = \pi(R^2 - x^2).$$

An infinitesimally thin slice of thickness dx has volume

$$dV = A(x) dx = \pi(R^2 - x^2) dx.$$

Integrate from $x = -R$ to $x = R$ to obtain the total volume:

$$V = \int_{-R}^R \pi(R^2 - x^2) dx.$$

Factor out π :

$$V = \pi \int_{-R}^R (R^2 - x^2) dx.$$

Evaluate the integral:

$$\int_{-R}^R R^2 dx = 2R^3, \quad \int_{-R}^R x^2 dx = \frac{2R^3}{3}.$$

Thus,

$$\int_{-R}^R (R^2 - x^2) dx = 2R^3 - \frac{2R^3}{3} = \frac{4R^3}{3}.$$

Substitute back:

$$V = \pi \cdot \frac{4R^3}{3} = \frac{4}{3}\pi R^3.$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$